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NONNEGATIVE SIGNED EDGE DOMINATION IN GRAPHS

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Abstract. A nonnegative signed edge dominating function of a graph $G = (V, E)$ is a function $f : E \to \{-1, 1\}$ such that $\sum_{e' \in N[e]} f(e') \geq 0$ for each $e \in E$, where $N[e]$ is the closed neighborhood of e. The weight of a nonnegative signed edge dominating function f is $\omega(f) = \sum_{e \in E} f(e)$. The nonnegative signed edge domination number $\gamma'_{ns}(G)$ of G is the minimum weight of a nonnegative signed edge dominating function of G. In this paper, we prove that for every tree T of order $n \geq 3$, $1 - \frac{n}{3} \leq \gamma'_{ns}(T) \leq \lfloor \frac{n-1}{3} \rfloor$. Also we present some sharp bounds for the nonnegative signed edge domination number. In addition, we determine the nonnegative signed edge domination number for the complete graph, and the complete bipartite graph $K_{n,n}$.

1. Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$ and the size $|E|$ of G is denoted by $m = m(G)$. For every vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The minimum and maximum degrees of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. Two edges e_1, e_2 of G are called adjacent if they are distinct and have a common end-vertex. For every edge $e \in E$, the open neighborhood $N_G(e) = N(e)$ is the set of all edges adjacent to e and its closed neighborhood is $N_G[e] = N[e] = N(e) \cup \{e\}$. If $X \subseteq V(G)$, then $G[X]$ is the induced subgraph. Any spanning subgraph of a graph G is referred to as a factor of G. A k-regular factor is called a k-factor. We write K_n for the complete graph of order n, $K_{p,q}$ for the complete bipartite graph with partite sets X and Y,

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where $|X| = p$ and $|Y| = q$, C_n for a cycle of length n and P_n for a path of length $n-1$. For a subset $S \subseteq E$ of edges of a graph G and a function $f : E \to \mathbb{R}$, we define $f(S) = \sum_{x \in S} f(x)$. For terminology and notation on graph theory not defined here, the reader is referred to [7, 8, 13].

A signed dominating function (SDF) on a graph G is a function $f\colon V \to \{-1, 1\}$ such that $\sum_{u \in N[v]} f(u) \geq 1$ for each vertex $v \in V$. The weight of an SDF is the sum of its function values over all vertices. The signed domination number of G, denoted by $\gamma_s(G)$, is the minimum weight of an SDF in G. The signed domination number was introduced by Dunbar et al. [6].

For a positive integer k, a signed edge k-dominating function (SEkDF) on a graph G is a function $f\colon E \to \{-1, 1\}$ such that $\sum_{e' \in N[e]} f(e') \geq k$ for each edge $e \in E$. The weight of an SEkDF is the sum of its function values over all edges. The signed edge k-domination number of G, denoted by $\gamma'_{sk}(G)$, is the minimum weight of an SEkDF in G. The signed edge k-domination number was introduced by Carney et al. [2]. The special case $k = 1$ was introduced and investigated in [15]. For more information the reader may also consult [3, 4, 10, 11, 14, 16].

A nonnegative signed dominating function (NNSDF) on a graph G is a function $f\colon V \to \{-1, 1\}$ such that $\sum_{x \in N[v]} f(x) \geq 0$ for each vertex $v \in V$. The weight of an NNSDF is the sum of its function values over all vertices. The nonnegative signed domination number of G, denoted by $\gamma_s^{NN}(G)$, is the minimum weight of an NNSDF in G. The nonnegative signed domination number was introduced by Huang et al. [9]. For more information the reader may also consult [1, 5].

A nonnegative signed edge dominating function (NNSEDF) on a graph G is a function $f\colon E \to \{-1, 1\}$ such that $\sum_{e' \in N[e]} f(e') \geq 0$ for each edge $e \in E$. The weight of an NNSEDF is the sum of its function values over all edges. The nonnegative signed edge domination number of G, denoted by $\gamma'_{ns}(G)$, is the minimum weight of an NNSEDF in G. A $\gamma'_{ns}(G)$-function is an NNSEDF on G of weight $\gamma'_{ns}(G)$. For an NNSEDF f, let $E_i = E_i(f) = \{e \in E : f(e) = i\}$ for $i = -1, 1$.

The aim of this paper, is to initiate the study of the nonnegative signed edge domination number. We prove that for every tree T of order $n \geq 3$, $1 - \frac{n}{3} \leq \gamma'_{ns}(T) \leq \left\lfloor \frac{n-1}{3} \right\rfloor$. Also we present some sharp bounds for the nonnegative signed edge domination number. In addition, we determine the nonnegative signed edge domination number for the complete graph, and the complete bipartite graph $K_{n,n}$.

We make use of the following results in this paper.

Observation 1.1. Let G be a connected graph of order $n \geq 2$. If f is an NNSEDF on G, then:

(a) $m = |E_{-1}| + |E_1|$;
(b) $\omega(f) = |E_1| - |E_{-1}|$.

Observation 1.2. If G is a connected graph of size $m \geq 1$, then $\gamma'_{ns}(G) \equiv m \pmod 2$.

Proposition 1.1. [1] *For any even graph G, $\gamma_s^{NN}(G) = \gamma_s(G)$.*

Proposition 1.2. [6] *For $n \geq 3$, $\gamma_s(C_n) = \frac{n}{3}$ when $n \equiv 0 \pmod 3$, $\gamma_s(C_n) = \left\lfloor \frac{n}{3} \right\rfloor + 1$ when $n \equiv 1 \pmod 3$, and $\gamma_s(C_n) = \left\lfloor \frac{n}{3} \right\rfloor + 2$ when $n \equiv 2 \pmod 3$.*

Proposition 1.3. [9] *For any path P_n, we have $\gamma_s^{NN}(P_n) = n - 2\left\lceil \frac{n}{3} \right\rceil$.*

Proposition 1.4. [9] *Let K_n be a complete graph. Then $\gamma_s^{NN}(K_n) = 0$ when n is even and $\gamma_s^{NN}(K_n) = 1$ when n is odd.*

The line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. It is easy to see that $L(K_{1,n}) = K_n$, $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$. The proof of the following result is straightforward and therefore omitted.

Observation 1.3. For any connected graph G of order $n \geq 3$, $\gamma'_{ns}(G) = \gamma_s^{NN}(L(G))$.

Using Observation 1.3, Propositions 1.1, 1.2, 1.3 and 1.4, we obtain the next results.

Corollary 1.1. *For $n \geq 1$, $\gamma'_{ns}(K_{1,n}) = 0$ when n is even and $\gamma'_{ns}(K_{1,n}) = 1$ when n is odd.*

Corollary 1.2. *For $n \geq 2$, $\gamma'_{ns}(P_n) = n - 1 - 2\left\lceil \frac{n-1}{3} \right\rceil$.*

Corollary 1.3. *For $n \geq 3$, $\gamma'_{ns}(C_n) = \frac{n}{3}$ when $n \equiv 0 \pmod 3$, $\gamma'_{ns}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor + 1$ when $n \equiv 1 \pmod 3$ and $\gamma'_{ns}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor + 2$ when $n \equiv 2 \pmod 3$.*

2. Trees

In this section we prove that for every tree T of order $n \geq 3$, $1 - \frac{n}{3} \leq \gamma'_{ns}(T) \leq \left\lfloor \frac{n-1}{3} \right\rfloor$.

A vertex of degree one is called a leaf, and its neighbor is called a support vertex. If v is a support vertex, then L_v will denote the set of all leaves adjacent to v. A support vertex v is called a strong support vertex if $|L_v| > 1$. A strong support vertex is said to be an end-strong support vertex if all its neighbors except one of them are leaves. For a vertex v in a rooted tree T, let $C(v)$ denotes the set of children of v, $D(v)$ denotes the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the depth of v, depth(v), is the largest distance from v to a vertex in $D(v)$. The maximal subtree at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v.

For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves.

Proposition 2.1. *For $r \geq s \geq 1$, $\gamma'_{ns}(S(r, s)) = 0$ when $r+s$ is odd and $\gamma'_{ns}(S(r, s)) = 1$ when $r + s$ is even.*

Proof. Let $S(r, s)$ be a double star whose central vertices are x, y with r pendant edges xx_i and s pendant edges yy_i. Since $S(1, 1) = P_4$, we have $\gamma'_{ns}(P_4) = 1$ by Corollary 1.2. Assume that f is a $\gamma'_{ns}(S(r, s))$-function. Consider the following two cases.

Case 1. $r + s$ is odd.

We may assume that r is odd and s is even (the case r is even and s is odd, is similar). Define $g : E(S(r,s)) \to \{-1,1\}$ by $g(xy) = 1$, $g(xx_i) = (-1)^i$ for $1 \le i \le r$ and $g(yy_j) = (-1)^j$ for $1 \le j \le s$. Obviously, g is an NNSEDF of $S(r,s)$ of weight 0 which implies $\gamma'_{ns}(S(r,s)) \le 0$. Now, we show that $\gamma'_{ns}(S(r,s)) = \omega(f) \ge 0$ in this case. Since $N[xy] = E(S(r,s))$, we have

$$\gamma'_{ns}(S(r,s)) = \omega(f) = f(E(S(r,s))) = f(N[xy]) \ge 0.$$

Hence $\gamma'_{ns}(S(r,s)) = 0$ when $r+s$ is odd.

Case 2. $r+s$ is even.

First let r and s be odd. Define $g : E(S(r,s)) \to \{-1,1\}$ by $g(xy) = -1$, $g(xx_i) = (-1)^{i+1}$ for $1 \le i \le r$ and $g(yy_j) = (-1)^{j+1}$ for $1 \le j \le s$. Obviously, g is an NNSEDF of $S(r,s)$ of weight 1 and hence $\gamma'_{ns}(S(r,s)) \le 1$. Now let r and s be even. Define $g : E(S(r,s)) \to \{-1,1\}$ by $g(xy) = 1$, $g(xx_i) = (-1)^i$ for $1 \le i \le r$ and $g(yy_j) = (-1)^j$ for $1 \le j \le s$. Obviously, g is an NNSEDF of $S(r,s)$ of weight 1 and hence $\gamma'_{ns}(S(r,s)) \le 1$. Now, we show that $\gamma'_{ns}(S(r,s)) = \omega(f) \ge 1$ when $r+s$ is even. Since $N[xy] = E(S(r,s))$, we have $\omega(f) = f(N[xy]) \ge 0$. By Observation 1.2, $\gamma'_{ns}(S(r,s)) = \omega(f) \equiv n \pmod 2$. Hence $\gamma'_{ns}(S(r,s)) \ge 1$ and $\gamma'_{ns}(S(r,s)) = 1$ when $r+s$ is even. This complete the proof. □

Let $r \ge 0$ be an integer and T_r be the tree obtained from the star $K_{1,2r+1}$ with central vertex x and leaves $x_1, x_2, \ldots, x_{2r+1}$ by adding exactly one pendant edge at x_i such that $x_i y_i \in E(T_r)$ for each $1 \le i \le r+1$ (Figure 1). Suppose $\mathcal{F} = \{T_r \mid r \ge 0\}$.

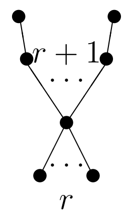

FIGURE 1. Family $\mathcal{F}$

Example 2.1. If $T \in \mathcal{F}$, then $\gamma'_{ns}(T) = 1 - \frac{|V(T)|}{3}$.

Proof. Let $T \in \mathcal{F}$. Then $T = T_r$ for some integer $r \ge 0$. To show that $\gamma'_{ns}(T) \le 1 - \frac{|V(T)|}{3}$, define $f : E(T) \to \{-1,1\}$ by $f(xx_i) = 1$ for each $1 \le i \le r+1$ and $f(e) = -1$ otherwise. Clearly, f is an NNSEDF of T of weight $1 - \frac{|V(T)|}{3}$ and so $\gamma'_{ns}(T) \le 1 - \frac{|V(T)|}{3}$. Now, we show that $\gamma'_{ns}(T) \ge 1 - \frac{|V(T)|}{3}$. Let f be a $\gamma'_{ns}(T)$-function. By definition, $f(N[x_i y_i]) = f(xx_i) + f(x_i y_i) \ge 0$ for each $1 \le i \le r+1$. This implies that

$$\gamma'_{ns}(T) = \omega(f) = \sum_{i=1}^{r+1} f(N[x_i y_i]) + \sum_{i=r+2}^{2r+1} f(xx_i) \ge -r = 1 - \frac{|V(T)|}{3}.$$

Thus $\gamma'_{ns}(T) = 1 - \frac{|V(T)|}{3}$ and the proof is complete. □

The next result is an immediate consequence of Example 2.1.

Corollary 2.1. *For every integer $r \geq 0$, there exists a connected graph G such that $\gamma'_{ns}(G) = -r$.*

Theorem 2.1. *Let T be a tree of order $n \geq 2$. Then*

$$\gamma'_{ns}(T) \geq 1 - \frac{n}{3}.$$

Proof. The proof is by induction on n. If $\mathrm{diam}(T) \leq 3$, then T is a star or a double star and by Corollary 1.1 and Proposition 2.1, we have $\gamma'_{ns}(T) \geq 1 - \frac{n}{3}$ with equality if $T = K_{1,2}$. Hence the statement holds for all trees T with $\mathrm{diam}(T) \leq 3$ as well as all trees of order $n \leq 4$. Assume that T is an arbitrary tree of order $n \geq 5$ and $\mathrm{diam}(T) \geq 4$. Let f be a $\gamma'_{ns}(T)$-function. We proceed further with a series of claims that we may assume satisfied by the tree T and the NNSEDF f.

Claim 1. T has no non-pendant edge e with $f(e) = -1$.
Proof. Assume that $e = u_1u_2 \in E(T)$ is a non-pendant edge in T with $f(e) = -1$. Let $T - e = T_{u_1} \cup T_{u_2}$, where T_{u_i} is the component of $T - e$ containing u_i for $i = 1, 2$. Obviously, $\gamma'_{ns}(T) = f(E(T_{u_1})) + f(E(T_{u_2})) - 1$ and the function f, restricted to T_{u_i} is an NNSEDF and hence $\gamma'_{ns}(T_{u_i}) \leq f(E(T_{u_i}))$ for $i = 1, 2$. Clearly, $|V(T_{u_i})| \geq 2$ for each $i = 1, 2$. By the induction hypothesis we obtain

$$\gamma'_{ns}(T) \geq \gamma'_{ns}(T_{u_1}) + \gamma'_{ns}(T_{u_2}) - 1 \geq 1 - \frac{n}{3}.$$

Claim 2. T has no two pendant edges vu_1 and vu_2 with $f(vu_1) = 1$ and $f(vu_2) = -1$.
Proof. Let vu_1 and vu_2 be two pendant edges in T such that $f(vu_1) = 1$ and $f(vu_2) = -1$. Let $T' = T - \{u_1, u_2\}$. Since $|V(T)| \geq 5$, we have $|V(T')| \geq 3$. Clearly, the function f, restricted to T' is an NNSEDF on T', and by the induction hypothesis we have

$$\gamma'_{ns}(T) \geq \gamma'_{ns}(T') \geq 1 - \frac{n-2}{3} > 1 - \frac{n}{3}.$$

We conclude from Claim 2 that all pendant edges at a vertex are either -1 edges or positive edges. Let $v_1v_2 \ldots v_d$ be a diametral path in T chosen to maximize $\deg_T(v_2)$ and root T at v_d. Assume that $E(v)$ is the set of all edges incident to the vertex v. Since f is a $\gamma'_{ns}(T)$-function, we have $f(v) = \sum_{e \in E(v)} f(e) \geq 0$ for every support vertex v.

Claim 3. $\deg(v_2) = 2$.
Proof. Let $\deg(v_2) \geq 3$. Since v_2 is a support vertex, $f(v_2) = \sum_{e \in E(v_2)} f(e) \geq 0$. It follows that all pendant edges at v_2 are 1 edges. In particular $f(v_1v_2) = 1$. If there is no -1 pendant edge at v_3, then obviously the function f, restricted to $T' = T - v_1$ is an NNSEDF of T' and $\gamma'_{ns}(T) = \omega(f) = \omega(f|_{T'}) + 1$. By the induction hypothesis we have

$$\gamma'_{ns}(T) \geq 1 - \frac{n-1}{3} + 1 > 1 - \frac{n}{3}.$$

Let v_3z be a -1 pendant edge at v_3, and let $T' = T - \{v_1, z\}$. Obviously, the function f, restricted to $T' = T - \{v_1, z\}$ is an NNSEDF of T' and $\gamma'_{ns}(T) = \omega(f) = \omega(f|_{T'})$. By the induction hypothesis we have

$$\gamma'_{ns}(T) \geq 1 - \frac{n-2}{3} > 1 - \frac{n}{3}.$$

Claim 4. $\deg(v_3) = 2$.
Proof. Let $\deg(v_3) \geq 3$. By the choice of the diametral path, every support vertex adjacent to v_3 has degree 2. Clearly $f(v_2) \geq 0$. First let $f(v_2) = 2$. Then $f(v_1v_2) = f(v_2v_3) = 1$. If there is no -1 pendant edge at v_3, then the function f, restricted to $T' = T - v_1$ is an NNSEDF of T' of weight $\omega(f) - 1$ and it follows from the induction hypothesis that $\gamma'_{ns}(T) > 1 - \frac{n}{3}$. Hence, we assume that there is a -1 pendant edge at v_3, say v_3z. Then the function $f|_{T-\{v_1,z\}}$ is an NNSEDF of $T - \{v_1, z\}$ and by the induction hypothesis we obtain $\gamma'_{ns}(T) > 1 - \frac{n}{3}$. Now, let $f(v_2) = 0$. Then $f(v_1v_2) = -1$ and $f(v_2v_3) = 1$. First assume that there is no -1 pendant edge at v_3. If there is no -1 pendant edge at v_4, then the function f, restricted to $T' = T - \{v_1, v_2\}$ is an NNSEDF of T' of weight $\omega(f)$ and it follows from the induction hypothesis that $\gamma'_{ns}(T) > 1 - \frac{n}{3}$. Hence, we assume that there is a -1 pendant edge at v_4, say v_4z. Then the function $f|_{T-\{v_1,v_2,z\}}$ is an NNSEDF of $T - \{v_1, v_2, z\}$ and by the induction hypothesis we obtain $\gamma'_{ns}(T) \geq 1 - \frac{n}{3}$. Now assume that there is a -1 pendant edge at v_3, say v_3z. Then the function $f|_{T-\{v_1,v_2,z\}}$ is an NNSEDF of $T - \{v_1, v_2, z\}$ and by the induction hypothesis we obtain $\gamma'_{ns}(T) \geq 1 - \frac{n}{3}$.

Hence $\deg(v_2) = \deg(v_3) = 2$. We now return to the proof of the theorem. If there is no -1 pendant edge at v_4, then the function f, restricted to $T' = T - \{v_1, v_2\}$ is an NNSEDF of T' of weight at most $\omega(f)$ and it follows from the induction hypothesis that $\gamma'_{ns}(T) > 1 - \frac{n}{3}$. Hence we assume that there is at least one -1 pendant edge at v_4. If there are two -1 pendant edges at v_4, say v_4z, v_4z', then the function $f|_{T-\{v_1,v_2,v_3,z\}}$ is an NNSEDF of $T - \{v_1, v_2, v_3, z\}$ and by the induction hypothesis we obtain $\gamma'_{ns}(T) > 1 - \frac{n}{3}$. Hence assume that there is one -1 pendant edges at v_4, say v_4z. Then the function $f|_{T-\{v_1,v_2\}}$ is an NNSEDF of $T - \{v_1, v_2\}$ and by the induction hypothesis we obtain $\gamma'_{ns}(T) > 1 - \frac{n}{3}$. This completes the proof. □

Example 2.1 shows that Theorem 2.1 is sharp.

Theorem 2.2. *Let T be a tree of order $n \geq 3$. Then*

$$\gamma'_{ns}(T) \leq \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Proof. The proof is by induction on n. If $\operatorname{diam}(T) \leq 3$, then T is a star or a double star and by Corollary 1.1 and Proposition 2.1, we have $\gamma'_{ns}(T) \leq \lfloor \frac{n-1}{3} \rfloor$. If $n = 5$ and $\operatorname{diam}(T) = 4$ or $n = 6$ and $\operatorname{diam}(T) = 5$, then T is path and the result follows by Corollary 1.2. Let $n = 6$ and $\operatorname{diam}(T) = 4$. Assume that $v_1v_2v_3v_4v_5$ is diametral path in T. Then T has exactly one pendant edge at v_2 (resp. v_4) or one pendant edge at v_3. If T has exactly one pendant edge at v_2, then the function

$f : E(T) \to \{-1,1\}$ by $f(v_1v_2) = f(v_2v_3) = f(v_3v_4) = 1$ and $f(e) = -1$ otherwise, is an NNSEDF of T of weight 1. If T has exactly one pendant edge at v_3, then the function $f : E(T) \to \{-1,1\}$ by $f(v_2v_3) = f(v_3v_4) = 1$ and $f(e) = -1$ otherwise, is an NNSEDF of T of weight -1. Hence, the statement is true for all trees of order $n \leq 6$. Assume that T is an arbitrary tree of order $n \geq 7$ and $\mathrm{diam}(T) \geq 4$. We proceed further with a series of claims that we may assume satisfied by the tree T.

Claim 1. T has no end-strong support vertex of degree at least 4.
Proof. Let T have an end-strong support vertex w of degree at least 4 and let w_1, w_2, w_3 be three leaves adjacent to w. Now let $T' = T - \{w_1, w_2\}$. Then for any $\gamma'_{ns}(T')$-function f, $f(N[w_3w]) \geq 0$. Now any $\gamma'_{ns}(T)$-function f, can be extended to an NNSEDF g of T as follows, $g(ww_1) = 1$, $g(ww_2) = -1$ and $g(e) = f(e)$ for $e \in E(T')$. It follows from the induction hypothesis that

$$\gamma'_{ns}(T) \leq \omega(g) = \omega(f) \leq \left\lfloor \frac{n-3}{3} \right\rfloor \leq \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Let $v_1v_2\ldots v_d$ be a diametral path in T chosen to maximize $\deg_T(v_2)$ and root T in v_d. By Claim 1, v_2 and any support vertex adjacent to v_3, except v_4, has degree 2 or 3.

Claim 2. $\deg(v_2) = 2$.
Proof. Let $\deg(v_2) = 3$ and $v_1' \in N(v_2) - \{v_1, v_3\}$. If $\deg(v_3) = 2$, then let $T' = T - T_{v_2}$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF of T by assigning 1 to v_1v_2, v_2v_3 and -1 to $v_1'v_2$. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \leq \omega(g) = \omega(f) + 1 \leq \left\lfloor \frac{n-4}{3} \right\rfloor + 1 = \lfloor \frac{n-1}{3} \rfloor.$$

If v_3 is adjacent to a leaf w, then let $T' = T - T_{v_2}$. Hence v_3 is a support vertex in T' and for any $\gamma'_{ns}(T')$-function f, $f(v_3) \geq 0$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF of T by assigning 1 to v_1v_2, v_2v_3 and -1 to $v_1'v_2$. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \leq \omega(g) = \omega(f) + 1 \leq \left\lfloor \frac{n-4}{3} \right\rfloor + 1 = \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Now let v_3 be adjacent to a support vertex w_2 not in $\{v_2, v_4\}$. First let $\deg(w_2) = 2$ and let w_1 be the leaf adjacent to w_2. Let $T' = T - \{v_1, v_1', v_2\}$. Since $f(N[w_2v_3]) \geq 0$, we have $f(v_3) \geq -1$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF g of T as follows, $g(v_1'v_2) = -1$, $g(v_1v_2) = g(v_2v_3) = 1$ and $g(e) = f(e)$ for $e \in E(T')$. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \leq \omega(g) = \omega(f) + 1 \leq \left\lfloor \frac{n-4}{3} \right\rfloor + 1 = \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Hence let any support vertex adjacent to v_3, except v_4, have degree 3. Assume that $N(v_3) - \{v_2, v_4\} = \{u_1, u_2, \ldots, u_k\}$. Let x_i, x_i' be the leaves adjacent to u_i for $1 \leq i \leq k$. Let $T' = T - (\{v_1, v_1', v_2\} \cup \{u_i, x_i, x_i' \mid 1 \leq i \leq k\})$. Hence v_3 is a leaf in T' and for any $\gamma'_{ns}(T')$-function f, $f(v_3) \geq -1$. Now any $\gamma'_{ns}(T')$-function

f, can be extended to an NNSEDF g of T as follows, $g(v_1'v_2) = g(x_i'u_i) = -1$, $g(v_1v_2) = g(v_2v_3) = g(x_iu_i) = g(u_iv_3) = 1$, for $1 \le i \le k$, and $g(e) = f(e)$ for $e \in E(T')$. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \le \omega(g) = \omega(f) + k + 1 \le \left\lfloor \frac{n - 1 - (3k+3)}{3} \right\rfloor + k + 1 = \left\lfloor \frac{n-1}{3} \right\rfloor.$$

By Claim 1 and 2, v_2 and any support vertex adjacent to v_3, except v_4, has degree 2.

Claim 3. $\deg(v_3) = 2$.

Proof. Let $\deg(v_3) \ge 3$. If v_3 is adjacent to a leaf w, then let $T' = T - T_{v_2}$. Hence v_3 is a support vertex in T' and for any $\gamma'_{ns}(T')$-function f, $f(v_3) \ge 0$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF of T by assigning 1 to v_2v_3, -1 to v_1v_2. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \le \omega(g) = \omega(f) \le \left\lfloor \frac{n-3}{3} \right\rfloor \le \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Hence let any vertex adjacent to v_3, except v_4, be a support vertex. Assume that $N(v_3) - \{v_2, v_4\} = \{u_1, u_2, \ldots, u_k\}$. Let x_i be the leaf adjacent to u_i for $1 \le i \le k$. Let $T' = T - (\{v_1, v_2\} \cup \{u_i, x_i \mid 1 \le i \le k\})$. Hence v_3 is a leaf in T' and for any $\gamma'_{ns}(T')$-function f, $f(v_3) \ge -1$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF g of T as follows, $g(v_1v_2) = g(x_iu_i) = -1$, $g(v_2v_3) = g(u_iv_3) = 1$, for $1 \le i \le k$, and $g(e) = f(e)$ for $e \in E(T')$. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \le \omega(g) = \omega(f) \le \left\lfloor \frac{n - 1 - (2k+2)}{3} \right\rfloor < \left\lfloor \frac{n-1}{3} \right\rfloor.$$

By Claim 1, 2 and 3, v_2, and any support vertex with depth 2 of v_4, except v_5, has degree 2.

Claim 4. $\deg(v_4) = 2$.

Proof. Let $\deg(v_4) \ge 3$. If v_4 is adjacent to a leaf w, then let $T' = T - T_{v_3}$. Hence v_4 is a support vertex in T' and for any $\gamma'_{ns}(T')$-function f, $f(v_4) \ge 0$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF g of T by assigning 1 to v_2v_3, v_3v_4, -1 to v_1v_2. Then by the induction hypothesis we obtain $\gamma'_{ns}(T) \le \omega(g) = \omega(f) + 1 \le \lfloor \frac{n-4}{3} \rfloor + 1 = \lfloor \frac{n-1}{3} \rfloor$. Now let v_4 be adjacent to a vertex w such that $\deg(w) = 2$. Let $T' = T - T_{v_3}$. Since $f(N[wv_4]) \ge 0$, we have $f(v_4) \ge -1$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF g of T by assigning 1 to v_2v_3, v_3v_4, -1 to v_1v_2. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \le \omega(g) = \omega(f) + 1 \le \left\lfloor \frac{n-4}{3} \right\rfloor + 1 = \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Hence let any vertex adjacent to v_4, except v_3, v_5, be a strong support vertex. Assume that $N(v_4) - \{v_3, v_5\} = \{u_1, u_2, \ldots, u_k\}$. Let x_i, x_i' be the leaves adjacent to u_i for $1 \le i \le k$. Let $T' = T - (\{v_1, v_2, v_3\} \cup \{u_i, x_i, x_i' \mid 1 \le i \le k\})$. Hence v_4 is a leaf in T' and for any $\gamma'_{ns}(T')$-function f, $f(v_4) \ge -1$. Now any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF g of T as follows, $g(v_1v_2) = g(x_i'u_i) = -1$,

$g(v_2v_3) = g(v_3v_4) = g(x_iu_i) = g(u_iv_3) = 1$, for $1 \le i \le k$, and $g(e) = f(e)$ for $e \in E(T')$. Then by the induction hypothesis we obtain

$$\gamma'_{ns}(T) \le \omega(g) = \omega(f) + k + 1 \le \left\lfloor \frac{n - 1 - (3k+3)}{3} \right\rfloor + k + 1 = \left\lfloor \frac{n-1}{3} \right\rfloor.$$

We now return to the proof of the theorem. Assume that $T' = T - T_{v_3}$. Then $f(v_4) \ge -1$ and any $\gamma'_{ns}(T')$-function f, can be extended to an NNSEDF of T by assigning -1 to v_1v_2 and 1 to v_2v_3, v_3v_4. Thus

$$\gamma'_{ns}(T) \le \omega(f) + 1 \le \left\lfloor \frac{n-1-3}{3} \right\rfloor + 1 = \left\lfloor \frac{n-1}{3} \right\rfloor.$$

This complete the proof. □

Corollary 1.2 shows that Theorem 2.2 is sharp for $n \not\equiv 2 \pmod 3$.

3. Bounds on $\gamma'_{ns}(G)$

In this section we present basic properties of $\gamma'_{ns}(G)$ and sharp bounds on the nonnegative signed edge domination number of a graph.

Theorem 3.1. *If G is a graph of size m, maximum degree Δ and minimum degree δ, then*

$$\gamma'_{ns}(G) \ge \frac{2m(\delta - \Delta)}{2\Delta - 1}.$$

Proof. Let f be a $\gamma'_{ns}(G)$-function and define $g : E \to \{0, 2\}$ by $g(e) = f(e) + 1$ for each $e \in E$. We have

$$\begin{aligned}\sum_{e\in E} g(N[e]) &\ge \sum_{e=uv\in E} (f(N[e]) + \deg(u) + \deg(v) - 1)\\ &\ge 2m\delta + \sum_{e=uv\in E} (f(N[e]) - 1)\\ &\ge 2m\delta - m = m(2\delta - 1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{e\in E} g(N[e]) &= \sum_{e=uv\in E} (\deg(u) + \deg(v) - 1)g(e)\\ &\le \sum_{e\in E} (2\Delta - 1)g(e)\\ &= (2\Delta - 1)g(E).\end{aligned}$$

Thus $g(E) \ge \frac{m(2\delta-1)}{2\Delta-1}$. Since $f(E) = g(E) - m$, we have

$$\gamma'_{ns}(G) \ge \frac{2m(\delta - \Delta)}{2\Delta - 1}.$$ □

For some special cases we can improve Theorem 3.1.

Theorem 3.2. *Let G be a graph of size m, maximum degree Δ and minimum degree δ. If* $\deg(x)$ *is odd for each vertex x or if* $\deg(x)$ *is even for each vertex x, then*

$$\gamma'_{ns}(G) \geq \frac{m(2\delta - 2\Delta + 1)}{2\Delta - 1}.$$

Proof. Let f be a $\gamma'_{ns}(G)$-function and define $g : E \to \{0, 2\}$ by $g(e) = f(e) + 1$ for each $e \in E$. Since $\deg(x)$ is odd for each vertex x or $\deg(x)$ is even for each vertex x, we observe that $f(N[e])$ is odd for each edge e, and therefore $f(N[e]) \geq 1$. As in the proof of Theorem 3.1, it follows that

$$\sum_{e \in E} g(N[e]) \geq 2m\delta + \sum_{e=uv \in E} (f(N[e]) - 1) \geq 2m\delta.$$

Using the upper bound

$$\sum_{e \in E} g(N[e]) \leq (2\Delta - 1)g(E),$$

from the proof of Theorem 3.1, we obtain analogously the desired result. □

Corollary 3.1. *If G is an r-regular graph of size m with $r \geq 1$, then $\gamma'_{ns}(G) \geq \frac{m}{2r-1}$.*

For $r = 1$ and the complete graphs K_4 and K_5 Corollary 3.1 is sharp. In addition, if $n \equiv 0, 1 \pmod 3$, then the cycle C_n shows that Corollary 3.1 is sharp for $r = 2$ too.

Next we present a sharp upper bound on the nonnegative signed edge domination number for some special regular graphs.

Theorem 3.3. *Let $p \geq 1$ be an integer, and let G be a $(2p + 1)$-regular graph with a p-factor. If G is of order n, then $\gamma'_{ns}(G) \leq \frac{n}{2}$.*

Proof. Let H be a p-factor of G. Define the function $f : E(G) \to \{-1, 1\}$ by $f(e) = -1$ for $e \in E(H)$ and $f(e) = 1$ otherwise. Then $f(N[e]) = 3$ for $e \in E(H)$ and $f(N[e]) = 1$ otherwise. Therefore f is an NNSEDF on G of weight

$$\frac{(2p + 1)n}{2} - pn = \frac{n}{2}$$

and thus $\gamma'_{ns}(G) \leq \frac{n}{2}$. □

Using the well-known result by Katerinis [12], that an r-regular graph with a 1-factor has a k-factor for all $k \in \{1, 2, \ldots, r\}$, Theorem 3.3 leads to the following corollary.

Corollary 3.2. *Let $p \geq 1$ be an integer, and let G be a $(2p + 1)$-regular graph with a 1-factor. If G is of order n, then $\gamma'_{ns}(G) \leq \frac{n}{2}$.*

Now we determine the nonnegative signed edge domination number for complete graphs, and complete bipartite graphs $K_{n,n}$.

Theorem 3.4. *For $n \geq 3$, $\gamma'_{ns}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. First let $n = 2p + 1$ for an integer $p \geq 1$. If $n = 3$, then the desired result follows from Corollary 1.3. Let now $p \geq 2$ and $u_1, u_2, \ldots, u_{2p+1}$ be the vertex set of $G = K_{2p+1}$. Let $H_1 = G[\{u_1, u_2, \ldots, u_{p+1}\}]$ and $H_2 = G[\{u_{p+2}, u_{p+3}, \ldots, u_{2p+1}\}]$. Define the function $f : E(G) \to \{-1, 1\}$ by $f(e) = -1$ for $e \in E(H_1) \cup E(H_2)$ and $f(e) = 1$ otherwise. Then $f(N[e]) = 2p - 2(p-1) - 1 = 1$ for $e \in E(H_1)$, $f(N[e]) = 2p + 2 - 2(p-2) - 1 = 5$ for $e \in E(H_2)$ and $f(N[e]) = 2p - p - (p-1) = 1$ otherwise. Therefore f is an NNSEDF on K_{2p+1} and thus

$$\gamma'_{ns}(K_{2p+1}) \leq p(p+1) - \frac{p(p+1)}{2} - \frac{p(p-1)}{2} = p.$$

Next we will show that $\gamma'_{ns}(K_{2p+1}) \geq p$ for $p \geq 2$. Let f be a $\gamma'_{ns}(G)$-function, and let H be the subgraph with vertex set $V(G)$ and edge set $E_{-1} = E_{-1}(f)$. We will show that $|E_{-1}| \leq p^2$. Suppose to the contrary that $|E_{-1}| \geq p^2 + 1$. Let $d_1 \geq d_2 \geq \cdots \geq d_{2p+1}$ be the degree sequence of H. Then $2|E_{-1}| = \sum_{i=1}^{2p+1} d_i \geq 2p^2 + 2$ and so $2p \geq d_1 \geq p$. Since $\sum_{e' \in N[e]} f(e') \geq 0$ for each edge $e \in E(G)$, we observe that

$$d_H(x) + d_H(y) \leq 2p - 1, \text{ when } e = xy \in E_1 \tag{3.1}$$

and

$$d_H(x) + d_H(y) \leq 2p, \text{ when } e = xy \in E_{-1}. \tag{3.2}$$

If $d_1 = 2p$, then we obtain the contradiction $d_1 + d_2 \geq 2p + 1$. Let now $d_1 = 2p - k$ for an integer $1 \leq k \leq p$, and assume that $d_H(u_1) = 2p - k$. Let $u_1 y \in E_1$. If $d_H(y) \geq k$, then $d_H(u_1) + d_H(y) \geq 2p$, a contradiction to (1). Therefore $d_H(x) \leq k - 1$ for $x \in V(H) - N_H[u_1]$. If $d_H(y) \geq k + 1$ for $y \in N_H(u_1)$, then $d_H(u_1) + d_H(y) \geq 2p + 1$, a contradiction to (2). Therefore $d_H(x) \leq k$ for $x \in N_H(u_1)$. Since $|N_H(u_1)| = 2p - k$, we deduce that

$$2p^2 + 2 \leq \sum_{i=1}^{2p+1} d_i \leq k(2p-k) + 2p - k + (k-1)k = 2pk + 2p - 2k.$$

This implies

$$(p-1)^2 = p^2 - 2p + 1 \leq k(p-1) - p,$$

and hence we obtain the contradiction

$$p - 1 \leq k - \frac{p}{p-1}.$$

Altogether, we see that $|E_{-1}| \leq p^2$ and so $\gamma'_{ns}(K_{2p+1}) \geq \frac{(2p+1)2p}{2} - 2p^2 = p$.

Second let $n = 2p$ for an integer $p \geq 2$. It is a part of mathematical folklore that the complete graph K_{2p} is 1-factorable, and therefore K_{2p} has a $(p-1)$-factor. Hence it follows from Theorem 3.3 that $\gamma'_{ns}(K_{2p}) \leq p$.

Next we will show that $\gamma'_{ns}(K_{2p}) \geq p$. Let f be a $\gamma'_{ns}(G)$-function, and let H be the subgraph with vertex set $V(G)$ and edge set $E_{-1} = E_{-1}(f)$. We will show that $|E_{-1}| \leq p^2 - p$. Suppose to the contrary that $|E_{-1}| \geq p^2 - p + 1$. Let $d_1 \geq d_2 \geq$

$\cdots \geq d_{2p}$ be the degree sequence of H. Then $2|E_{-1}| = \sum_{i=1}^{2p} d_i \geq 2p^2 - 2p + 2$ and so $2p - 1 \geq d_1 \geq p$. Since $\sum_{e' \in N[e]} f(e') \geq 0$ for each edge $e \in E(G)$, we observe that

$$d_H(x) + d_H(y) \leq 2p - 2, \text{ when } e = xy \in E_1 \tag{3.3}$$

and

$$d_H(x) + d_H(y) \leq 2p - 1, \text{ when } e = xy \in E_{-1}. \tag{3.4}$$

If $d_1 = 2p-1$, then we obtain the contradiction $d_1+d_2 \geq 2p$. Let now $d_1 = 2p-k$ for an integer $2 \leq k \leq p$, and assume that $d_H(u_1) = 2p - k$. Let $u_1y \in E_1$. If $d_H(y) \geq k - 1$, then $d_H(u_1) + d_H(y) \geq 2p - 1$, a contradiction to (3). Therefore $d_H(x) \leq k - 2$ for $x \in V(H) - N_H[u_1]$. If $d_H(y) \geq k$ for $y \in N_H(u_1)$, then $d_H(u_1) + d_H(y) \geq 2p$, a contradiction to (4). Therefore $d_H(x) \leq k-1$ for $x \in N_H(u_1)$. Since $|N_H(u_1)| = 2p-k$, we deduce that

$$2p^2 - 2p + 2 \leq \sum_{i=1}^{2p} d_i \leq (k-1)(2p-k) + 2p - k + (k-2)(k-1) = 2pk - 3k + 2.$$

This leads to the contradiction

$$p \leq k - \frac{k}{2(p-1)}.$$

Altogether, we see that $|E_{-1}| \leq p^2 - p$ and so $\gamma'_{ns}(K_{2p}) \geq \frac{2p(2p-1)}{2} - 2p^2 + 2p = p$. □

Theorem 3.5. *For $n \geq 2$, $\gamma'_{ns}(K_{n,n}) = n$.*

Proof. Let $X = \{u_1, u_2, \ldots, u_n\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be a bipartition of $G = K_{n,n}$.

First let $n = 2p + 1$ for an integer $p \geq 1$. Clearly, $K_{2p+1,2p+1}$ has p-factor, and thus Theorem 3.3 implies $\gamma'_{ns}(K_{2p+1,2p+1}) \leq 2p + 1$.

Next we will show that $\gamma'_{ns}(K_{2p+1,2p+1}) \geq 2p + 1$. Let f be a $\gamma'_{ns}(G)$-function, and let H be the subgraph with vertex set $V(G)$ and edge set $E_{-1} = E_{-1}(f)$. We will show that $|E_{-1}| \leq 2p^2 + p$. Suppose to the contrary that $|E_{-1}| \geq 2p^2 + p + 1$. Let $d_1 \geq d_2 \geq \ldots \geq d_{4p+2}$ be the degree sequence of H. Then $2|E_{-1}| = \sum_{i=1}^{4p+2} d_i \geq 4p^2 + 2p + 2$ and so $2p + 1 \geq d_1 \geq p + 1$. Since $\sum_{e' \in N[e]} f(e') \geq 0$ for each edge $e \in E(G)$, we observe that

$$d_H(x) + d_H(y) \leq 2p, \text{ when } e = xy \in E_1 \tag{3.5}$$

and

$$d_H(x) + d_H(y) \leq 2p + 1, \text{ when } e = xy \in E_{-1}. \tag{3.6}$$

Let now $d_1 = 2p+1-k$ for an integer $0 \leq k \leq p$, and assume, without loss of generality, that $d_H(u_1) = 2p + 1 - k$. If $d_H(y) \geq k$ for $u_1y \in E_1$, then $d_H(u_1) + d_H(y) \geq 2p + 1$, a contradiction to (5). Therefore $d_H(x) \leq k - 1$ for $x \in Y - N_H(u_1)$. If $d_H(y) \geq k + 1$ for $y \in N_H(u_1)$, then $d_H(u_1) + d_H(y) \geq 2p + 2$, a contradiction to (6). Therefore $d_H(x) \leq k$ for $x \in N_H(u_1)$. We deduce that

$$|E_{-1}| \leq k(k-1) + (2p+1-k)k = 2pk,$$

a contradiction to $|E_{-1}| \geq 2p^2 + p + 1$.

Altogether, we see that $|E_{-1}| \leq 2p^2 + p$ and so

$$\gamma'_{ns}(K_{2p+1,2p+1}) \geq (2p+1)^2 - 2(2p^2 + p) = 2p + 1.$$

Second let $n = 2p$ for an integer $p \geq 1$, and let $B_1 = G[\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}]$ and $B_2 = G[\{u_{p+1}, u_{p+2}, \ldots, u_{2p}, v_{p+1}, v_{p+2}, \ldots, v_{2p}\}]$ be two induced subgraphs of G. In addition, let H_2 be a $(p-1)$-factor of B_2. Define the function $f : E(G) \to \{-1, 1\}$ by $f(e) = -1$ for $e \in E(B_1) \cup E(H_2)$ and $f(e) = 1$ otherwise. Then $f(N[e]) = 2p - 2(p-1) - 1 = 1$ for $e \in E(B_1)$, $f(N[e]) = 2p + 2 - 2(p-2) - 1 = 5$ for $e \in E(H_2)$, $f(N[e]) = 2p + 1 - 2(p-1) = 3$ for $e \in E(B_2) - E(H_2)$ and $f(N[e]) = 2p - p - (p-1) = 1$ otherwise. Therefore f is an NNSEDF on $K_{2p,2p}$ of weight

$$2p^2 + p - p^2 - p(p-1) = 2p$$

and thus $\gamma'_{ns}(K_{2p,2p}) \leq 2p$.

Next we will show that $\gamma'_{ns}(K_{2p,2p}) \geq 2p$. Let f be a $\gamma'_{ns}(G)$-function, and let H be the subgraph with vertex set $V(G)$ and edge set $E_{-1} = E_{-1}(f)$. We will show that $|E_{-1}| \leq 2p^2 - p$. Suppose to the contrary that $|E_{-1}| \geq 2p^2 - p + 1$. Let $d_1 \geq d_2 \geq \ldots \geq d_{4p}$ be the degree sequence of H. Then $2|E_{-1}| = \sum_{i=1}^{4p} d_i \geq 4p^2 - 2p + 2$ and so $2p \geq d_1 \geq p$. Since $\sum_{e' \in N[e]} f(e') \geq 0$ for each edge $e \in E(G)$, we observe that

$$d_H(x) + d_H(y) \leq 2p - 1, \text{ when } e = xy \in E_1 \tag{3.7}$$

and

$$d_H(x) + d_H(y) \leq 2p, \text{ when } e = xy \in E_{-1}. \tag{3.8}$$

Let now $d_1 = 2p - k$ for an integer $0 \leq k \leq p$, and assume, without loss of generality, that $d_H(u_1) = 2p - k$. If $d_H(y) \geq k$ for $u_1 y \in E_1$, then $d_H(u_1) + d_H(y) \geq 2p$, a contradiction to (7). Therefore $d_H(x) \leq k - 1$ for $x \in Y - N_H(u_1)$. If $d_H(y) \geq k + 1$ for $y \in N_H(u_1)$, then $d_H(u_1) + d_H(y) \geq 2p + 1$, a contradiction to (8). Therefore $d_H(x) \leq k$ for $x \in N_H(u_1)$. We deduce that

$$|E_{-1}| \leq k(k-1) + (2p-k)k = 2pk - k,$$

a contradiction to $|E_{-1}| \geq 2p^2 - p + 1$.

Altogether, we see that $|E_{-1}| \leq 2p^2 - p$ and so $\gamma'_{ns}(K_{2p,2p}) \geq (2p)^2 - 2(2p^2 - p) = 2p$. □

Theorems 3.4 and 3.5 show that Theorem 3.3 is sharp.

Proposition 3.1. *Let G be a graph of size $m \geq 1$. If u and v are two adjacent vertices, then*

$$\gamma'_{ns}(G) \geq \deg(u) + \deg(v) - m - 1.$$

Proof. Let f be a $\gamma'_{ns}(G)$-function. By definition $f(N[uv]) \geq 0$, and the least possible weight for f will now be achieved if $f(e) = -1$ for each $e \in E(G) - N[uv]$. Thus

$$\begin{aligned}\gamma'_{ns}(G) &\geq f(N[uv]) - |E(G) - N[uv]| \geq -(m - (\deg(u) + \deg(v) - 1)) \\ &= \deg(u) + \deg(v) - m - 1.\end{aligned}$$ □

The trees of the family $\mathcal{F}$ show that Proposition 3.1 is sharp. Choosing u as a vertex of maximum degree in Proposition 3.1, we obtain the following corollary.

Corollary 3.3. *If G is a graph of size $m \geq 1$, then*

$$\gamma'_{ns}(G) \geq \Delta + \delta - m - 1.$$

Corollary 1.1 demonstrates that Propositions 3.1 and Corollary 3.3 are sharp when n is even.

Theorem 3.6. *Let G be a connected graph of size $m \geq 1$. Then $\gamma'_{ns}(G) \geq 2 - m$ with equality if and only if G is isomorphic to P_2 or P_3.*

Proof. If $m = 1$, then $\gamma'_{ns}(G) = 1 = 2 - m$. If $m \geq 2$, then $\Delta \geq 2$ and the desired result follows from Corollary 3.3.

Assume now that $\gamma'_{ns}(G) = 2 - m$, and let f be a $\gamma'_{ns}(G)$-function. This implies that G has exactly one edege e with $f(e) = 1$ and $m - 1$ edges $e_1, e_2, \ldots, e_{m-1}$ such that $f(e_i) = -1$ for $1 \leq i \leq m - 1$. Suppose that $m \geq 3$, and let, without loss of generality, e_1 be adjacent to e. Since G is connected, there exists an edge, say e_2, adjacent to e or to e_1. If e_2 is adjacent to e, then we obtain the contradiction $\sum_{e' \in N[e]} f(e') \leq -1$, and if e_2 is adjacent to e_1, then we obtain the contradiction $\sum_{e' \in N[e_1]} f(e') \leq -1$. This implies that $m \leq 2$, and thus G is isomorphic to P_2 or P_3.

Conversely, if G is isomorphic to P_2 or P_3, then $\gamma'_{ns}(G) = 2 - m$. □

Using Observation 1.2 and Theorem 3.6, we obtain the next result immediately.

Corollary 3.4. *If G is a connected graph of size $m \geq 3$, then $\gamma'_{ns}(G) \geq 4 - m$.*

Remark 3.1. If $\Delta \geq 5$ or $\Delta \geq 4$ and $\delta \geq 2$, then Corollary 3.3 implies that $\gamma'_{ns}(G) \geq 5 - m$ and therefore $\gamma'_{ns}(G) \geq 6 - m$ by Observation 1.2.

In the case that $\Delta = 4$ and $\delta = 1$, we have $\gamma'_{ns}(K_{1,4}) = 0 = 4 - m(K_{1,4})$ and therefore equality in the inequality of Corollary 3.4. Proposition 3.1 shows that the star $K_{1,4}$ is the only graph with equality in the inequality $\gamma'_{ns}(G) \geq 4 - m$ in the case that $\Delta = 4$.

Corollaries 1.2 and 1.3 imply the next result.

Proposition 3.2. *Let G be a connected graph of size $m \geq 3$ with $\Delta(G) = 2$. Then $\gamma'_{ns}(G) = 4 - m$ if and only if G is isomorphic to C_3, P_4 or P_5.*

The graphs $K_{1,3}$ and $T_1 \in \mathcal{F}$ are further examples with equality in the inequality $\gamma'_{ns}(G) \geq 4 - m$ of Corollary 3.4.

Theorem 3.7. *Let G be a graph of size m and minimum degree $\delta \geq 2$. Then*

$$\gamma'_{ns}(G) \leq m - 2\delta + 2.$$

Proof. Let $v \in V$ be a vertex, $t = \delta - 1$ and $u_1, u_2, \ldots, u_t \in N(v)$. Define $f : E \to \{-1, 1\}$ by $f(vu_i) = -1$ for $1 \leq i \leq t$ and $f(e) = 1$ otherwise. Then $f(N[vw]) \geq -t + 1 + \deg(w) - 1 \geq \delta - t \geq 0$ for $w \in N(v)$. Let $e = wz$ such that $w, z \neq v$. Then $f(N[wz]) \geq 0$ when $\delta = 2$ and $f(N[wz]) \geq \deg(w) + \deg(z) - 5 \geq 2\delta - 5 > 0$ when $\delta \geq 3$. Therefore f is an NNSEDF on G of weight $m - 2t$ and so $\gamma'_{ns}(G) \leq m - 2t = m - 2\delta + 2$. □

Proposition 3.3. *Let G be a connected graph, different from C_5, of order $n \geq 5$ with* diam(G) = 2. *Then*

$$\gamma'_{ns}(G) \leq m - 4.$$

Proof. If $\delta(G) \geq 3$, then the result is immediate by Theorem 3.7. Henceforth, we assume $\delta(G) \leq 2$. First let $\delta = 1$, $v_1 \in V$ be a vertex of minimum degree and $v_1v_2 \in E(G)$. Since diam(G) = 2, for every vertex $w \in V - \{v_1, v_2\}$, $w \in N(v_2)$. Let $w \in N(v_2) - \{v_1\}$. Define $f : E(G) \to \{-1, 1\}$ by $f(v_2w) = f(v_1v_2) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, wv_2\}$. Clearly, f is an NNSEDF of G of weight at most $m - 4$ and hence $\gamma'_{ns}(G) \leq m - 4$. Hence let $\delta = 2$. Let $v_1 \in V$ be a vertex of minimum degree and $v_2, v_3 \in N(v_1)$. If $\deg(v_2) \geq 3$ and $\deg(v_3) \geq 3$, then define $f : E(G) \to \{-1, 1\}$ by $f(v_1v_2) = f(v_1v_3) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, v_1v_3\}$. Clearly, f is an NNSEDF of G of weight at most $m - 4$ and hence $\gamma'_{ns}(G) \leq m - 4$. Hence let $\deg(v_2) = 2$ or $\deg(v_3) = 2$. We may assume that $\deg(v_2) = 2$. Since diam(G) = 2 and $n \geq 5$, we observe that $\deg(v_3) \geq 3$ and let $w \in N(v_3) - N(v_2)$. Define $f : E(G) \to \{-1, 1\}$ by $f(v_1v_2) = f(wv_3) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, wv_3\}$. Clearly, f is an NNSEDF of G of weight at most $m - 4$ and hence $\gamma'_{ns}(G) \leq m - 4$. This complete the proof. □

Proposition 3.4. *Let G be a connected graph of order $n \geq 5$ with* diam(G) = 3. *Then*

$$\gamma'_{ns}(G) \leq m - 4.$$

Proof. If $\delta(G) \geq 3$, then the result is immediate by Theorem 3.7. Henceforth, we assume $\delta(G) \leq 2$. Consider two cases.

Case 1. $\delta = 2$.

Let $v_1v_2v_3v_4$ be a diametral path in G. If $\deg(v_2) \geq 3$ or $\deg(v_3) \geq 3$, then define $f : E(G) \to \{-1, 1\}$ by $f(v_1v_2) = f(v_3v_4) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, v_3v_4\}$. Clearly, f is an NNSEDF of G of weight at most $m - 4$ and hence $\gamma'_{ns}(G) \leq m - 4$. Hence let $\deg(v_2) = \deg(v_3) = 2$. Since diam(G) = 3, for every vertex $w \in V - \{v_1, v_2, v_3, v_4\}$, $w \in N(v_1) \cup N(v_4)$. If $\deg(v_1) = \deg(v_4) = 2$, then $G = C_6$ and by Corollary 1.3, $\gamma'_{ns}(G) \leq m - 4$. Hence let $\deg(v_1) \geq 3$ or $\deg(v_4) \geq 3$. We may assume that $\deg(v_1) \geq 3$ and $w \in N(v_1) - \{v_2\}$. Define $f : E(G) \to \{-1, 1\}$

by $f(v_1w) = f(v_2v_3) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1w, v_2v_3\}$. Clearly, f is an NNSEDF of G of weight at most $m-4$ and hence $\gamma'_{ns}(G) \leq m-4$.

Case 2. $\delta = 1$.

Let $v_1 \in V$ be a vertex of minimum degree and $v_1v_2 \in E(G)$. First let $\deg(v_2) = 2$ and $v_3 \in N(v_2) - \{v_1\}$. Since diam(G) = 3, for every vertex $w \in V - \{v_1, v_2, v_3\}$, $w \in N(v_3)$. Let $w \in N(v_3) - \{v_2\}$. Define $f : E(G) \to \{-1, 1\}$ by $f(v_3w) = f(v_1v_2) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, wv_3\}$. Clearly, f is an NNSEDF of G of weight at most $m-4$ and hence $\gamma'_{ns}(G) \leq m-4$. Now let $\deg(v_2) \geq 3$. If $\deg(v_2) \geq 4$ and $v_3 \in N(v_2) - \{v_1\}$, then define $f : E(G) \to \{-1, 1\}$ by $f(v_1v_2) = f(v_2v_3) - 1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, v_2v_3\}$. Clearly, f is an NNSEDF of G of weight at most $m-4$ and hence $\gamma'_{ns}(G) \leq m-4$. Hence let $\deg(v_2) = 3$ and $v_3, v'_3 \in N(v_2) - \{v_1\}$. Since diam(G) = 3, for every vertex $w \in V - \{v_1, v_2, v_3, v'_3\}$, $w \in N(v_3) \cup N(v'_3)$. Let $w \in N(v_3) - \{v_2\}$. Define $f : E(G) \to \{-1, 1\}$ by $f(v_3w) = f(v_1v_2) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, wv_3\}$. Clearly, f is an NNSEDF of G of weight at most $m-4$ and hence $\gamma'_{ns}(G) \leq m-4$. This complete the proof. □

Proposition 3.5. *If G is a connected graph of order $n \geq 5$ with* diam(G) ≥ 4*, then* $\gamma'_{ns}(G) \leq m-4$.

Proof. Let $v_1v_2 \ldots v_d$ be a diametral path in G. Define $f : E(G) \to \{-1, 1\}$ by $f(v_1v_2) = f(v_4v_5) = -1$ and $f(e) = 1$ for $e \in E(G) - \{v_1v_2, v_4v_5\}$. Clearly, f is an NNSEDF of G of weight at most $m-4$ and hence $\gamma'_{ns}(G) \leq m-4$. □

Theorem 3.8. *Let G be a connected graph of order $n \geq 3$ and size m. Then $\gamma'_{ns}(G) = m-2$ if and only if $G \cong P_3, P_4, C_3, C_4, C_5$, or $K_{1,3}$.*

Proof. Clearly, if $G \cong P_3, P_4, C_3, C_4, C_5$, or $K_{1,3}$, then $\gamma'_{ns}(G) = m-2$. Conversely, let G be a connected graph of size $m \geq 2$ and let $\gamma'_{ns}(G) = m-2$. By Propositions 3.3, 3.4 and 3.5, $n \leq 4$ or $G = C_5$ and by Theorem 3.7, $\delta \leq 2$. The case $G = C_5$ is obvious by Corollary 1.3. Let $n \leq 4$ and $\delta \leq 2$. If $\delta = 2$, we must have $G = C_3, C_4$ and $C_4 + e$. If $G = C_3, C_4$, we are done by Corollary 1.3. Let $G = C_4 + e$ and $V(C_4 + e) = \{v_1, v_2, v_3, v_4\}$, where $e = v_1v_3$. Define $f : E(C_4 + e) \to \{-1, 1\}$ by $f(v_1v_2) = f(v_3v_4) = -1$ and $f(e) = 1$ otherwise. Clearly, f is an NNSEDF of $C_4 + e$ with weight 1. Thus $G \neq C_4 + e$. Let $\delta = 1$. It is easy to see that the only graphs satisfying the conditions are P_3, P_4 or $K_{1,3}$. This completes the proof. □

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