

## ON NORMALIZED SIGNLESS LAPLACIAN RESOLVENT ENERGY

Š. B. BOZKURT ALTINDAĞ<sup>1</sup>, I. MILOVANOVIĆ<sup>2</sup>, E. MILOVANOVIĆ<sup>2</sup>, AND M. MATEJIĆ<sup>2</sup>

ABSTRACT. Let  $G$  be a simple connected graph with  $n$  vertices. Denote by  $\mathcal{L}^+(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2}$  the normalized signless Laplacian matrix of graph  $G$ , where  $Q(G)$  and  $D(G)$  are the signless Laplacian and diagonal degree matrices of  $G$ , respectively. The eigenvalues of matrix  $\mathcal{L}^+(G)$ ,  $2 = \gamma_1^+ \geq \gamma_2^+ \geq \dots \geq \gamma_n^+ \geq 0$ , are normalized signless Laplacian eigenvalues of  $G$ . In this paper, we introduce the normalized signless Laplacian resolvent energy of  $G$  as  $ERNS(G) = \sum_{i=1}^n \frac{1}{3-\gamma_i^+}$ . We also obtain some lower and upper bounds for  $ERNS(G)$  as well as its relationships with other energies and signless Kemeny's constant.

### 1. INTRODUCTION

Let  $G = (V, E)$ ,  $V = \{v_1, v_2, \dots, v_n\}$ , be a simple connected graph with  $n$  vertices and  $m$  edges, where  $|V| = n$  and  $|E| = m$ . Denote by  $d_i$  the degree of the vertex  $v_i$  of  $G$ ,  $i = 1, 2, \dots, n$ . If  $v_i$  and  $v_j$  are two adjacent vertices of  $G$ , then we denote this by  $i \sim j$ .

Let  $A(G)$  be the adjacency matrix of  $G$ . Eigenvalues of  $A(G)$ ,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , are said to be the (ordinary) eigenvalues of  $G$  [11]. Then the energy of the graph  $G$  is defined as [15]

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

Various properties and bounds on  $E(G)$  may be found in the monographs [19, 22] and references cited therein.

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In line with concept of graph energy, the resolvent energy of  $G$  is put forward in [18] as

$$ER(G) = \sum_{i=1}^n \frac{1}{n - \lambda_i}.$$

For the basic properties and bounds of  $ER(G)$ , the reader may refer to [1, 13, 34, 35].

Let  $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$  denote the diagonal degree matrix of  $G$ . The Laplacian and signless Laplacian matrices of  $G$  are, respectively, defined as  $L(G) = D(G) - A(G)$  and  $Q(G) = D(G) + A(G)$ . Denote by  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} > \mu_n = 0$  and  $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$  the eigenvalues of  $L(G)$  and  $Q(G)$ , respectively [26]. Recently, Laplacian resolvent and signless Laplacian resolvent energies of  $G$  are, respectively, introduced as [7]

$$RL(G) = \sum_{i=1}^n \frac{1}{n + 1 - \mu_i}$$

and

$$RQ(G) = \sum_{i=1}^n \frac{1}{2n - 1 - q_i}.$$

Since graph  $G$  is connected, the matrix  $D(G)^{-1/2}$  is well defined. Then, the normalized Laplacian matrix of  $G$  is defined by [10]

$$\mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} = I_n - R(G),$$

where  $I_n$  is the  $n \times n$  unity matrix and  $R(G)$  is the Randić matrix [2]. The following properties for the normalized Laplacian eigenvalues,  $\gamma_1^- \geq \gamma_2^- \geq \dots \geq \gamma_{n-1}^- > \gamma_n^- = 0$ , are valid [36]

$$(1.1) \quad \sum_{i=1}^{n-1} \gamma_i^- = n \quad \text{and} \quad \sum_{i=1}^{n-1} (\gamma_i^-)^2 = n + 2R_{-1}(G),$$

where

$$R_{-1}(G) = \sum_{i \sim j} \frac{1}{d_i d_j},$$

is a kind of topological index of  $G$  called as general Randić index [8, 31].

The matrix  $\mathcal{L}^+(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2} = I_n + R(G)$  is defined to be the normalized signless Laplacian matrix of  $G$  [10]. Some well known identities concerning the normalized signless Laplacian eigenvalues,  $\gamma_1^+ \geq \gamma_2^+ \geq \dots \geq \gamma_n^+ \geq 0$ , are [9]

$$(1.2) \quad \sum_{i=1}^n \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^n (\gamma_i^+)^2 = n + 2R_{-1}(G).$$

For  $i = 1, 2, \dots, n$ , the following relations (see [14, 24]) exist

$$(1.3) \quad \gamma_i^- = 1 - \rho_{n-i+1} \quad \text{and} \quad \gamma_i^+ = 1 + \rho_i.$$

Here,  $1 = \rho_1 \geq \rho_2 \geq \dots \geq \rho_n$  are the Randić eigenvalues of  $G$  [2, 24].

Motivated by the definitions of graph resolvent energies  $ER$ ,  $RL$  and  $RQ$  and considering the fact that  $\gamma_i^- \leq 2$ ,  $1 \leq i \leq n$ , Sun and Das [33] defined the normalized Laplacian resolvent energy of  $G$  as

$$ERN(G) = \sum_{i=1}^n \frac{1}{3 - \gamma_i^-}.$$

Since the property  $\gamma_i^+ \leq 2$ ,  $1 \leq i \leq n$ , is also satisfied by the normalized signless Laplacian eigenvalues, we now introduce the normalized signless Laplacian resolvent energy of  $G$  as follows

$$ERNS(G) = \sum_{i=1}^n \frac{1}{3 - \gamma_i^+}.$$

Notice that in the case of bipartite graph the normalized Laplacian and normalized signless Laplacian eigenvalues coincide [3]. From hence, for bipartite graphs,  $ERN(G)$  is equal to  $ERNS(G)$ .

Before we proceed, let us recall another graph invariant closely related to normalized Laplacian eigenvalues and so called Kemeny's constant. It is defined as [6]

$$K(G) = \sum_{i=1}^{n-1} \frac{1}{\gamma_i^-}.$$

For more information on  $K(G)$ , see [21, 27].

Since for connected non-bipartite graphs  $\gamma_i^+ > 0$  for  $i = 1, 2, \dots, n$ , [4], very recently, in an analogous manner with Kemeny's constant, signless Kemeny's constant of connected non-bipartite graphs is considered as [28]

$$K^+(G) = \sum_{i=1}^n \frac{1}{\gamma_i^+}.$$

In [28], it is also emphasized that  $K(G)$  coincides with  $K^+(G)$  in the case of bipartite graphs.

In this paper, we obtain some lower and upper bounds for  $ERNS(G)$  as well as its relationships with other energies and  $K^+(G)$ .

## 2. LEMMAS

We now recall some known results on graph spectra and analytical inequalities that will be used in our main results.

**Lemma 2.1** ([14]). *For any connected graph  $G$ , the largest normalized signless Laplacian eigenvalue is  $\gamma_1^+ = 2$ .*

**Lemma 2.2** ([14]). *Let  $G$  be a graph of order  $n \geq 2$  with no isolated vertices. Then*

$$\gamma_2^+ = \gamma_3^+ = \dots = \gamma_n^+ = \frac{n-2}{n-1}$$

*if and only if  $G \cong K_n$ .*

**Lemma 2.3** ([4]). *If  $G$  is a connected non-bipartite graph of order  $n$ , then  $\gamma_i^+ > 0$  for  $i = 1, 2, \dots, n$ .*

**Lemma 2.4** ([3]). *If  $G$  is a bipartite graph, then the eigenvalues of  $\mathcal{L}$  and  $\mathcal{L}^+$  coincide.*

**Lemma 2.5** ([23]). *Let  $G$  be a connected graph of order  $n$ . Then  $\gamma_2^- \geq 1$ , the equality holds if and only if  $G$  is a complete bipartite graph.*

**Lemma 2.6** ([12]). *Let  $G$  be a connected graph with  $n > 2$  vertices. Then  $\gamma_2^- = \gamma_3^- = \dots = \gamma_{n-1}^-$  if and only if  $G \cong K_n$  or  $G \cong K_{p,q}$ .*

**Lemma 2.7** ([10]). *Let  $G$  be a bipartite graph with  $n$  vertices. Then for  $i = 1, 2, \dots, n$ ,  $\gamma_i^- + \gamma_{n-i+1}^- = 2$ .*

**Lemma 2.8** ([24]). *For any connected graph  $G$ , the largest Randić eigenvalue is  $\rho_1 = 1$ .*

**Lemma 2.9** ([2]). *Let  $G$  be a graph with  $n$  vertices and Randić matrix  $R(G)$ . Then*

$$\operatorname{tr} \left( R(G)^2 \right) = 2R_{-1}$$

and

$$\operatorname{tr} \left( R(G)^3 \right) = 2 \sum_{i \sim j} \frac{1}{d_i d_j} \left( \sum_{k \sim i, k \sim j} \frac{1}{d_k} \right).$$

**Lemma 2.10** ([30]). *Let  $x = (x_i)$  and  $a = (a_i)$  be two sequences of positive real numbers,  $i = 1, 2, \dots, n$ . Then for any  $r \geq 0$*

$$(2.1) \quad \sum_{i=1}^n \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left( \sum_{i=1}^n x_i \right)^{r+1}}{\left( \sum_{i=1}^n a_i \right)^r}.$$

*Equality holds if and only if  $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_n}{a_n}$ .*

**Lemma 2.11** ([20]). *Let  $a = (a_i)$  and  $p = (p_i)$  be two sequences of positive real numbers such that  $\sum_{i=1}^n p_i = 1$  and  $0 < r \leq a_i \leq R < +\infty$ ,  $i = 1, 2, \dots, n$ ,  $r, R \in \mathbb{R}$ . Then*

$$(2.2) \quad \sum_{i=1}^n p_i a_i \sum_{i=1}^n \frac{p_i}{a_i} \leq \frac{1}{4} \left( \sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}} \right)^2.$$

*Equality holds if and only if  $R = a_1 = a_2 = \dots = a_n = r$ .*

### 3. LOWER AND UPPER BOUNDS ON $ERNS(G)$

In this section, we establish some lower and upper bounds for  $ERNS(G)$ .

**Theorem 3.1.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then, for any real  $\alpha$ , such that  $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$ ,*

$$(3.1) \quad ERNS(G) \geq 1 + \frac{1}{3 - \alpha} + \frac{(n - 2)^2}{2n - 4 + \alpha}.$$

*If  $\alpha = \frac{n-2}{n-1}$ , equality holds if and only if  $G \cong K_n$ .*

*Proof.* By arithmetic-harmonic mean inequality [29], we have

$$\sum_{i=3}^n a_i \sum_{i=3}^n \frac{1}{a_i} \geq (n - 2)^2,$$

where  $a_i > 0$ ,  $i = 3, 4, \dots, n$ , are arbitrary real numbers. For  $a_i = 3 - \gamma_i^+$ ,  $i = 3, 4, \dots, n$ , the above inequality transforms into

$$\sum_{i=3}^n (3 - \gamma_i^+) \sum_{i=3}^n \frac{1}{3 - \gamma_i^+} \geq (n - 2)^2,$$

that is

$$\sum_{i=1}^n \frac{1}{3 - \gamma_i^+} \geq \frac{1}{3 - \gamma_1^+} + \frac{1}{3 - \gamma_2^+} + \frac{(n - 2)^2}{\sum_{i=3}^n (3 - \gamma_i^+)}.$$

Then, it follows from the above, (1.2) and Lemma 2.1 that

$$(3.2) \quad ERNS(G) \geq 1 + \frac{1}{3 - \gamma_2^+} + \frac{(n - 2)^2}{2n - 4 + \gamma_2^+}.$$

Now, consider the function defined as follows

$$f(x) = \frac{1}{3 - x} + \frac{(n - 2)^2}{2n - 4 + x}.$$

It can be easily seen that  $f$  is increasing for  $x \geq \frac{n-2}{n-1}$ . Then for any real  $\alpha$ ,  $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$ ,

$$f(\gamma_2^+) \geq f(\alpha) = \frac{1}{3 - \alpha} + \frac{(n - 2)^2}{2n - 4 + \alpha}.$$

Based on this inequality and (3.2), we obtain the lower bound (3.1). Equality in (3.1) holds if and only if

$$\gamma_2^+ = \alpha \quad \text{and} \quad \gamma_3^+ = \dots = \gamma_n^+.$$

If  $\alpha = \frac{n-2}{n-1}$ , then from the above and Lemma 2.2, one can easily conclude that the equality in (3.1) holds if and only if  $G \cong K_n$ . □

**Corollary 3.1.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$ERNS(G) \geq 1 + \frac{(n - 1)^2}{2n - 1}.$$

*Equality holds if and only if  $G \cong K_n$ .*

Considering the techniques in Theorem 3.1 with Lemmas 2.1, 2.4 and 2.6, we obtain the following result for bipartite graphs.

**Theorem 3.2.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then, for any real  $\alpha$ , such that  $\gamma_2^+ = \gamma_2^- \geq \alpha \geq 1$*

$$ERNS(G) = ERN(G) \geq \frac{4}{3} + \frac{1}{3-\alpha} + \frac{(n-3)^2}{2n-7+\alpha}.$$

If  $\alpha = 1$ , equality holds if and only if  $G \cong K_{p,q}$ ,  $p+q = n$ .

In [5], it was obtained that

$$(3.3) \quad \gamma_2^+ = \gamma_2^- \geq 1 + \sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}.$$

From Theorem 3.2 and (3.3), we directly have the following.

**Corollary 3.2.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$(3.4) \quad ERNS(G) = ERN(G) \geq \frac{4}{3} + \frac{1}{2 - \sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}} + \frac{(n-3)^2}{2n-6 + \sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}}.$$

From Theorem 3.2 and Lemma 2.5, we have the following result. It was proven in Theorem 3.8 of [33].

**Corollary 3.3** ([33]). *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$(3.5) \quad ERNS(G) = ERN(G) \geq \frac{n}{2} + \frac{1}{3}.$$

Equality holds if and only if  $G \cong K_{p,q}$ .

*Remark 3.1.* Note that the lower bound (3.4) is stronger than the lower bound (3.5).

**Theorem 3.3.** *Let  $G$  be a connected graph with  $n \geq 3$  vertices. Then*

$$(3.6) \quad ERNS(G) \geq \frac{1}{3} \left( n + 2 + \frac{(n-2)^2}{2(n-1-R_{-1}(G))} \right).$$

Equality holds if and only if  $G \cong K_n$  or  $G \cong K_{p,q}$ ,  $p+q = n$ .

*Proof.* Suppose  $G$  is a connected non-bipartite graph with  $n \geq 3$  vertices. Then, by Lemmas 2.1 and 2.3,  $\gamma_1^+ = 2$  and  $\gamma_i^+ > 0$ ,  $i = 2, 3, \dots, n$ . For  $r = 1$  the inequality (2.1) transforms into

$$(3.7) \quad \sum_{i=2}^n \frac{x_i^2}{a_i} \geq \frac{\left( \sum_{i=2}^n x_i \right)^2}{\sum_{i=2}^n a_i}.$$

Setting  $x_i = \gamma_i^+$ ,  $a_i = \gamma_i^+(3 - \gamma_i^+)$ ,  $i = 2, 3, \dots, n$ , in (3.7) and using (1.2) and Lemma 2.1, we have

$$(3.8) \quad \sum_{i=2}^n \frac{(\gamma_i^+)^2}{\gamma_i^+(3 - \gamma_i^+)} \geq \frac{\left(\sum_{i=2}^n \gamma_i^+\right)^2}{\sum_{i=2}^n \gamma_i^+(3 - \gamma_i^+)} = \frac{(n-2)^2}{2(n-1 - R_{-1}(G))}.$$

On the other hand, from the above and Lemma 2.1, we also have

$$(3.9) \quad \begin{aligned} \sum_{i=2}^n \frac{(\gamma_i^+)^2}{\gamma_i^+(3 - \gamma_i^+)} &= \sum_{i=2}^n \frac{\gamma_i^+}{3 - \gamma_i^+} = \sum_{i=2}^n \frac{\gamma_i^+ - 3 + 3}{3 - \gamma_i^+} \\ &= -(n-1) + 3(ERNS(G) - 1) \\ &= 3ERNS(G) - n - 2. \end{aligned}$$

From (3.8) and (3.9), the inequality (3.6) is obtained.

Equality in (3.8) holds if and only if

$$\frac{1}{3 - \gamma_2^+} = \frac{1}{3 - \gamma_3^+} = \dots = \frac{1}{3 - \gamma_n^+},$$

that is  $\gamma_2^+ = \gamma_3^+ = \dots = \gamma_n^+$ . By Lemma 2.2, when  $G$  is non-bipartite graph, equality in (3.6) holds if and only if  $G \cong K_n$ .

Now, suppose  $G$  is a connected bipartite graph with  $n \geq 3$  vertices. Then, by Lemmas 2.1 and 2.4,  $\gamma_1^+ = 2$  and  $\gamma_2^+ \geq \gamma_3^+ \geq \dots \geq \gamma_{n-1}^+ > \gamma_n^+ = 0$ . The inequality (2.1) can be considered as

$$\sum_{i=2}^{n-1} \frac{x_i^2}{a_i} \geq \frac{\left(\sum_{i=2}^{n-1} x_i\right)^2}{\sum_{i=2}^{n-1} a_i}.$$

Taking  $x_i = \gamma_i^+$ ,  $a_i = \gamma_i^+(3 - \gamma_i^+)$ ,  $i = 2, 3, \dots, n - 1$ , in the above inequality and considering (1.2), we get

$$(3.10) \quad \sum_{i=2}^{n-1} \frac{(\gamma_i^+)^2}{\gamma_i^+(3 - \gamma_i^+)} \geq \frac{\left(\sum_{i=2}^{n-1} \gamma_i^+\right)^2}{\sum_{i=2}^{n-1} \gamma_i^+(3 - \gamma_i^+)} = \frac{(n-2)^2}{2(n-1 - R_{-1}(G))}.$$

Observe that

$$\begin{aligned} \sum_{i=2}^{n-1} \frac{(\gamma_i^+)^2}{\gamma_i^+(3 - \gamma_i^+)} &= \sum_{i=2}^{n-1} \frac{\gamma_i^+}{3 - \gamma_i^+} = \sum_{i=2}^{n-1} \frac{\gamma_i^+ - 3 + 3}{3 - \gamma_i^+} \\ &= -(n-2) + 3\left(ERNS(G) - 1 - \frac{1}{3}\right) = \\ &= 3ERNS(G) - n - 2. \end{aligned}$$

From the above and inequality (3.10) we arrive at (3.6).

Equality in (3.10) holds if and only if

$$\frac{1}{3 - \gamma_2^+} = \frac{1}{3 - \gamma_3^+} = \dots = \frac{1}{3 - \gamma_{n-1}^+},$$

that is when  $\gamma_2^+ = \gamma_3^+ = \dots = \gamma_{n-1}^+$ . Since  $G$  is a bipartite graph, by Lemmas 2.4 and 2.6, equality in (3.6) holds if and only if  $G \cong K_{p,q}$ ,  $p + q = n$ .  $\square$

**Corollary 3.4.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{1}{3} \left( n + 2 + \frac{(n - 2)^2}{2(n - 1 - R_{-1}(G))} \right).$$

*Equality holds if and only if  $G \cong K_{p,q}$ ,  $p + q = n$ .*

**Theorem 3.4.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$(3.11) \quad ERNS(G) \leq 1 + \frac{4n - 5 - (n - 1)(\gamma_2^+ + \gamma_n^+)}{(3 - \gamma_2^+)(3 - \gamma_n^+)}.$$

*Equality holds if and only if  $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$ , for  $i = 2, 3, \dots, n$ .*

*Proof.* For every  $i = 2, 3, \dots, n$ , the following inequalities are valid

$$(3.12) \quad \begin{aligned} (3 - \gamma_2^+ - 3 + \gamma_i^+)(3 - \gamma_n^+ - 3 + \gamma_i^+) &\leq 0, \\ (3 - \gamma_i^+)^2 + (3 - \gamma_2^+)(3 - \gamma_n^+) &\leq (6 - \gamma_2^+ - \gamma_n^+)(3 - \gamma_i^+), \\ (3 - \gamma_i^+) + \frac{(3 - \gamma_2^+)(3 - \gamma_n^+)}{3 - \gamma_i^+} &\leq 6 - \gamma_2^+ - \gamma_n^+. \end{aligned}$$

After summation of (3.12) over  $i$ ,  $i = 2, 3, \dots, n$ , we obtain

$$\sum_{i=2}^n (3 - \gamma_i^+) + (3 - \gamma_2^+)(3 - \gamma_n^+) \sum_{i=2}^n \frac{1}{3 - \gamma_i^+} \leq (6 - \gamma_2^+ - \gamma_n^+) \sum_{i=2}^n 1,$$

that is

$$(3.13) \quad 2n - 1 + (3 - \gamma_2^+)(3 - \gamma_n^+)(ERNS(G) - 1) \leq (6 - \gamma_2^+ - \gamma_n^+)(n - 1),$$

from which (3.11) is obtained.

Equality in (3.12) holds if and only if  $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$ , for every  $i = 2, 3, \dots, n$ , which implies that equality in (3.11) holds if and only if  $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$ , for every  $i = 2, 3, \dots, n$ .  $\square$

**Corollary 3.5.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$(3.14) \quad ERNS(G) \leq 1 + \frac{\left( (n - 1)(6 - \gamma_2^+ - \gamma_n^+) \right)^2}{4(2n - 1)(3 - \gamma_2^+)(3 - \gamma_n^+)}.$$

*Equality holds if and only if  $G \cong K_n$ .*



*Proof.* After applying the arithmetic-geometric mean inequality, AM-GM, on (3.13) we obtain

$$2\sqrt{(2n - 1)(3 - \gamma_2^+)(3 - \gamma_n^+)(ERNS(G) - 1)} \leq (6 - \gamma_2^+ - \gamma_n^+)(n - 1),$$

from which (3.14) is obtained. □

The proof of the next theorem is fully analogous to that of Theorem 3.4, thus omitted.

**Theorem 3.5.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$ERNS(G) = ERN(G) \leq \frac{4}{3} + \frac{2(n - 2)}{(3 - \gamma_2^+)(3 - \gamma_{n-1}^+)}.$$

*Equality holds if and only if  $G \cong K_{p,q}$ ,  $p + q = n$ .*

**Theorem 3.6.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$(3.15) \quad ERNS(G) \leq \frac{1}{3} \left( n + 2 + \frac{\left( (n - 2)(6 - \gamma_2^+ - \gamma_n^+) \right)^2}{8(n - 1 - R_{-1}(G))(3 - \gamma_2^+)(3 - \gamma_n^+)} \right).$$

*Equality holds if and only if  $G \cong K_n$ .*

*Proof.* Setting  $p_i = \frac{\gamma_i^+}{n-2}$ ,  $a_i = \frac{1}{3-\gamma_i^+}$ ,  $i = 2, 3, \dots, n$ ,  $R = \frac{1}{3-\gamma_2^+}$ ,  $r = \frac{1}{3-\gamma_n^+}$  in (2.2), we have that

$$\sum_{i=2}^n \left( \frac{\gamma_i^+}{n-2} \right) \left( \frac{1}{3-\gamma_i^+} \right) \sum_{i=2}^n \left( \frac{\gamma_i^+}{n-2} \right) (3 - \gamma_i^+) \leq \frac{1}{4} \left( \sqrt{\frac{3 - \gamma_n^+}{3 - \gamma_2^+}} + \sqrt{\frac{3 - \gamma_2^+}{3 - \gamma_n^+}} \right)^2.$$

Considering this with (1.2) and (3.9) and Lemma 2.1, we obtain that

$$\frac{2(n - 1 - R_{-1}(G))}{(n - 2)^2} (3ERNS(G) - n - 2) \leq \frac{1}{4} \left( \frac{(6 - \gamma_2^+ - \gamma_n^+)^2}{(3 - \gamma_2^+)(3 - \gamma_n^+)} \right).$$

From the above result, we arrive at the upper bound (3.15). The equality in (3.15) holds if and only if

$$\frac{1}{3 - \gamma_2^+} = \frac{1}{3 - \gamma_3^+} = \dots = \frac{1}{3 - \gamma_n^+},$$

that is

$$\gamma_2^+ = \gamma_3^+ = \dots = \gamma_n^+.$$

Thus, in view of Lemma 2.2, we conclude that the equality in (3.15) holds if and only if  $G \cong K_n$ . □

Using the techniques in Theorem 3.6 with Lemmas 2.1, 2.4, 2.6, 2.7 and 2.11, we have the following.

**Theorem 3.7.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$ERNS(G) = ERN(G) \leq \frac{1}{3} \left( n + 2 + \frac{2(n-2)^2}{(n-1 - R_{-1}(G))(3 - \gamma_2^+)(3 - \gamma_{n-1}^+)} \right).$$

*Equality holds if and only if  $G \cong K_{p,q}$ ,  $p + q = n$ .*

#### 4. RELATIONS BETWEEN $ERNS(G)$ AND OTHER ENERGIES

One of the chemically/mathematically most important graph spectrum-based invariants in graph theory is the concept of graph energy introduced in [15]. Due to the evident success of graph energy, a number of graph energies and energy-like graph invariants have been put forward in the literature. We first recall some of them.

For a graph  $G$ , in full analogy with the graph energy [15], Randić (normalized Laplacian or normalized signless Laplacian) energy is defined as [2, 8, 17]

$$RE(G) = \sum_{i=1}^n |\rho_i|,$$

where  $1 = \rho_1 \geq \rho_2 \geq \dots \geq \rho_n$  are the Randić eigenvalues of  $G$  [2, 24].

In analogous manner with Laplacian energy-like invariant [25], Laplacian incidence energy is introduced as [32]

$$LIE(G) = \sum_{i=1}^{n-1} \sqrt{\gamma_i^-}$$

and by analogy with incidence energy [16], the Randić (normalized) incidence energy is put forward in [9, 14] as

$$I_RE(G) = \sum_{i=1}^n \sqrt{\gamma_i^+}.$$

Here,  $\gamma_1^- \geq \gamma_2^- \geq \dots \geq \gamma_{n-1}^- > \gamma_n^- = 0$  and  $2 = \gamma_1^+ \geq \gamma_2^+ \geq \dots \geq \gamma_n^+ \geq 0$  are, respectively, the normalized Laplacian and normalized signless Laplacian eigenvalues of  $G$  [10, 14]. Note that  $LIE$  is equal to  $I_RE$ , for bipartite graphs [3].

Now, we are ready to give some relationships between  $ERNS(G)$  and other energies emphasized in the above.

**Theorem 4.1.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$(4.1) \quad ERNS(G) \geq 1 + \frac{(RE(G) - 1)^2}{4R_{-1} - 2 \sum_{i \sim j} \frac{1}{d_i d_j} \left( \sum_{k \sim i, k \sim j} \frac{1}{d_k} \right) - 1}.$$

*Equality is achieved for  $G \cong K_n$ .*

*Proof.* For  $x_i = |\gamma_i^+ - 1|$  and  $a_i = \frac{1}{3-\gamma_i^+}$ ,  $i = 2, 3, \dots, n$ , the inequality (3.7) becomes

$$(4.2) \quad \sum_{i=2}^n (\gamma_i^+ - 1)^2 (3 - \gamma_i^+) \geq \frac{\left(\sum_{i=2}^n |\gamma_i^+ - 1|\right)^2}{\sum_{i=2}^n \frac{1}{3-\gamma_i^+}}.$$

From (1.3) and Lemmas 2.8 and 2.9, we have

$$\begin{aligned} \sum_{i=2}^n (\gamma_i^+ - 1)^2 (3 - \gamma_i^+) &= \sum_{i=2}^n \rho_i^2 (2 - \rho_i) \\ &= 2 \sum_{i=2}^n \rho_i^2 - \sum_{i=2}^n \rho_i^3 \\ &= 2(2R_{-1} - 1) - \left(2 \sum_{i \sim j} \frac{1}{d_i d_j} \left(\sum_{k \sim i, k \sim j} \frac{1}{d_k}\right) - 1\right) \\ (4.3) \quad &= 4R_{-1} - 2 \sum_{i \sim j} \frac{1}{d_i d_j} \left(\sum_{k \sim i, k \sim j} \frac{1}{d_k}\right) - 1. \end{aligned}$$

Then by (4.2) and (4.3) and Lemma 2.1, we get that

$$4R_{-1} - 2 \sum_{i \sim j} \frac{1}{d_i d_j} \left(\sum_{k \sim i, k \sim j} \frac{1}{d_k}\right) - 1 \geq \frac{(RE(G) - 1)^2}{ERNS(G) - 1}.$$

From the above, the inequality (4.1) follows. One can easily check that the equality in (4.1) is achieved for  $G \cong K_n$ . □

**Theorem 4.2.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$(4.4) \quad ERNS(G) \geq \frac{1}{3} \left( n + 2 + \frac{(IRE(G) - \sqrt{2})^2}{2n - 1} \right).$$

*Equality holds if and only if  $G \cong K_n$ .*

*Proof.* Taking  $x_i = \sqrt{\gamma_i^+}$ ,  $a_i = 3 - \gamma_i^+$ ,  $i = 2, 3, \dots, n$ , in (3.7)

$$\sum_{i=2}^n \frac{\gamma_i^+}{3 - \gamma_i^+} \geq \frac{\left(\sum_{i=2}^n \sqrt{\gamma_i^+}\right)^2}{\sum_{i=2}^n (3 - \gamma_i^+)}.$$

Considering this with (1.2) and (3.9) and Lemma 2.1

$$3ERNS(G) - n - 2 \geq \frac{(IRE(G) - \sqrt{2})^2}{2n - 1}.$$

From the above we obtain (4.4). Equality in (4.4) holds if and only if

$$\frac{\sqrt{\gamma_2^+}}{3 - \gamma_2^+} = \frac{\sqrt{\gamma_3^+}}{3 - \gamma_3^+} = \dots = \frac{\sqrt{\gamma_n^+}}{3 - \gamma_n^+},$$

that is if and only if

$$\left(\sqrt{\gamma_i^+} - \sqrt{\gamma_j^+}\right) \left(3 + \sqrt{\gamma_i^+ \gamma_j^+}\right) = 0, \quad i \neq j,$$

which implies that equality in (4.4) holds if and only if  $G \cong K_n$ . □

**Theorem 4.3.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{1}{3} \left( n + 2 + \frac{(LIE(G) - \sqrt{2})^2}{2(n-2)} \right).$$

*Equality holds if  $G \cong K_{p,q}$ ,  $p + q = n$ .*

### 5. RELATIONSHIPS BETWEEN $ERNS(G)$ AND $K^+(G)$

In this section, we present some relationships between  $ERNS(G)$  and  $K^+(G)$ .

**Theorem 5.1.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$(5.1) \quad ERNS(G) \geq \frac{3}{2} - K^+(G) + \frac{3(n-1)^2}{2(n-1-R_{-1}(G))}.$$

*Equality holds if and only if  $G \cong K_n$ .*

*Proof.* The arithmetic-harmonic mean inequality can be considered as [29]

$$(5.2) \quad \sum_{i=2}^n a_i \sum_{i=2}^n \frac{1}{a_i} \geq (n-1)^2.$$

For  $a_i = \gamma_i^+(3 - \gamma_i^+)$ ,  $i = 2, 3, \dots, n$ , the above inequality transforms into

$$(5.3) \quad \sum_{i=2}^n \gamma_i^+(3 - \gamma_i^+) \sum_{i=2}^n \frac{1}{\gamma_i^+(3 - \gamma_i^+)} \geq (n-1)^2.$$

From the above, (1.2) and Lemma 2.1,

$$(5.4) \quad \sum_{i=2}^n \frac{1}{\gamma_i^+(3 - \gamma_i^+)} \geq \frac{(n-1)^2}{2(n-1-R_{-1}(G))}.$$

On the other hand, by Lemma 2.1, we have that

$$\begin{aligned} \sum_{i=2}^n \frac{1}{\gamma_i^+(3 - \gamma_i^+)} &= \frac{1}{3} \left( \sum_{i=2}^n \frac{1}{\gamma_i^+} + \sum_{i=2}^n \frac{1}{3 - \gamma_i^+} \right) \\ &= \frac{1}{3} \left( K^+(G) - \frac{1}{2} + ERNS(G) - 1 \right) \\ &= \frac{1}{3} \left( K^+(G) - \frac{3}{2} + ERNS(G) \right) \end{aligned}$$

Combining this with (5.4) we arrive at (5.1). Equality in (5.3) holds if and only if  $\gamma_2^+(3 - \gamma_2^+) = \gamma_3^+(3 - \gamma_3^+) = \dots = \gamma_n^+(3 - \gamma_n^+)$ . Suppose  $i \neq j$ . Then, from the identity  $\gamma_i^+(3 - \gamma_i^+) = \gamma_j^+(3 - \gamma_j^+)$ , follows that  $(\gamma_i^+ - \gamma_j^+)(3 - \gamma_i^+ - \gamma_j^+) = 0$ . Thus, we conclude

that equality in (5.3) holds if and only if  $\gamma_2^+ = \gamma_3^+ = \dots = \gamma_n^+$ . Having this in mind and Lemma 2.2, we conclude that equality in (5.1) holds if and only if  $G \cong K_n$ .  $\square$

Using the similar idea in Theorem 5.1 with Lemmas 2.1, 2.4 and 2.6, we get the following.

**Theorem 5.2.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{11}{6} - K(G) + \frac{3(n-2)^2}{2(n-1-R_{-1}(G))}.$$

*Equality holds if and only if  $G \cong K_{p,q}$ ,  $p + q = n$ .*

**Theorem 5.3.** *Let  $G$  be a connected non-bipartite graph with  $n \geq 3$  vertices. Then*

$$(5.5) \quad ERNS(G) \geq \frac{1}{3} \left( n + 2 + \frac{2(n-1)^2}{6K^+(G) - 2n - 1} \right).$$

*Equality holds if and only if  $G \cong K_n$ .*

*Proof.* For  $a_i = \frac{\gamma_i^+}{3-\gamma_i^+}$ ,  $i = 2, 3, \dots, n$ , the inequality (5.2) transforms into

$$(5.6) \quad \sum_{i=2}^n \frac{\gamma_i^+}{3-\gamma_i^+} \sum_{i=2}^n \frac{3-\gamma_i^+}{\gamma_i^+} \geq (n-1)^2.$$

On the other hand, by Lemma 2.1, we have that

$$\sum_{i=2}^n \frac{3-\gamma_i^+}{\gamma_i^+} = 3 \left( K^+(G) - \frac{1}{2} \right) - (n-1) = 3K^+(G) - n - \frac{1}{2}.$$

From the above, (3.9) and (5.6), we obtain (5.5). Equality in (5.6) holds if and only if

$$\frac{\gamma_2^+}{3-\gamma_2^+} = \frac{\gamma_3^+}{3-\gamma_3^+} = \dots = \frac{\gamma_n^+}{3-\gamma_n^+}.$$

Suppose  $i \neq j$ . Then equality in (5.6) holds if and only if  $\frac{\gamma_i^+}{3-\gamma_i^+} = \frac{\gamma_j^+}{3-\gamma_j^+}$ , that is if and only if  $\gamma_2^+ = \gamma_3^+ = \dots = \gamma_n^+$ . By Lemma 2.2, equality in (5.5) holds if and only if  $G \cong K_n$ .  $\square$

Considering the same idea in Theorem 5.3 together with Lemma 2.1, 2.4 and 2.6, we have the following.

**Theorem 5.4.** *Let  $G$  be a connected bipartite graph with  $n \geq 3$  vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{1}{3} \left( n + 2 + \frac{2(n-2)^2}{6K(G) - 2n + 1} \right).$$

*Equality holds if and only if  $G \cong K_{p,q}$ ,  $p + q = n$ .*

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<sup>1</sup>YENIKENT KARDELEN KONUTLARI, SELÇUKLU, 42070  
KONYA, TURKEY  
Email address: srf\_burcu\_bozkurt@hotmail.com

<sup>2</sup>FACULTY OF ELECTRONIC ENGINEERING,  
UNIVERSITY OF NIŠ  
Email address: igor@elfak.ni.ac.rs  
Email address: ema@elfak.ni.ac.rs  
Email address: marjan.matejicelfak.ni.ac.rs