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ON GENERALIZED LAGRANGE-BASED APOSTOL-TYPE AND RELATED POLYNOMIALS

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Abstract. In this article, we introduce a new class of generalized polynomials, ascribed to the new families of generating functions and identities concerning Lagrange, Hermite, Miller-Lee, and Laguerre polynomials and of their associated forms. It is shown that the proposed method allows the derivation of sum rules involving products of generalized polynomials and addition theorems. We develop a point of view based on generating relations, exploited in the past, to study some aspects of the theory of special functions. The possibility of extending the results to include generating functions involving products of Lagrange-based unified Apostol-type and other polynomials is finally analyzed.

1. Introduction

The Lagrange polynomials in several variables, which are known as the Chan-Chyan-Srivastava polynomials [2] are defined by means of the following generating function

$$\prod_{j=1}^{r}(1-x_jt)^{-\alpha_j}=\sum_{n=0}^{\infty}g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)t^n, \tag{1.1}$$

where $\alpha_j\in\mathbb{C}$, $j=1,\dots,r$, $|t|<\min\{|x_1|^{-1},\dots,|x_r|^{-1}\}$, and are represented by

$$\begin{aligned}g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,x_2,\dots,x_r)=&\sum_{k_{r-1}=0}^{n}\cdots\sum_{k_2}^{k_3}\sum_{k_1}^{k_2}(\alpha_1)_{k_1}\\&\times(\alpha_2)_{k_2-k_1}\cdots(\alpha_{r-1})_{k_{r-1}-k_{r-2}}(\alpha_r)_{n-k_{r-1}}\end{aligned}$$

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$$\times \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-k_1}}{(k_2-k_1)!}\cdots\frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!}\frac{x_r^{n-k_r-1}}{(n-k_r-1)!}, \tag{1.2}$$

where $(\lambda)_0 := 1$ and $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$, $n \in \mathbb{N} := \{1,2,3,\dots\}$.

Altin and Erkus [1] presented a multivariable extension of the so called Lagrange-Hermite polynomials generated by (see [1, page 239, (1.2)]):

$$\prod_{j=1}^{r}(1-x_jt^j)^{-\alpha_j} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots x_r)t^n, \tag{1.3}$$

where $\alpha_j \in \mathbb{C}$, $j = 1,\dots,r$, $|t| < \min\{|x_1|^{-1}, |x_2|^{-\frac{1}{2}}, \dots, |x_r|^{-\frac{1}{r}}\}$ and

$$h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{k_1+2k_2+\cdots+rk_r=n}(\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}.$$

The special case when $r = 2$ in (1.3) is essentially a case which corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli et al. [3]

$$(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)t^n. \tag{1.4}$$

The multi-variable (Erkus-Srivastava) polynomials $U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ defined by the following generating function, (see [5, page 268, (3))]:

$$\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j} = \sum_{n=0}^{\infty} U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)t^n, \tag{1.5}$$

where $\alpha_j \in \mathbb{C}$, $j = 1,\cdots,r$, $l_j \in \mathbb{N}$, $j = 1,\dots,r$, $|t| < \min\{|x_1|^{-1},\dots,|x_r|^{-1}\}$, are a unification (and generalization) of several known families of multivariable polynomials including (for example) the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ defined by (1.1) (see, for details, [5]). It is evident that the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ and the Lagrange-Hermite polynomials $h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ follow as the special cases of the Erkus-Srivastava polynomials $U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ when $l_j = 1$, $j = 1,\dots,r$.

The generating function (1.5) yields the following explicit representation (see [5, page 268, (4)]):

$$U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) = \sum_{l_1k_1+\cdots+l_rk_r=n}(\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!},$$

which, in the special case when $l_j = 1$, $j = 1,\dots,r$, corresponds to (1.2).

Recently, Ozarslan [14] introduced the following unification of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Explicitly Ozarslan studied the following generating function:

$$f_{a,b}^{(\alpha)}(x;t,a,b) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}, \tag{1.6}$$

where

$$\left|t + b\ln\left(\frac{\beta}{\alpha}\right)\right| < 2\pi, \quad k \in \mathbb{N}_0, \quad a, b \in \mathbb{R}\backslash\{0\}, \quad \alpha, \beta \in \mathbb{C}.$$

For $\alpha = 1$ in (1.6), we get

$$f_{a,b}(x;t,a,b) = \frac{2^{1-k}t^k}{\beta^b e^t - a^b} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}(x;k,a,b)\frac{t^n}{n!}, \tag{1.7}$$

where

$$\left|t + b\ln\left(\frac{\beta}{\alpha}\right)\right| < 2\pi, \quad k \in \mathbb{N}_0, \quad a, b \in \mathbb{R}\backslash\{0\}, \quad \alpha, \beta \in \mathbb{C}.$$

From (1.6) and (1.7), we have

$$P_{n,\beta}^{(1)}(x;k,a,b) = P_{n,\beta}(x;k,a,b), \quad n \in \mathbb{N},$$

which is defined by Ozden and Simsek [16]. Now Ozden et al. [15] introduced many properties of these polynomials. We give some specific special cases.

1. By substituting $a = b = k = 1$ and $\beta = \lambda$ into (1.6), one has the Apostol-Bernoulli polynomials $P_{n,\beta}^{(\alpha)}(x;1,1,1) = P_{n,\lambda}^{(\alpha)}(x;1,1,1)$, which are defined by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \quad |t + \log\lambda| < 2\pi, \tag{1.8}$$

(see for details [6–13] and also, the references cited in each of these earlier works).

For $\lambda = \alpha = 1$ in (1.8), the result reduces to

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi,$$

where $B_n(x)$ denotes the classical Bernoulli polynomials (see from example [17, 18], see also the references cited in each of these earlier works).

2. If we substitute $b = 1$, $k = 0$, $a = -1$ and $\beta = \lambda$ into (1.6), we have the Apostol-Euler polynomials $P_{n,\lambda}^{(\alpha)}(x;0,-1,1) = E_n^{(\alpha)}(x,\lambda)$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \quad |t + \log\lambda| < \pi, \tag{1.9}$$

(see for details [6–13] and also the references cited in each of these earlier works).

For $\lambda = \alpha = 1$ in (1.9), the result reduces to

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi,$$

where $E_n(x)$ denotes the classical Euler polynomials (see from example [14–18] and also the references cited in each of these earlier works).

3. By substituting $b = \alpha = 1$, $k = 1$, $a = -1$ and $\beta = \lambda$ into (1.6), one has the Apostol-Genocchi polynomials $P_{n,\beta}^{(1)}(x; 1, -1, 1) = \frac{1}{2}G_n(x; \lambda)$, which is defined by means of the following generating function

$$\frac{2t}{\lambda e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x; \lambda)\frac{t^n}{n!}, \quad |t + \log \lambda| < \pi,$$

(see for details [6–18] and also the references cited in each of these earlier works).

4. By substituting $x = 0$ in the generating function (1.6), we obtain the corresponding unification of the generating functions of Bernoulli, Euler and Genocchi numbers of higher order. Thus, we have

$$P_{n,\beta}^{(\alpha)}(0; k, a, b) = P_{n,\beta}^{(\alpha)}(k, a, b), \quad n \in \mathbb{N}.$$

The generalized Stirling numbers of the second kinds $S(n, \nu, a, b, \beta)$ of order ν are defined in [16] as follows:

$$\sum_{n=0}^{\infty} S(n, \nu, a, b, \beta)\frac{t^n}{n!} = \frac{(\beta^b e^t - a^b)^\nu}{\nu!}. \tag{1.10}$$

On setting $\beta = \lambda$, $a = b = 1$, (1.10) reduces to

$$\sum_{n=0}^{\infty} S(n, \nu, \lambda)\frac{t^n}{n!} = \frac{(\lambda e^t - 1)^\nu}{\nu!}.$$

The outline of this paper is as follows. In Section 2, we introduce the Lagrange-based unified Apostol-type polynomials and investigate some properties. In Section 3, we introduce Miller-Lee polynomials and derive some relationship between Lagrange-based unified Apostol-type polynomials. In Section 4, we introduce Laguerre polynomials and obtain some properties of Laguerre and Lagrange-based unified Apostol-type polynomials.

2. Lagrange-Based Unified Apostol-type Polynomials

In this section, we connect the Lagrange polynomials in several variables with Hermite and Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. The resulting formulae allow a considerable unification of various special results that appear in the literature.

Definition 2.1. The Lagrange-based unified Apostol-type polynomials $T_{n,\beta,k}^{(\alpha_1,\cdots,\alpha_r)}(x|x_1, \ldots x_r; a, b)$ in several variables by means of the following generating function:

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)e^{xt}\prod_{j=1}^{r}(1 - x_j t^j)^{-\alpha_j} = \sum_{n=0}^{\infty} T_{n,\beta,k}^{(\alpha_1,\cdots,\alpha_r)}(x|x_1, \ldots x_r; a, b)t^n, \tag{2.1}$$

which, for ordinary case $r = 2$, (2.1) reduces to the Lagrange-based unified Apostol-type Hermite polynomials

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right) e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n. \tag{2.2}$$

In particular, when $x_1 = x_2 = 0$, (2.2) reduces to unified Apostol-type polynomials $Y_{n,\beta}(x;k,a,b)$ defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right) e^{xt} = \sum_{n=0}^{\infty} Y_{n,\beta}(x;k,a,b)\frac{t^n}{n!}.$$

The Lagrange-based unified Apostol-type Hermite polynomials of order α are defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n. \tag{2.3}$$

Note that

$${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b) = \sum_{m=0}^{n} h_m^{(\alpha_1,\alpha_2)}(x_1,x_2)\frac{Y_{n-m,\beta}^{(\alpha)}(x;k,a,b)}{(n-m)!}.$$

On setting $\alpha = x = 0$ in (2.3), the result reduces to (1.4). For $\alpha = 1$, (2.3) reduces to (2.2).

The Lagrange-based unified Apostol-type polynomials of order α are defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_Yg_{n.\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n. \tag{2.4}$$

Thus, we have

$${}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b) = \sum_{m=0}^{n} g_m^{(\alpha_1,\alpha_2)}(x_1,x_2)\frac{Y_{n-m,\beta}^{(\alpha)}(x;k,a,b)}{(n-m)!}.$$

On taking $x = 0$ in (2.4), the result reduces to

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} (1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x_1,x_2;k,a,b)t^n,$$

where

$${}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(0|x_1,x_2;k,a,b) = {}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x_1,x_2;k,a,b).$$

Remark 2.1. By substituting $a = b = k = 1$ and $\beta = \lambda$ in (2.3), we get the generalized Lagrange-based Apostol-Bernoulli polynomials by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_BH_{n,\lambda}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n. \tag{2.5}$$

For $\lambda = 1$ in (2.5), the result reduces to the known result of Khan and Pathan [8] as follows

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt}(1 - x_1 t)^{-\alpha_1}(1 - x_2 t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_B H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2)t^n.$$

Remark 2.2. If we substitute $b = \alpha = 1$, $k = 0$, $a = -1$ and $\beta = \lambda$ into (2.3), we get the generalized Lagrange-based Apostol-Euler polynomials by means of the following generating function

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt}(1 - x_1 t)^{-\alpha_1}(1 - x_2 t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_E H_{n,\lambda}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2)t^n.$$

Remark 2.3. By substituting $b = \alpha = 1$, $k = 1$, $a = -\frac{1}{2}$ and $\beta = \frac{\lambda}{2}$ into (2.3), we obtain the generalized Lagrange-based Apostol-Genocchi polynomials by means of the following generating function

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt}(1 - x_1 t)^{-\alpha_1}(1 - x_2 t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_G H_{n,\lambda}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2)t^n.$$

Theorem 2.1. *The following summation formula for Lagrange-Hermite polynomials* $h_n^{(\alpha_1,\alpha_2)}(x_1, x_2)$ *holds true:*

$$h_n^{(\alpha_1,\alpha_2)}(x_1, x_2) = \sum_{m=0}^{n} {}_H Y_{n-m,\beta}^{(\alpha_1,\alpha_2)}(x|x_1, x_2; k, a, b)\frac{(-x)^m}{m!}. \tag{2.6}$$

Proof. For $\alpha = 0$ in (2.3), we have

$$\begin{aligned} e^{-xt}\sum_{n=0}^{\infty} {}_H Y_{n,\beta}^{(\alpha_1,\alpha_2)}(x|x_1, x_2; k, a, b)t^n =&(1 - x_1 t)^{-\alpha_1}(1 - x_2 t^2)^{-\alpha_2} \\ &\times \sum_{m=0}^{\infty}\frac{(-x)^m t^m}{m!}\sum_{n=0}^{\infty} {}_H Y_{n,\beta}^{(\alpha_1,\alpha_2)}(x|x_1, x_2; k, a, b)t^n \\ &= \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1, x_2)t^n. \end{aligned}$$

Replacing n by $n - m$ in L. H. S, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_H Y_{n-m,\beta}^{(\alpha_1,\alpha_2)}(x|x_1, x_2; k, a, b)\frac{(-x)^m}{m!}t^n = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1, x_2)t^n.$$

Comparing the coefficients of t^n on both sides, we get (2.6). □

Theorem 2.2. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials* ${}_H Y_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2; k, a, b)$ *holds true:*

$${}_H Y_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x + y|x_1, x_2; k, a, b) = \sum_{m=0}^{n} {}_H Y_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1, x_2; k, a, b)\frac{y^m}{m!}. \tag{2.7}$$

Proof. From (2.3), we have

$$\begin{aligned}
&\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+y)t}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\
&=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n\\
&=\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}e^{yt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\
&=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty} y^m \frac{t^m}{m!}.
\end{aligned}$$

Replacing n by $n-m$ in above equation and comparing the coefficients of t^n on both sides, we get the result (2.7). □

Theorem 2.3. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$$\begin{aligned}
&{}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1+\alpha_2,\alpha_3+\alpha_4)}(x+y|x_1,x_2;k,a,b)\\
&=\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_3)}(x|x_1,x_2;k,a,b)\,{}_HY_{m,\beta}^{(\gamma|\alpha_2,\alpha_4)}(y|x_1,x_2;k,a,b).
\end{aligned}\tag{2.8}$$

Proof. By using definition (2.3), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1+\alpha_2,\alpha_3+\alpha_4)}(x+y|x_1,x_2;k,a,b)t^n\\
&=\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha+\gamma} e^{(x+y)t}(1-x_1t)^{-\alpha_1-\alpha_2}(1-x_2t^2)^{-\alpha_3-\alpha_4}\\
&=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty} {}_HY_{m,\beta}^{(\gamma|\alpha_3,\alpha_4)}(y|x_1,x_2;k,a,b)t^m.
\end{aligned}$$

Replacing n by $n-m$ in above equation, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_3)}(x|x_1,x_2;k,a,b)\,{}_HY_{m,\beta}^{(\gamma|\alpha_2,\alpha_4)}(y|x_1,x_2;k,a,b)\right)t^n.
\end{aligned}$$

Comparing the coefficients of t^non both sides, we get the result (2.8). □

Theorem 2.4. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)=\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x-z|x_1,x_2;k,a,b)\frac{z^m}{m!}.\tag{2.9}$$

Proof. By exploiting the generating function (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x-z)t}e^{zt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x-z|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty} z^m\frac{t^m}{m!}.$$

Replacing n by $n-m$ in above equation, we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x-z|x_1,x_2;k,a,b)\frac{z^m}{m!}\right)t^n.$$

Comparing the coefficients of t^n on both sides, we get the result (2.9). □

Theorem 2.5. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$$\begin{aligned}{}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b) &= \sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(z|x_1,x_2;k,a,b)\\ &\times \frac{{}_HY_{m,\beta}^{(\gamma|0,0)}(x-z;k,a,b)}{m!}.\end{aligned}\tag{2.10}$$

Proof. Going back to the generating function (2.3), we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n$$
$$=\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{zt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\gamma} e^{(x-z)t}$$
$$=\sum_{n=0}^{\infty} {}_HY_{n}^{(\alpha|\alpha_1,\alpha_2)}(z|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty}\frac{{}_HY_{m,\beta}^{(\gamma|0,0)}(x-z;k,a,b)t^m}{m!}.$$

Replacing n by $n-m$ in above equation and comparing the coefficients of t^n on both sides, we get the result (2.10). □

Theorem 2.6. *The following implicit summation formula for the Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$$ {}_HY_{n,\beta}^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2;k,a,b) = \sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)h_m^{(\beta_1,\beta_2)}(x_1,x_2).\tag{2.11}$$

Proof. Using definition (2.3), we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2;k,a,b)t^n$$

$$
\begin{aligned}
&=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1-\beta_1}(1-x_2t^2)^{-\alpha_2-\beta_2}\\
&=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty} h_m^{(\beta_1,\beta_2)}(x_1,x_2)t^m.
\end{aligned}
$$

Replacing n by $n-m$ in above equation, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2;k,a,b)t^n\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)h_m^{(\beta_1,\beta_2)}(x_1,x_2)t^n.
\end{aligned}
$$

Comparing the coefficients of t^n on both sides, we get the result (2.11). □

Theorem 2.7. *There is the following relation between the Apostol-type Stirling numbers of second kind and Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$*:*

$$
\begin{aligned}
&a^{b\alpha}\alpha!\sum_{r=0}^{n} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)}{r!}\\
&=\begin{cases} 0, & \text{for } n<k\alpha,\\ 2^{(1-k)\alpha}h_{n-k\alpha}^{(\alpha_1,\alpha_2)}(x|x_1,x_2), & \text{for } n\geq k\alpha,\end{cases}
\end{aligned}
\tag{2.12}
$$

with $\alpha\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ *and* $k\in\mathbb{N}$ *fixed.*

Proof. By using equation (2.3) and (1.10), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n\\
&=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\
&=\frac{2^{(1-k)\alpha}t^{k\alpha}}{a^{b\alpha}\left(\left(\frac{\beta}{a}\right)^b e^t-1\right)^{\alpha}}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\
&=\frac{2^{(1-k)\alpha}t^{k\alpha}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{a^{b\alpha}\alpha!\sum\limits_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^r}{r!}},\\
&\left(\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n\right)\left(a^{b\alpha}\alpha!\sum_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^r}{r!}\right)\\
&=2^{(1-k)\alpha}t^{k\alpha}\sum_{n=0}^{\infty}h_n^{(\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{n!},
\end{aligned}
$$

$$a^{b\alpha}\alpha!\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}{}_{H}Y_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)S\left(r,\alpha,\left(\frac{\beta}{a}\right)^{b}\right)\frac{1}{r!}\right)t^n$$
$$=2^{(1-k)\alpha}\sum_{n=0}^{\infty}h_{n-k\alpha}^{(\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{(n-k\alpha)!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result (2.12). □

Theorem 2.8. *There is the following relation between the Apostol-type Stirling numbers of second kind and Lagrange-based unified Apostol-type Hermite polynomials* ${}_{H}Y_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$*:*

$$\gamma!\sum_{r=0}^{n}{}_{H}Y_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{S(r,\gamma,a,b,\beta)}{r!}$$
$$=\begin{cases}0, & \text{for } n<k\gamma,\\ 2^{(1-k)\gamma}{}_{H}Y_{n-k\gamma,\beta}^{(\alpha-\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b), & \text{for } n\geq k\gamma,\end{cases} \tag{2.13}$$

with $\gamma\in\mathbb{N}_0$ *and* $k\in\mathbb{N}$ *fixed.*

Proof. From (2.3) and (1.10), we have

$$\sum_{n=0}^{\infty}{}_{H}Y_{n,\beta}^{(\alpha-\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n$$
$$=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha-\gamma}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\left(\frac{\beta^b e^t-a^b}{2^{1-k}t^k}\right)^{\gamma},$$
$$2^{(1-k)\gamma}\sum_{n=0}^{\infty}{}_{H}Y_{n,\beta}^{(\alpha-\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^{n+k\gamma}$$
$$=\gamma!\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}{}_{H}Y_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{S(r,\gamma,a,b,\beta)}{r!}\right)t^n.$$

Comparing the coefficients of t^n on both sides, we get the desired result (2.13). □

Theorem 2.9. *The following implicit summation formula involving the Lagrange-based unified Apostol-type Hermite polynomials* ${}_{H}Y_{n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *and Lagrange-based unified Apostol-type polynomials* ${}_{Y}g_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$$\sum_{m=0}^{n}{}_{H}Y_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{(\gamma)_m y^m}{m!}=\sum_{m=0}^{n}{}_{Y}g_{n-m,\beta}^{(\alpha|\gamma,\alpha_2)}(x|y,x_2;k,a,b)\frac{x_1^m(\alpha_1)_m}{m!}. \tag{2.14}$$

Proof. We first start with the generating function (2.3). On multiplying both the sides by $(1-yt)^{-\gamma}$ and interpreting the result using (2.4) and series expansion of $(1-yt)^{-\gamma}$, we get the required result (2.14). □

3. Lagrange-Based Unified Apostol-Type Miller-Lee Polynomials

The definitions (2.3) and (2.4) can be exploited in a number of ways and provide a useful tool to frame known and new generating functions in the following way.

As a first example, we set $\alpha = \alpha_2 = 0$, $\alpha_1 = m+1$, $x_1 = 1$ in (2.3) to get

$$e^{xt}(1-t)^{-m-1} = \sum_{n=0}^{\infty} G_n^{(m)}(x)t^n, \quad |t| < 1,$$

where $G_n^{(m)}(x)$ are called the Miller-Lee polynomials (see [4, page 21, (1.11)]).

Another example is the definition of Lagrange-based Apostol-type Hermite-Miller-Lee polynomials ${}_YHG_{n,\beta}^{(m,\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ given by the following generating function

(3.1)
$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}\frac{(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} {}_YHG_{n,\beta}^{(m,\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{t^n}{n!},$$

which for $\alpha = 0$ reduces to

$$\frac{(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{(1-t)^{m+1}}e^{xt} = \sum_{n=0}^{\infty} {}_HG_n^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{n!},$$

where ${}_HG_n^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)$ are called the Lagrange-based Hermite-Miller-Lee polynomials.

Putting $\alpha_1 = \alpha_2 = 0$ into (3.1) gives

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} \frac{e^{xt}}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)\frac{t^n}{n!}, \tag{3.2}$$

where ${}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)$ are called the Apostol-type Miller-Lee polynomials.

Theorem 3.1. *The following relationship between Lagrange-based unified Apostol-type Hermite polynomials ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$, Apostol-type Miller-Lee polynomials ${}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)$ and Miller-Lee polynomials $G_n^{(m)}(x)$ holds true:*

$$\begin{aligned} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b) =& \sum_{r=0}^{n}\binom{n}{r} Y_{n-r,\beta}^{(\alpha)}(k,a,b)G_r^{(m)}(x) \\ =& n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(-\alpha_2)_r(x_2)^r}{r!}\,{}_HY_{n-2r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b). \end{aligned} \tag{3.3}$$

Proof. For $x_1 = 1$ and $\alpha_1 = m+1$ in (2.3) and using (3.2), we have

$$\begin{aligned} \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-t)^{-m-1} =& \sum_{n=0}^{\infty} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)\frac{t^n}{n!} \\ =& (1-x_2t^2)^{\alpha_2}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n, \end{aligned}$$

which on using binomial expansion takes the form

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(k,a,b)\frac{t^n}{n!}\sum_{r=0}^{\infty} G_r^{(m)}(x)t^r = \sum_{r=0}^{\infty}\frac{(-\alpha_2)_r(x_2)^r t^{2r}}{r!} \times \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.$$

Replacing n by $n-r$ in above equation, we have

$$\sum_{n=0}^{\infty} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} Y_{n-r,\beta}^{(\alpha)}(k,a,b)G_r^{(m)}(x)\frac{t^n}{(n-r)!}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]}\frac{(-\alpha_2)_r(x_2)^r}{r!} {}_HY_{n-2r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.$$

Finally, comparing the coefficients of t^n, we get (3.3). □

Remark 3.1. Equation (3.3) is obviously a series representation of the Apostol-type Miller-Lee polynomials ${}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)$ linking Lagrange-based unified Apostol-type Hermite polynomials and Miller-Lee polynomials.

Theorem 3.2. *The following relationship holds true:*

(3.4)
$${}_HY_{n,\beta}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b) = \sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2;k,a,b)G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r.$$

Proof. On replacing x by $x+y$ and α_1 by α_1+m+1, respectively in (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+y)t}(1-x_1t)^{-m-1}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$= \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n,$$

which can be written as

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2;k,a,b)t^n\sum_{r=0}^{\infty} G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r t^r$$
$$= \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n.$$

Now replacing n by $n-r$ in the left hand side of the above equation, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2;k,a,b)G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r t^n$$
$$= \sum_{n=0}^{\infty} {}_HY_{n}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n.$$

Finally, comparing the coefficients of t^n, we get the result (3.4). □

Theorem 3.3. *The following relationship holds true:*

$$\begin{aligned}&\sum_{r=0}^{n} Y_{n-r,\beta}^{(\alpha)}(k,a,b)_{H}G_{r}^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{r!}{(n-r)!}\\
&=\sum_{r=0}^{n}(\alpha_1)_r x_1^r{}_{H}Y_{n-r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b).\end{aligned}\tag{3.5}$$

Proof. For $\alpha_1 = m+1$ and $x_1 = 1$ in (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-t)^{-m-1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.$$

Multiplying both sides by $(1-x_1t)^{-\alpha_1}$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(k,a,b)\frac{t^n}{n!}\sum_{r=0}^{\infty} {}_{H}G_{r}^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)t^r\\
&=7(1-x_1t)^{-\alpha_1}\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n\\
&=\sum_{r=0}^{\infty}(\alpha_1)_r x_1^r\frac{t^r}{r!}\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.\end{aligned}$$

Now replacing n by $n-r$ in the above equation, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{r=0}^{n} Y_{n-r,\beta}^{(\alpha)}(k,a,b)_{H}G_{r}^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{(n-r)!}\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{n}(\alpha_1)_r x_1^r{}_{H}Y_{n-r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)\frac{t^n}{r!}.\end{aligned}$$

Comparing the coefficients of t^n on both sides, we get the result (3.5). □

4. Lagrange-Based Unified Apostol-Type Laguerre Polynomials

In this section, we shall be interested in the connection between the Lagrange-based unified Apostol-type Hermite polynomials ${}_{H}Y_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ and Laguerre polynomials $L_n^{(m)}(x)$.

For $x_2 = 0$, $x_1 = -1$, $\alpha_1 = -m$ and $\alpha_2 = 0$ in equation (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1+t)^m = \sum_{n=0}^{\infty} {}_{L}Y_{n,\beta}^{(\alpha|m)}(x;k,a,b)\frac{t^n}{n!},\tag{4.1}$$

where ${}_{H}Y_{n,\beta}^{(\alpha|-m,0)}(x|-1,0;k,a,b) = {}_{L}Y_{n,\beta}^{(\alpha|m)}(x;k,a,b)$ are called the generalized Laguerre-based unified Apostol-type polynomials.

When $\alpha = 0$ in (4.1), ${}_{Y}L_{n,\beta}^{(\alpha|m)}(x;k,a,b)$ reduces to ordinary Laguerre polynomials $L_n^{(m)}(x)$ as follows (see [19])

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right).$$

Theorem 4.1. *The following relationship between Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *and Laguerre polynomials* $L_n^{(m)}(x)$ *holds true:*

$$\sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)L_r^{(m-r)}(y) = \sum_{r=0}^{n}(\alpha)_r x_1^r {}_HY_{n-r,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)\frac{1}{r!}. \tag{4.2}$$

Proof. Replacing x by $x+y$ and setting $x_1=-1$, $\alpha_1=-m$ in (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+y)t}(1+t)^m(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)t^n.$$

Multiplying both sides $(1-x_1t)^{-\alpha_1}$, we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+y)t}(1+t)^m(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$=(1-x_1t)^{-\alpha_1}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)t^n,$$
$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \sum_{r=0}^{\infty} L_r^{(m-r)}(y)t^r$$
$$=\sum_{r=0}^{\infty}\frac{(\alpha)_r(x_1)^r t^r}{r!}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)t^n.$$

Replacing n by $n-r$ in above equation, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)L_r^{(m-r)}(y)t^n$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}(\alpha)_r x_1^r {}_HY_{n-r,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)\frac{t^n}{r!}.$$

Comparing the coefficients of t^n on both sides, we get (4.2). □

Theorem 4.2. *The following relationship holds true:*

$$\sum_{k=0}^{n} Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)L_k^{(m-k)}(y)\frac{1}{(n-k)!} = {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b). \tag{4.3}$$

Proof. Replacing x by $x+y$ and setting $x_1=-1$, $\alpha_1=-m$ and $\alpha_2=0$ in equation (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+y)t}(1+t)^m = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b)t^n,$$

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}\sum_{k=0}^{\infty} L_k^{(m-k)}(y)t^k = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b)t^n.$$

Replacing n by $n-k$ in the left hand side of the above equation, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)L_k^{(m-k)}(y)\frac{t^n}{(n-k)!} = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b)t^n.$$

Comparing the coefficients of t^n on both sides, we get (4.3). □

Theorem 4.3. *The following relationship holds true:*

$$\begin{aligned}&\sum_{k=0}^{n} {}_HY_{n-k,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)(-x_1)^k L_k^{(m-k)}(y/x_1)\\ &={}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b).\end{aligned} \tag{4.4}$$

Proof. Replacing α_1 by $-m+\alpha_1$ and $x \to x-y$ in (2.3), we have

$$\begin{aligned}&\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x-y)t}(1-x_1t)^{m-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\ &=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b)t^n,\\ &\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \sum_{k=0}^{\infty}(-x_1)^k t^k L_k^{(m-k)}(y/x_1)\\ &=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b)t^n.\end{aligned}$$

Replacing n by $n-k$ in the left hand side of the above equation, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_HY_{n-k,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)(-x_1)^k L_k^{(m-k)}(y/x_1)t^n\\ &=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b)t^n.\end{aligned}$$

Comparing the coefficients of t^n on both sides, we get (4.4). □

Theorem 4.4. *The following relationship holds true:*

$$\sum_{k=0}^{n} L_k^{(m-k)}(y)\frac{Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)}{(n-k)!} = {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b). \tag{4.5}$$

Proof. For $x_1=-1$, $\alpha_1=-m$, $\alpha_2=0$, $x \longrightarrow x-y$ in (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x-y)t}(1+t)^m = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x-y|-1,0;k,a,b)t^n,$$

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}\sum_{k=0}^{\infty} L_k^{(m-k)}(-y)t^k = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x-y|-1,0;k,a,b)t^n.$$

Replacing n by $n-k$ in the left hand side of the above equation, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)L_{k}^{(m-k)}(-y)\frac{t^n}{(n-k)!}=\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha|-m,0)}(x-y|-1,0;k,a,b)t^n.$$

Finally, replacing y by $-y$ and comparing the coefficients of t^n, we get (4.5). □

Theorem 4.5. *The following relationship holds true:*

(4.6)

$$\sum_{r=0}^{n}\binom{n}{r} {}_{L}Y_{n-r,\beta}^{(\alpha|m)}(x;k,a,b){}_{L}Y_{r,\beta}^{(\gamma|k)}(y;k,a,b)=n!{}_{H}Y_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b).$$

Proof. Replacing α by $\alpha+\gamma$, $x \to x+y$ and setting $x_1=-1$, $\alpha_1=-m-k$, $\alpha_2=0$ in (2.4), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha+\gamma} e^{(x+y)t}(1+t)^{m+k}$$
$$=\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b)t^n,$$
$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1+t)^{m}\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\gamma} e^{yt}(1+t)^{k}$$
$$=\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(-x-y|-1,x_2;k,a,b)t^n,$$
$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1+t)^{m}\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\gamma} e^{yt}(1+t)^{k}$$
$$=\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,x_2;k,a,b)t^n,$$

which leads directly to

$$\sum_{n=0}^{\infty} {}_{L}Y_{n,\beta}^{(\alpha|m)}(x;k,a,b)\frac{t^n}{n!}\sum_{r=0}^{\infty} {}_{L}Y_{r,\beta}^{(\gamma|k)}(y;k,a,b)\frac{t^r}{r!}$$
$$=\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b)t^n.$$

Replacing n by $n-r$ in the left hand side of the above equation, we have

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} {}_{L}Y_{n-r,\beta}^{(\alpha|m)}(x;k,a,b){}_{L}Y_{r,\beta}^{(\gamma|k)}(y;k,a,b)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty} {}_{H}Y_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b)t^n.$$

Now comparing the coefficients of t^n, we get (4.6). □

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