ON THE $\Omega$ CURVATURE TENSOR OF A GENERALIZED SASAKIAN-SPACE-FORM

U. C. DE\textsuperscript{1} AND P. MAJHI\textsuperscript{1}

Abstract. The object of the present paper is to study $\xi$-$\Omega$ flat, $\phi$-$\Omega$ flat generalized Sasakian-space-forms. Besides these, we consider generalized Sasakian-space-forms satisfying $P(\xi, X) \cdot \Omega = 0$, $\Omega(\xi, X) \cdot P = 0$ and $\Omega(\xi, X) \cdot \Omega = 0$. As a consequence we obtain several important results. Finally, illustrative examples are given.

1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor $R$ of the manifold. It is well known that the sectional curvatures of a manifold determine the curvature tensor completely. A Riemannian manifold with constant sectional curvature $c$ is known as real space-form and its curvature tensor is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}.$$  

A Sasakian manifold with constant $\phi$-sectional curvature becomes a Sasakian-space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame Alegre, Blair and Carriazo introduced the notion of generalized Sasakian-space-form in 2004 [1]. In this connection it should be mentioned that in 1989 Olszak [22] studied generalized complex-space-form and proved its existence. A generalized Sasakian-space-form is defined as follows.

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that $M$ is a generalized Sasakian-space-form if there exist three functions $f_1$, $f_2$, $f_3$ on $M$ such that the

---

Key words and phrases. Generalized Sasakian-space-form, $\xi$-$\Omega$ flat, $\phi$-$\Omega$, Einstein, $\eta$-Einstein.


Received: June 09, 2017.

Accepted: December 13, 2017.
curvature tensor $R$ is given by
\begin{equation}
R(X,Y)Z = f_1 \{g(Y,Z)X - g(X,Z)Y\} \\
+ f_2 \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
+ f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \}
\end{equation}
for any vector fields $X$, $Y$, $Z$ on $M$. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [1], the authors cited several examples of generalized Sasakian-space-forms. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form. In [19], Kim studied conformally flat generalized Sasakian-space-forms under the assumption that the characteristic vector field $\xi$ is Killing and he classified locally symmetric generalized Sasakian-space-forms. Also he proved some geometric properties of generalized Sasakian-space-forms which depend on the nature of the functions $f_1$, $f_2$ and $f_3$. Generalized Sasakian-space-forms have also been studied in [2–5, 8–12, 14, 15, 20, 24] and many others. In [5], the authors studied trans-Sasakian generalized Sasakian-space-forms and its particular cases. Moreover in [13] the authors studied certain curvature conditions on generalized Sasakian-space-forms. On the other hand recently Mantica and Suh [21] introduced $Q$ curvature tensor and study its relativistic significance. In the present paper we generalize the results of [13].

The present paper is organized as follows.

After preliminaries in Section 3, we consider $\xi$-$Q$ flat generalized Sasakian-space-forms. Section 4 is devoted to study $\phi$-$Q$ flat generalized Sasakian-space-forms. Sections 5, 6 and 7 deal with generalized Sasakian-space-forms satisfying $P(\xi,X) \cdot Q = 0$, $Q(\xi,X) \cdot P = 0$ and $Q(\xi,X) \cdot Q = 0$ respectively. Finally, illustrative examples are given to verify the results of sections 3 and 4.

2. Preliminaries

In an almost contact metric manifold we have [6–7]
\begin{align}
\phi^2(X) &= -X + \eta(X)\xi, \quad \phi \xi = 0, \\
\eta(\xi) &= 1, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \\
g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), \\
g(\phi X, Y) &= -g(X,\phi Y), \quad g(\phi X, X) = 0, \\
g(\phi X, \xi) &= 0.
\end{align}
Again for a $(2n+1)$-dimensional generalized Sasakian-space-form we have [1]
\begin{equation}
R(X,Y)Z = f_1 \{g(Y,Z)X - g(X,Z)Y\} \\
+ f_2 \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
+ f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \}
\end{equation}
where the conformal curvature tensor is called conformally flat if the conformal curvature vanishes for curvature tensor introduced by Ishii [16]. A Riemannian manifold point of the manifold, then the conformal curvature tensor reduces to conharmonic curvature tensor, is the scalar curvature of the manifold. A Riemannian manifold is defined by \[25\]

\[R = \text{Riemann-Christoffel curvature tensor}\]

simply connected locally symmetric space is symmetric. Every symmetric space is semisymmetric, but the converse is not true, in particular, if the \(0, p\)-Riemann-Christoffel curvature tensor \(R\) is parallel, i.e., when \[\nabla T = 0,\]

in which the case \(n = 1\), then \(M\) is said to be locally symmetric. This property justifies the name given to such manifolds locally they are symmetric with respect to each of their points. If each geodesic symmetry \(s\), \(p\) \(\in M\), is a global isometry of \(M\), then \(M\) is called symmetric space. Thus \(\nabla R = 0\) for every symmetric space and conversely, every complete and simply connected locally symmetric space is symmetric.

A Riemannian manifold \((M^{2n+1}, g)\) is said to be semisymmetric if its curvature tensor \(R\) is satisfies \(R(X, Y).R = 0\), \(X, Y \in \chi(M)\), where \(R(X, Y)\) acts on \(R\) as a derivation [11]. Every symmetric space is semisymmetric, but the converse is not true, in general.

Let \((M^{2n+1}, g), n \geq 1\), be a Riemannian manifold. The conformal curvature tensor \(C\) is defined by [25]

\[C(X, Y)Z = R(X, Y)Z - \frac{1}{2n - 1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
+ \frac{r}{2n(2n - 1)} [g(Y, Z)X - g(X, Z)Y],\]

where \(S\) is the Ricci tensor, \(Q\) is the Ricci operator defined by \(S(X, Y) = g(QX, Y)\) and \(r\) is the scalar curvature of the manifold \(M\). If the scalar curvature \(r\) vanishes at each point of the manifold, then the conformal curvature tensor reduces to conharmonic curvature tensor introduced by Ishii [16]. A Riemannian manifold \((M^{2n+1}, g), n \geq 1\), is called conformally flat if the conformal curvature vanishes for \(n > 1\). If \(n = 1\), then the conformal curvature tensor \(C\) vanishes identically. Moreover a manifold of constant
sectional curvature is conformally flat. In [19] conformally flat generalized Sasakian space form have been studied by Kim. Also De and Sarkar [8] studied projective curvature tensor of generalized Sasakian-space-forms. Moreover in [23], Shukla et al. studied concircular curvature tensor on generalized Sasakian-space-forms.

After the conformal curvature tensor, the projective curvature tensor is an important tensor from the differential geometric point of view. Let $M$ be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of $M$ and a domain in Euclidean space such that any geodesic of Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 1$, $M$ is locally projectively flat if and only if the well-known projective curvature tensor $P$ vanishes. The projective curvature tensor is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y],$$

(2.16)

where $S$ is the Ricci tensor of $M$.

A transformation in an $(2n+1)$ dimensional Riemannian manifold $M$, which transforms every geodesic circle of $M$ into a geodesic circle of $M$, is said to be a concircular transformation ([18,26]). A concircular transformation is always a conformal transformation [18]. Here, we mean a geodesic circle by a curve in $M$ whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformation is a generalization of inversive geometry in the sense that the change of metric is more general than induced by a circle preserving diffeomorphism. An important invariant of concircular transformation is the concircular curvature tensor $\mathcal{Z}$, defined by [26]

$$\mathcal{Z}(X,Y)W = R(X,Y)W - \frac{r}{2n(2n+1)}[g(Y,W)X - g(X,W)Y],$$

(2.17)

for all $X,Y,W \in \chi(M)$, where $R$ is the Riemannian curvature tensor and $r$ is the scalar curvature with respect to the Levi-Civita connection.

In a recent paper Mantica and Suh [21] introduced a new curvature tensor of type (1,3) in an $n$-dimensional Riemannian manifold $(M^n, g)$, $n > 2$, denoted by $\mathcal{Q}$ and defined by

$$\mathcal{Q}(X,Y)W = R(X,Y)W - \frac{\psi}{(n-1)}[g(Y,W)X - g(X,W)Y],$$

(2.18)

where $\psi$ is an arbitrary scalar function. Such a tensor $\mathcal{Q}$ is known as Q-curvature tensor. The notion of Q tensor is also suitable to reinterpret some differential structures on a Riemannian manifold. Mantica and Suh [21] have studied pseudo-Q-symmetric Riemannian manifolds. If $\psi = \frac{r}{(2n+1)}$, then $\mathcal{Q}$ curvature tensor reduces to concircular curvature tensor.

Let $M$ be an almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$. At each point $p \in M$, decompose the tangent space
\( T_pM \) into direct sum \( T_pM = \phi(T_pM) \oplus \{ \xi_p \} \), where \( \{ \xi_p \} \) is the 1-dimensional linear subspace of \( T_pM \) generated by \( \{ \xi_p \} \). Thus the conformal curvature tensor \( C \) is a map

\[
C : T_pM \times T_pM \times T_pM \rightarrow \phi(T_pM) \oplus \{ \xi_p \}, \quad p \in M.
\]

It may be natural to consider the following particular cases.

1. \( C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow L(\xi_p) \), i.e., the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero.
2. \( C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M)) \), i.e., the projection of the image of \( C \) in \( L(\xi_p) \) is zero.

This condition is equivalent to

\[
(2.19) \quad C(X,Y)\xi = 0.
\]

3. \( C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \rightarrow L(\xi_p) \), i.e., when \( C \) is restricted to \( \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \), the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero. This condition is equivalent to

\[
(2.20) \quad \phi^2C(\phi X, \phi Y)\phi Z = 0.
\]

A \( K \)-contact manifold satisfying (2.19) and (2.20) are called \( \xi \)-conformally flat and \( \phi \)-conformally flat respectively. A \( K \)-contact manifold satisfying the cases (1), (2) and (3) are considered in [27], [28] and [29] respectively.

**Definition 2.1.** A generalized Sasakian-space-form \( (M^{(2n+1)}, g) \), \( n > 1 \), is said to be \( \xi \)-flat if \( \Omega(X,Y)\xi = 0 \) on \( M \).

**Definition 2.2.** A generalized Sasakian-space-form \( (M^{(2n+1)}, g) \), \( n > 1 \), is said to be \( \phi \)-flat if \( g(\Omega(\phi X, \phi Y)\phi Z, \phi W) = 0 \) on \( M \).

### 3. \( \xi \)-\( \Omega \) Flat Generalized Sasakian-space-forms

In this section we characterize \( \xi \)-\( \Omega \) flat generalized Sasakian-space-forms.

From (2.18) we have

\[
(3.1) \quad \Omega(X,Y)\xi = R(X,Y)\xi - \frac{\psi}{2n}[\eta(Y)X - \eta(X)Y],
\]

for any for any vector fields \( X \) and \( Y \) in \( T(M) \). Using (2.9) in (3.1) implies

\[
(3.2) \quad \Omega(X,Y)\xi = \left[ \frac{\psi}{2n} - (f_1 - f_3) \right][\eta(Y)X - \eta(X)Y].
\]

Thus we can state the following.

**Theorem 3.1.** A \( (2n + 1) \)-dimensional generalized Sasakian-space-form \( M(f_1, f_2, f_3) \) is \( \xi \)-\( \Omega \) flat if and only if \( \psi = 2n(f_1 - f_3) \).

For \( \psi = \frac{r}{2n+1} \), \( \Omega \) curvature tensor reduces to concircular curvature tensor. Thus in view of Theorem 3.1 and making use of (2.14) we obtain the following.

**Corollary 3.1.** A \( (2n + 1) \)-dimensional generalized Sasakian-space-form \( M(f_1, f_2, f_3) \) is \( \xi \)-concircularly flat if and only if \( f_3 = \frac{3}{1-2n} f_2 \).
In [23], Shukla et al. proved that a generalized Sasakian-space-form is concircularly flat if and only if
\[ f_3 = \frac{3}{1-2n} f_2. \]
Since concircularly flat manifold implies \( \xi \)-concircularly flat, therefore from Corollary 3.1 we can mention the following.

**Remark 3.1.** A \((2n+1)\)-dimensional generalized Sasakian-space-form \( M(f_1, f_2, f_3) \) is concircularly flat if and only if
\[ f_3 = \frac{3}{1-2n} f_2. \]

Therefore Theorem 3.1 of [23] is a particular case of Corollary 3.1.

If \( f_1 = \frac{c+3}{4}, \) \( f_2 = \frac{c-1}{4} \) and \( f_3 = \frac{c-1}{4}, \) then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form. So \( f_1 - f_3 = \frac{c+3}{4} - \frac{c-1}{4} = 1. \)

Thus we have the following.

**Corollary 3.2.** A \((2n + 1)\)-dimensional Sasakian-space-form is \( \xi \)-Q flat if and only if \( \psi = 2n. \)

4. \( \phi \)-Q Flat Generalized Sasakian-space-forms

Suppose \( M \) be a \((2n + 1)\)-dimensional, \( n > 1, \) \( \phi \)-Q flat generalized Sasakian-space-form. Then \( g(\phi X, \phi Y)\phi Z, \phi W) = 0, \) for any vector fields \( X, Y, Z \) and \( W \in T(M), \) which implies

\[ g(R(\phi X, \phi Y)\phi Z, \phi W) - \frac{\psi}{2n} [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)] = 0. \] (4.1)

Using (2.1) in (2.6) we obtain

\[
g(R(\phi X, \phi Y)\phi Z, \phi W) = f_1 [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]
+ f_2 [g(\phi X, Z)g(Y, \phi W) - g(\phi Y, Z)g(X, \phi W)
+ 2g(\phi X, Y)g(Z, \phi W)].
\] (4.2)

Using (4.2) in (4.1) yields

\[
\left( f_1 - \psi \right) \frac{2n}{2n} [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]
+ f_2 [g(\phi X, Z)g(Y, \phi W) - g(\phi Y, Z)g(X, \phi W) + 2g(\phi X, Y)g(Z, \phi W)] = 0.
\] (4.3)

Contracting \( Y \) and \( Z \) in (4.3) gives

\[ (2n-1) \left( f_1 - \psi \right) \frac{2n}{2n} [2g(\phi X, \phi W) - g(\phi^2 X, \phi^2 W)] + f_2 [g(\phi X, \phi W) + 2g(\phi X, \phi W)] = 0. \] (4.4)

Using (2.1) in (4.4) we have

\[ (f_1 - \psi) \frac{2n}{2n} [2g(\phi X, \phi W) - g(X, W) + \eta(X)\eta(W)] + 3f_2 g(\phi X, \phi W) = 0. \] (4.5)

Again using (2.3) in (4.5) we have

\[ (2n-1) \left( f_1 - \psi \right) \frac{2n}{2n} + 3f_2 \left( \phi X, \phi W \right) = 0, \] (4.6)
which yields,

\[(4.7) \quad f_1 - \frac{\psi}{2n} = - \frac{3f_2}{2n-1}.\]

From (4.7) and (4.3) we have
\[
- \frac{3f_2}{2n-1} [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]
+ f_2 [g(\phi X, Z)g(Y, \phi W) - g(\phi Y, Z)g(X, \phi W) + 2g(\phi X, Y)g(Z, \phi W)] = 0.
\]

Replacing \(Y\) by \(\phi Y\) and using (2.1) in (4.8) implies
\[
- \frac{3f_2}{2n-1} [-g(Y, \phi Z)g(\phi X, \phi W) + g(\phi X, \phi Z)g(Y, \phi W)]
+ f_2 [g(\phi X, Z)g(\phi Y, \phi W) + g(Y, Z)g(X, \phi W)]
- g(X, \phi W)\eta(Y)\eta(Z) + 2g(\phi X, \phi Y)g(Z, \phi W) = 0.
\]

Substituting \(Y\) by \(\xi\) in (4.9)
\[(4.10) \quad f_2 g(X, \phi W)\eta(Z) = 0,
\]
which implies
\[f_2 = 0.\]

Hence we have the following.

**Theorem 4.1.** If a \((2n + 1)\)-dimensional \(n > 1\) generalized Sasakian-space-form \(M(f_1, f_2, f_3)\) is \(\phi\)-\(\mathcal{Q}\) flat, then \(f_2 = 0\).

In [19], U. K. Kim proved that for a \((2n + 1)\)-dimensional generalized Sasakian-space-form the following hold.

(i) If \(n > 1\), then \(M\) is conformally flat if and only if \(f_2 = 0\).

(ii) If \(M\) is conformally flat and \(\xi\) is Killing, then \(M\) is locally symmetric and has constant \(\phi\)-sectional curvature.

In the view of the first part of above theorem we have the following.

**Corollary 4.1.** In a \((2n + 1)\)-dimensional \(n > 1\) generalized Sasakian-space-form \(\phi\)-\(\mathcal{Q}\) flat and conformally flat are equivalent.

Again, in view of the second part of the above theorem we have the following.

**Corollary 4.2.** A \(\phi\)-\(\mathcal{Q}\) flat \((2n + 1)\)-dimensional \(n > 1\) generalized Sasakian-space-form \(M(f_1, f_2, f_3)\) with \(\xi\) as a Killing vector field is locally symmetric and has constant \(\phi\)-sectional curvature.

Suppose \(f_2 = 0\). Then from (4.3) we have
\[(4.11) \quad g(\mathcal{Q}(\phi X, \phi Y)\phi Z, \phi W) = \left(f_1 - \frac{\psi}{2n}\right) [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].\]
Substituting $f_2 = 0$ in (4.7) yields

\begin{equation}
(4.12) \quad f_1 = \frac{\psi}{2n}.
\end{equation}

Combining the equations (4.11) and (4.12) we can state the following.

**Theorem 4.2.** A $(2n + 1)$-dimensional $n > 1$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $\phi$-$\Theta$ flat if and only if $f_2 = 0$.

5. **Generalized Sasakian-space-forms Satisfying $P(\xi, X) \cdot \Theta = 0$**

In this section we characterize generalized Sasakian-space-forms satisfying $P(\xi, X) \cdot \Theta = 0$, where $P$ is the projective curvature tensor. Suppose a $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian-space-form satisfies $(P(\xi, X) \cdot \Theta)(Y, Z)U = 0$, for any vector fields $X, Y, Z$ and $U \in T(M)$. Then we have

\begin{equation}
(5.1) \quad P(\xi, X)\Theta(Y, Z)U - \Theta(P(\xi, X)Y, Z)U - \Theta(Y, P(\xi, X)Z)U - \Theta(Y, Z)P(\xi, X)U = 0.
\end{equation}

Using (2.10) and (2.16) we have

\begin{equation}
P(\xi, X)\Theta(Y, Z)U = \left[ \frac{f_3 - 3f_2}{2n} - f_3 \right] [g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)\eta(X)\xi] \\
- \frac{\psi}{2n} \left[ \frac{f_3 - 3f_2}{2n} - f_3 \right] g(Z, U)[g(X, Y)\xi - \eta(X)\eta(Y)\xi] \\
+ \frac{\psi}{2n} \left[ \frac{f_3 - 3f_2}{2n} - f_3 \right] g(Y, U)[g(X, Z)\xi - \eta(X)\eta(Z)\xi].
\end{equation}

Again using (2.10), (2.16) and (2.18) we obtain

\begin{equation}
\Theta(P(\xi, X)Y, Z)U = \left[ \frac{f_3 - 3f_2}{2n} - f_3 \right] \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] \\
\times [g(X, Y) - \eta(X)\eta(Y)][g(Z, U)\xi - \eta(U)\xi].
\end{equation}

Making use of (2.10) and (2.16) we have

\begin{equation}
\Theta(Y, P(\xi, X)Z)U = \left[ \frac{f_3 - 3f_2}{2n} - f_3 \right] \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] \\
\times [g(X, Z) - \eta(X)\eta(Z)][\eta(U)Y - g(Y, U)\xi].
\end{equation}

Similarly using (2.10), (2.16) and (2.18) we have

\begin{equation}
\Theta(Y, Z)P(\xi, X)U = \left[ \frac{f_3 - 3f_2}{2n} - f_3 \right] \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] \\
\times [g(X, U) - \eta(X)\eta(U)][\eta(Z)Y - \eta(Y)Z].
\end{equation}

Substituting (5.2)-(5.5) in (5.1) yields

\begin{equation}
\left[ \frac{f_3 - 3f_2}{2n} - f_3 \right] [g(X, R(Y, Z)U)\xi - \eta(X)\eta(R(Y, Z)U)\xi]
\end{equation}
\[
\begin{align*}
\frac{\psi}{2n} & \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] g(Z, U)[g(X, Y)\xi - \eta(X)\eta(Y)\xi] \\
+ \frac{\psi}{2n} & \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] g(Y, U)[g(X, Z)\xi - \eta(X)\eta(Z)\xi] - \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] \\
\times & \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] [g(X, Y) - \eta(X)\eta(Y)][g(Z, U)\xi - \eta(U)\eta(Z)] - \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] \\
\times & \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] [g(X, Z) - \eta(X)\eta(Z)][\eta(U)Y - g(Y, U)\xi] - \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right]
\end{align*}
\] (5.6)

\[
\times \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] [g(X, U) - \eta(X)\eta(U)][\eta(Z)Y - \eta(Y)\eta(Z)] = 0.
\]

Taking inner product with \( \xi \) in (5.6) implies

\[
\begin{align*}
& \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] g(X, R(Y, Z)U) - \eta(X)\eta(R(Y, Z)U) - \frac{\psi}{2n} \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] \\
\times & g(Z, U)[g(X, Y) - \eta(X)\eta(Y)] + \frac{\psi}{2n} \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] g(Y, U)[g(X, Z) - \eta(X)\eta(Z)] \\
- & \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] [g(X, Y) - \eta(X)\eta(Y)][g(Z, U) - \eta(U)\eta(Z)] \\
- & \left[ \frac{f_3 - 3 f_2}{2n} - f_3 \right] \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] [g(X, Z) - \eta(X)\eta(Z)]
\end{align*}
\] (5.7)

\[
\times \left[ \eta(U)\eta(Y) - g(Y, U) \right] = 0.
\]

Putting \( X = Y = e_i, \) where \( \{e_i, \xi\}, 1 \leq i \leq 2n, \) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i, \) we get

\[
\begin{align*}
& \frac{(1 - 2n)f_3 - 3 f_2}{2n} [S(Z, U) - 2n(f_1 - f_3)g(Z, U)] \\
+ & 2n \left( f_1 - f_3 - \frac{\psi}{2n} \right) \eta(U)\eta(Z) = 0.
\end{align*}
\] (5.8)

Therefore, either

\[
(1 - 2n)f_3 - 3 f_2 = 0
\]

or

\[
(5.9) \quad S(Z, U) = 2n(f_1 - f_3)g(Z, U) - 2n \left( f_1 - f_3 - \frac{\psi}{2n} \right) \eta(U)\eta(Z).
\]

In the second case, comparing this equation with (2.7) for \( Z \) and \( U \) orthogonal to \( \xi \) we get

\[
(5.10) \quad 2n(f_1 - f_3) = 2nf_1 + 3 f_2 - f_3.
\]
It follows that
\begin{equation}
(5.11) \quad f_3 = \frac{3}{1 - 2n} f_2.
\end{equation}
Then as \( f_3 = \frac{3}{1 - 2n} f_2 \), from (2.7), \( S(X,Y) = (2n f_1 + 3 f_2 - f_3)g(X,Y) \) and \( M \) is an Einstein manifold. Conversely, if \( \frac{f_3 - 3f_2}{f_2} - f_3 = 0 \), that is, \( f_3 = \frac{(1 - 2n)}{3} f_2 \), then in view of equations (5.2)-(5.5) we have \( P(\xi, X) \cdot Q = 0 \).

In view of the above results we can state the following.

**Theorem 5.1.** A \((2n+1)\)-dimensional \( n > 1 \) generalized Sasakian-space-form satisfies \( P(\xi, X) \cdot Q = 0 \) if and only if \( f_3 = \frac{3}{(1 - 2n)} f_2 \). In such a case it is an Einstein manifold.

For \( \psi = \frac{r}{(2n+1)} \), the \( Q \) curvature tensor reduces to the concircular curvature tensor. Now putting \( Z = U = e_i \) in (5.9), where \( \{e_i, \xi\}, 1 \leq i \leq 2n \), is the orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i \), we get \( r = 2n(2n + 1)(f_1 - f_3) - 2n \left[ (f_1 - f_3) - \frac{\psi}{2n} \right] \). Using \( \psi = \frac{r}{(2n+1)} \), then \( (2n + 1)\psi = 2n(2n + 1)(f_1 - f_3) - 2n(f_1 - f_3) + \psi \), which implies \( \frac{\psi}{2n} = (f_1 - f_3) \) and hence the manifold reduces to an Einstein manifold.

Thus we have following.

**Theorem 5.2.** A \((2n+1)\)-dimensional \( n > 1 \) generalized Sasakian-space-form satisfies \( P(\xi, X) \cdot Z = 0 \) if and only if \( f_3 = \frac{3}{(1 - 2n)} f_2 \). In such a case it is an Einstein manifold.

**Remark 5.1.** The above theorem has been proved by De and Yildiz in [13].

6. **Generalized Sasakian-space-forms Satisfying \( Q(\xi, X) \cdot P = 0 \)**

Suppose a \((2n+1)\)-dimensional \( n > 1 \) generalized Sasakian-space-form satisfies \( Q(\xi, X) \cdot P)(Y, Z)U = 0 \), for any vector fields \( X, Y, Z \) and \( U \in T(M) \). Then
\begin{equation}
(6.1) \quad Q(\xi, X)P(Y, Z)U - P(Q(\xi, X)Y, Z)U - P(Y, Q(\xi, X)Z)U - P(Y, Z)Q(\xi, X)U = 0.
\end{equation}

Now using (2.10), (2.16) and (2.18) we have
\begin{equation}
(6.2) \quad Q(\xi, X)P(Y, Z)U = \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \left[ g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X \right] - \frac{1}{2n} \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \left[ g(X, Y)\xi - \eta(Y)X \right] S(Z, U)
\end{equation}
\begin{equation}
+ \frac{1}{2n} \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \left[ g(X, Z)\xi - \eta(Z)X \right] S(Y, U).
\end{equation}

Using (2.10), (2.16) and (2.18) we obtain
\begin{equation}
(6.3) \quad P(Q(\xi, X)Y, Z)U = \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \left[ g(X, Y)P(\xi, Z)U - \eta(Y)P(X, Z)U \right].
\end{equation}
Again using (2.10), (2.16) and (2.18) we get

\[
P(Y, Q(\xi, X)Z)U = \left[ f_1 - f_3 - \frac{\psi}{2n} \right] [g(X, Z)P(Y, \xi)U - \eta(Z)P(Y, X)U].
\]

Finally, using (2.10), (2.16) and (2.18), we have

\[
P(Y, Z)Q(\xi, X)U = \left[ f_1 - f_3 - \frac{\psi}{2n} \right] [g(X, U)P(Y, Z)\xi - \eta(U)P(Y, Z)X].
\]

Using (6.2)-(6.5) in (6.1) yields

\[
\begin{align*}
&\left[ f_1 - f_3 - \frac{\psi}{2n} \right] [g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X] \\
&- \frac{1}{2n} \left[ f_1 - f_3 - \frac{\psi}{2n} \right] [g(X, Y)\xi - \eta(Y)X]S(Z, U) \\
&+ \frac{1}{2n} [f_1 - f_3 - \frac{\psi}{2n}] [g(X, Z)\xi - \eta(Z)X]S(Y, U) \\
&- \left[ f_1 - f_3 - \frac{\psi}{2n} \right] [g(X, Y)P(\xi, Z)U - \eta(Y)P(X, Z)U] \\
&- \left[ f_1 - f_3 - \frac{\psi}{2n} \right] [g(X, Z)P(Y, \xi)U - \eta(Z)P(Y, X)U] \\
&- \left[ f_1 - f_3 - \frac{\psi}{2n} \right] [g(X, U)P(Y, Z)\xi - \eta(U)P(Y, Z)X] = 0.
\end{align*}
\]

Putting \( X = Y = e_i \), where \( \{e_i, \xi\} \), \( 1 \leq i \leq 2n \), is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i \), we have

\[
\begin{align*}
&\left[ f_1 - f_3 - \frac{\psi}{2n} \right] \left\{ (f_1 - f_3) \{ \eta(Z)g(U, W) - g(Z, U)\eta(W) \} \right. \\
&- \frac{1}{2n} \{ S(Z, U)\eta(W) + \eta(Z)S(U, W) \} \\
&- (2n - 1)(f_1 - f_3) \{ g(Z, U)\eta(W) - \eta(U)g(Z, W) \} \\
&+ \frac{2n}{2n} - 1 S(Z, U)\eta(W) - (2n - 1)(f_1 - f_3)\eta(U)g(Z, W) \\
&\left. + \eta(U)S(Z, W) + \frac{1}{2n} S(Z, W)\eta(U) - \frac{r}{2n} g(Z, W)\eta(U) \right] = 0.
\end{align*}
\]

Putting \( W = \xi \) in (6.7) we have

\[
\begin{align*}
&\left[ f_1 - f_3 - \frac{\psi}{2n} \right] \left\{ \left( 1 - \frac{1}{n} \right) S(Z, U) - 2(n - 1)(f_1 - f_3)g(Z, U) \\
&- \left\{ (2n + 1)(f_1 - f_3) + \frac{r}{2n} \right\} \eta(Z)\eta(U) \right] = 0.
\end{align*}
\]
Therefore, either $f_1 - f_3 - \frac{\psi}{2n} = 0$ or

\[
\left(1 - \frac{1}{n}\right)S(Z, U) - 2(n - 1)(f_1 - f_3)g(Z, U)
\]

(6.9) $\{- (2n + 1)(f_1 - f_3) + \frac{r}{2n}\} \eta(Z) \eta(U) = 0.$

In the second case, substituting $U = Z = \xi$ we get $-(2n + 1)(f_1 - f_3) = \frac{r}{2n}$ and hence
\[
S(Z, U) = 2(n - 1)(f_1 - f_3)g(Z, U).
\]

In this case $M$ is an Einstein manifold.

Conversely, if $f_1 - f_3 - \frac{\psi}{2n} = 0$, then in view of equation (6.2)-(6.5) we have $Q(\xi, X) \cdot P = 0.$

Thus we can state the following.

**Theorem 6.1.** A $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form satisfies $Q(\xi, X) \cdot P = 0$ if and only if $f_1 - f_3 - \frac{\psi}{2n} = 0.$ In such a case $M$ is an Einstein manifold.

In particular, if $\psi = \frac{r}{(2n+1)}$, then the $Q$ curvature tensor reduces to the concircular curvature tensor. Thus in view of (2.14), $f_1 - f_3 - \frac{\psi}{2n} = 0$ reduces to $3f_2 + (2n-1)f_3 = 0.$ Thus we can state the following.

**Theorem 6.2.** A $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form satisfies $Q(\xi, X) \cdot Z = 0$ if and only if $f_3 = \frac{3}{2n-1}f_2.$ In such a case $M$ is an Einstein manifold.

**Remark 6.1.** The above theorem has been proved by De and Yildiz in [13].

7. **Generalized Sasakian-space-forms Satisfying $Q(\xi, X) \cdot Q = 0$**

Suppose a $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form satisfies $(Q(\xi, X) \cdot Q)(Y, Z)U = 0,$ for any vector fields $X,$ $Y,$ $Z$ and $U \in T(M).$ Then

\[
Q(\xi, X)Q(Y, Z)U - Q(\xi, X)Q(Y, Z)U - Q(Y, Q(\xi, X)Z)U - Q(Y, Z)Q(\xi, X)U = 0.
\]

(7.1)

Now using (2.10), (2.16) and (2.18) we get

\[
Q(\xi, X)Q(Y, Z)U = \left[f_1 - f_3 - \frac{\psi}{2n}\right] [g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X]
\]

\[
- \frac{\psi}{2n} \left[f_1 - f_3 - \frac{\psi}{2n}\right] g(Z, U)[g(X, Y)\xi - \eta(Y)X]
\]

\[
+ \frac{\psi}{2n} \left[f_1 - f_3 - \frac{\psi}{2n}\right] g(Y, U)[g(X, Z)\xi - \eta(Z)X].
\]

(7.2)

Using (2.10), (2.16) and (2.18) in (7.2) we have

\[
Q(Q(\xi, X)Y, Z)U = \left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X, Y)[g(X, Z)\xi - \eta(U)Z]
\]
Again using (2.10), (2.16) and (2.18) in (7.3) we obtain

\[ Q(Y, Q(\xi, X)Z)U = \left[ f_1 - f_3 - \frac{\psi}{2n} \right]^2 g(X, Z)[\eta(U)Y - g(Y, U)\xi] \]

(7.4)

Finally, using (2.10), (2.16) and (2.18) in (7.4) we get

\[ Q(Y, Z)Q(\xi, X)U = \left[ f_1 - f_3 - \frac{\psi}{2n} \right]^2 g(X, U)[\eta(Y)X - \eta(X)Y] \]

(7.5)

Substituting (7.2)-(7.5) in (7.1) we have

\[ \left[ f_1 - f_3 - \frac{\psi}{2n} \right][g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X] \]

\[ - \frac{\psi}{2n} \left[ f_1 - f_3 - \frac{\psi}{2n} \right]g(Z, U)[g(X, Y)\xi - \eta(Y)X] + \frac{\psi}{2n} \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \]

\[ \times g(Y, U)[g(X, Z)\xi - \eta(Z)X] - \left[ f_1 - f_3 - \frac{\psi}{2n} \right]^2 g(X, Y)[g(X, Z)\xi - \eta(U)Z] \]

\[ + \left[ f_1 - f_3 - \frac{\psi}{2n} \right]\eta(Y)Q(X, Z)U - \left[ f_1 - f_3 - \frac{\psi}{2n} \right]^2 g(X, Z)[\eta(U)Y - g(Y, U)\xi] \]

\[ + \left[ f_1 - f_3 - \frac{\psi}{2n} \right]\eta(Z)Q(Y, X)U - \left[ f_1 - f_3 - \frac{\psi}{2n} \right]^2 g(X, U)[\eta(Y)X - \eta(X)Y] \]

(7.6)

\[ + \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \eta(U)Q(Y, Z)X = 0. \]

Taking inner product with \( \xi \) in (7.6) we obtain

\[ \left[ f_1 - f_3 - \frac{\psi}{2n} \right][g(X, R(Y, Z)U) - \eta(R(Y, Z)U)\eta(X)] \]

\[ - \frac{\psi}{2n} \left[ f_1 - f_3 - \frac{\psi}{2n} \right]g(Z, U)[g(X, Y) - \eta(Y)\eta(X)] + \frac{\psi}{2n} \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \]

\[ \times g(Y, U)[g(X, Z) - \eta(Z)\eta(X)] - \left[ f_1 - f_3 - \frac{\psi}{2n} \right]^2 g(X, Y)[g(U, Z) - \eta(U)\eta(Z)] \]
\begin{align*}
&+ \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \eta(Y) \eta(Q(X, Z)U) - \left[ f_1 - f_3 - \frac{\psi}{2n} \right]^2 g(X, Z)[\eta(U)\eta(Y) - g(Y, U)] \\
&= f_1 - f_3 - \frac{\psi}{2n} \eta(Z) \eta(Q(Y, X)U) + \left[ f_1 - f_3 - \frac{\psi}{2n} \right] \eta(U) \eta(Q(Y, Z)X) = 0.
\end{align*}

Putting \( X = Y = e_i \), where \( \{e_i, \xi\} \), \( 1 \leq i \leq 2n \), is the orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i \), we have

\begin{equation}
\left[ f_1 - f_3 - \frac{\psi}{2n} \right] [S(Z, U) - 2n(f_1 - f_3)g(Z, U) + 2 \left( f_1 - f_3 - \frac{\psi}{2n} \right) \eta(Z) \eta(U)] = 0.
\end{equation}

Therefore, either \( f_1 - f_3 - \frac{\psi}{2n} = 0 \) or

\[ S(Z, U) = 2n(f_1 - f_3)g(Z, U) - 2 \left( f_1 - f_3 - \frac{\psi}{2n} \right) \eta(Z) \eta(U). \]

In the second case, comparing this equation with (2.7) for \( Z \) and \( U \) orthogonal to \( \xi \), we get

\begin{equation}
2n(f_1 - f_3) = 2nf_1 + 3f_2 - f_3.
\end{equation}

It follows that

\begin{equation}
f_3 = \frac{3}{1 - 2n} f_2.
\end{equation}

So comparing the expression \( f_3 = \frac{3}{1 - 2n} f_2 \) with (2.8) we get

\begin{equation}
S(\xi, \xi) = 2n(f_1 - f_3) = 2nf_1 - 2 \left( f_1 - f_3 - \frac{\psi}{2n} \right).
\end{equation}

So also in this case \( f_1 - f_3 - \frac{\psi}{2n} = 0 \). Again if \( f_3 = \frac{3}{1 - 2n} f_2 \), then from (2.7), \( S(X, Y) = \frac{3}{1 - 2n} f_2 \), then in view of equations (7.2)-(7.5) we have \( Q(\xi, X) \cdot Q = 0 \). This leads to the following.

**Theorem 7.1.** A \((2n+1)\)-dimensional \( n > 1 \) generalized Sasakian-space-form satisfies \( Q(\xi, X) \cdot Q = 0 \) if and only if \( f_1 - f_3 - \frac{\psi}{2n} = 0 \). In such a case it is an Einstein manifold.

In particular, if \( \psi = \frac{r}{(2n+1)} \), then the \( Q \) curvature tensor reduces to the concircular curvature tensor. Thus in view of (2.14), \( f_1 - f_3 - \frac{\psi}{2n} = 0 \) reduces to \( 3f_2 + (2n-1)f_3 = 0 \). Thus we can state the following.

**Theorem 7.2.** A \((2n+1)\)-dimensional \( n > 1 \) generalized Sasakian-space-form satisfies \( \mathcal{Z}(\xi, X) \cdot \mathcal{Z} = 0 \) if and only if \( f_3 = \frac{3}{1 - 2n} f_2 \). In such a case it is an Einstein manifold.

**Remark 7.1.** The above theorem has been proved by De and Yildiz in [13].
8. Example

Example 8.1. Let \( N(a, b) \) be a generalized complex space-form, then the warped product \( M = \mathbb{R} \times_f N \) endowed with the almost contact metric structure \( (\phi, \xi, \eta, g_f) \) is a generalized Sasakian-space-form \( M(f_1, f_2, f_3) \) \([1]\) with

\[
  f_1 = \frac{a - (f')^2}{f^2}, \quad f_2 = \frac{b}{f^2}, \quad f_3 = \frac{a - (f')^2 + f''}{f},
\]

where \( f = f(t), t \in \mathbb{R} \), and \( f' \) denotes the derivative of \( f \) with respect to \( t \). If we choose \( a = 2, b = 0 \) and \( f(t) = t \) with \( t \neq 0 \), then \( f_1 = \frac{1}{t^2}, f_2 = 0 \) and \( f_3 = \frac{1}{t^2} \),

\[
  R(X, Y)Z = \frac{1}{t^2} \{ g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
  + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \}. \quad (8.1)
\]

From (8.1) it follows that

\[
  R(X, Y)\xi = 0. \quad (8.2)
\]

Moreover in this case \( \psi = 2n(f_1 - f_3) \) will be \( \psi = 2n \left( \frac{1}{t^2} - \frac{1}{t^2} \right) = 0 \). Thus from (3.1) we get

\[
  Q(X, Y)\xi = R(X, Y)\xi - \frac{\psi}{2n} [\eta(Y)X - \eta(X)Y] = 0. \quad (8.3)
\]

Thus the generalized Sasakian-space-form is \( \xi Q \) flat if and only if \( \psi = 2n \left( \frac{1}{t^2} - \frac{1}{t^2} \right) = 0 \). Hence, Theorem 3.1 is verified.

Example 8.2. In \([1]\), it was shown that the warped product \( \mathbb{R} \times_f \mathbb{C}^m \) is a generalized Sasakian-space-form with

\[
  f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2 + f''}{f},
\]

where \( f = f(t), t \in \mathbb{R} \), and \( f' \) denotes the derivative of \( f \) with respect to \( t \). If we choose \( m = 4 \) and \( f(t) = e^t \), then \( M(f_1, f_2, f_3) \) is a 5-dimensional conformally flat generalized Sasakian-space-form with \( f_1 = -1, f_2 = 0 \) and \( f_3 = 0 \). Therefore, the generalized Sasakian-space-form is \( \phi Q \) flat. Hence, Theorem 4.1 is verified.

Acknowledgements. The authors are thankful to the referee for his/her valuable suggestions and comments towards the improvement of the paper.

References


1Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata-700019, West Bengal, India

Email address: uc_de@yahoo.com
Email address: mpradipmajhi@gmail.com