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SOME NOVEL RESULTS ON THE EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION TO A KIND OF NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. This work investigates a fractional boundary value problem in the sense of Riemann-Liouville derivative and integral. We derive some novel results for the necessary and sufficient conditions for the existence and uniqueness of the positive solution. In this regard, some fixed-point theorems on cones are used. Also, a convergent successive sequence to find the solution to the problem is introduced. We derive the numerical scheme for the proposed problems. The correctness of the proposed results is verified with some illustrative examples.

1. Introduction

Fractional Calculus, which extends integer order calculus to arbitrary order calculus, has garnered attention from scientists recently. Fractional differential equations represent physical processes in science and engineering [20, 28, 29, 33]. Some recent applications of fractional-order operators can be seen in epidemiology [8, 18, 22, 31, 35], ecology [23], mechanics [17], psychology [25], chemical reactor theory [16], etc. Particularly, Fractional-order Boundary Value Problems (FBVPs) have been used to describe various real-world problems. In [14], the authors derived a Caputo-type boundary value problem representing a corneal shape model. The authors in [24] proposed a heat conduction model of fractional-order in terms of the Caputo-type boundary value problem. In [6], the authors proposed a boundary value problem related to the dynamics of glucose graph.

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Numerous publications addressing the existence, uniqueness, and multiplicity of positive solutions to fractional initial and boundary value issues have been written in recent decades (see [2–4, 7, 9, 10, 13, 30, 32, 40]). Bai in [5] derived positive solutions of a nonlocal fractional boundary value problem. In [1], the authors derived some novel simulations on the existence of a unique positive solution for FBVPs. In [34], an analysis of the existence and uniqueness of positive solutions to a coupled system of nonlinear FBVPs with anti-periodic boundary conditions has been given. In [26], the authors produced some novel findings for the existence and uniqueness of positive solutions to m-point FBVPs. In [21], the same results are produced for multi-point FBVPs with p-Laplacian operator. There have been some theoretical improvements in [36] on the existence of a unique positive solution for a class of nonlinear FBVPs with mixed-type boundary conditions. The positive solution of a nonlinear fractional q-difference equation with integral boundary conditions has been studied for existence and uniqueness in [19]. Some existence and stability results for nonlocal FBVPs were derived by Erturk et al. [15]. Bekri et al. [11] investigated some novel findings on the existence and uniqueness of a nonlinear q-difference FBVP. In [12], the analyses of existence and uniqueness on two Caputo-type FBVPs have been given.

In this study, we address the existence of positive solutions for the following FBVP and the uniqueness of each of those solutions.

$$\begin{aligned} &\mathfrak{D}^{\delta}_{0^+}(u(t) + \mathfrak{I}^{\epsilon}_{0^+}\Psi(t, u(t))) + \Phi(t, u(t)) = 0,\\ &\lim_{t\to 0} t^{\delta-3}u(t) = \lim_{t\to 0} t^{\delta-3}u'(t) = u'(1) = 0, \end{aligned} \tag{1.1}$$

where $2 < \delta \leq \epsilon \leq 3$, $t \in [0, 1]$ and $\mathfrak{D}^{\delta}_{0^+}$ is the standard Riemann-Liouville (R-L) fractional derivative of order δ and $\mathfrak{I}^{\epsilon}_{0^+}$ is the R-L fractional integral of order ϵ. Also, the functions Φ and Ψ have some properties which will be presented later.

This article is put together in the following way. In section 2, some necessary definitions are presented. Section 3 calculates the Green function of the problem and presents some properties of this function. In section 4, the main results about the existence and uniqueness of positive solutions of the proposed FBVP (1.1) are obtained. Section 5 gives illustrative examples verifying main results with numerical solutions. In section 6, we conclude our findings.

2. Fundamentals

Here we recall some fundamentals [20, 27–29, 33] used throughout the study.

Definition 2.1. The Riemann-Liouville (R-L) fractional integral is given by

$$\mathfrak{I}^{\delta}_{a^+} f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t - s)^{\delta-1} f(s)ds,$$

where Γ denotes the Gamma function and a is an arbitrary fixed initial point. The function f is considered locally integrable and δ is a real or complex number $\mathrm{Re}\,(\delta) > 0$.

Definition 2.2. The R-L fractional derivative of order $\delta > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathfrak{D}_{a^+}^{\delta} f(t) = \frac{1}{\Gamma(n-\delta)} \cdot \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\delta-1} f(s) ds,$$

where $n = [\delta] + 1$, considering right-hand is point-wise defined on $(0, +\infty)$.

Lemma 2.1 ([27])**.** *Let $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\delta > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then,*

$$\mathfrak{I}_{0^+}^{\delta} \mathfrak{D}_{0^+}^{\delta} u(t) = u(t) + C_1 t^{\delta-1} + C_2 t^{\delta-2} + \cdots + C_n t^{\delta-n},$$

where $n = [\delta] + 1$.

Throughout the paper, let $(E, \|\cdot\|)$ be a real Banach space and θ be a zero of E. A nonempty closed convex set P is a cone if satisfies the following conditions

i) $u \in P$, $\lambda \geq 0$ implies $\lambda u \in P$;
ii) $u_1 \leq u_2 \Leftrightarrow u_2 - u_1 \in P$.

Also, cone P is a normal cone if there exists $N \in \mathbb{R}$ such that for all $u_1, u_2 \in P$ with $\theta \leq u_1 \leq u_2$ we have $\|u_1\| \leq N\|u_2\|$ and N is called the normality constant.

For all $u_1, u_2 \in E$, write $u_1 \sim u_2$ (we say u_1 is equivalent with u_2) if there exist constants $\lambda, \mu > 0$ such that $\lambda u_1 \leq u_2 \leq \mu u_1$. If $h > \theta$, then $P_h = \{u \in P : u \sim h\}$. It is clear that $P_h \subset P$.

Definition 2.3. Let $\eta \in (0,1)$. An operator $T : P \to P$ is called η-concave if for all $\lambda \in (0,1)$ and $u \in P$ we have $T(\lambda u) \geq \lambda^{\eta} T(u)$. Also an operator $T : P \to P$ is called sub-homogeneous if for all $\lambda > 0$ and $u \in P$ we have $T(\lambda u) \geq \lambda T(u)$.

Now we recall some fixed point theorems.

Theorem 2.1 ([39])**.** *Let P be a normal cone in a real Banach space E, $T_1, T_2 : P \to P$ be an increasing η-concave operator and an increasing sub-homogeneous operator, respectively. If*

i) *for some $h > \theta$ we have $T_1 h \in P_h$ and $T_2 h \in P_h$;*
ii) *for some constant ρ_0 and all $u \in P$ we have $T_1 u \geq \rho_0 T_2 u$,*

then the operator $T = T_1 + T_2$ has unique fixed point. In the other words, the operator equation $u = T_1 u + T_2 u$ has unique solution $u^* \in P_h$. Moreover, for any initial value u_0, the succsessive sequence $u_{n+1} = T_1 u_n + T_2 u_n$, for $n = 0, 1, 2, \ldots$ converges to the u^*.

Theorem 2.2 ([37])**.** *Let P be a normal cone in real Banach space E, $T_1, T_2 : P \to P$ are respectively increasing and decrasing operator. Assume*

i) *for any $u \in P$ and $\lambda \in (0,1)$, there exist $\varphi_i(\lambda) \in (\lambda, 1)$, $i = 1, 2$ such that*

$$T_1(\lambda u) \geq \varphi_1(\lambda) T_1(u), \quad T_2(\lambda u) \leq \frac{1}{\varphi_2(\lambda)} T_2(u),$$

ii) *there exists $h_0 \in P_h$ such that $T_1 h_0 + T_2 h_0 \in P_h$.*

Then, the operator equation $u = T_1u + T_2u$ has unique solution $u^* \in P_h$. Moreover, for any initial values v_0, u_0 successive sequences

$$u_{n+1} = T_1u_n + T_2v_n, \quad v_{n+1} = T_1v_n + T_2u_n, \quad n = 0, 1, 2, \ldots$$

converge to $u^* \in P_h$.

3. Green Function and Bounds

We need to calculate the Green function of a desired operator for applying the fixed point theorems. In this section, in addition to calculate Green function, we also outline some properties of it which is used throughout this paper.

Lemma 3.1. *Suppose $g, h : [0, 1] \to [0, +\infty)$ be continuous functions, then the solution of the FBVP*

$$\mathfrak{D}^{\delta}_{0^+}[u(t) + \mathfrak{I}^{\epsilon}_{0^+}g(t)] + h(t) = 0, \tag{3.1}$$
$$\lim_{t\to 0} t^{\delta-3}u(t) = \lim_{t\to 0} t^{\delta-3}u'(t) = u'(1) = 0,$$

is expressed by

$$u(t) = \int_0^1 G_1(t, s)h(s)ds + \int_0^1 G_2(t, s)g(s)ds, \tag{3.2}$$

where

$$G_1(t, s) = \begin{cases} \frac{t^{\delta-1}(1-s)^{\delta-2}-(t-s)^{\delta-1}}{\Gamma(\delta)}, & 0 \le s \le t < 1, \\ \frac{t^{\delta-1}(1-s)^{\delta-2}}{\Gamma(\delta)}, & 0 \le t \le s \le 1, \end{cases} \tag{3.3}$$

$$G_2(t, s) = \begin{cases} \frac{(\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2}-(\delta-1)(t-s)^{\epsilon-1}}{(\delta-1)\Gamma(\epsilon)}, & 0 \le s \le t < 1, \\ \frac{(\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

Proof. Integrating the first equation of (3.1), follows

$$u(t) + \mathfrak{I}^{\epsilon}_{0^+}g(t) = -\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}h(s)ds + c_1t^{\delta-1} + c_2t^{\delta-2} + c_3t^{\delta-3}.$$

One can easily check that from the boundary conditions $\lim_{t\to 0} t^{\delta-3}u(t) = \lim_{t\to 0} t^{\delta-3}u'(t) = 0$, we have $c_2 = c_3 = 0$. By derivation from the above relation, we have

$$u'(t) = -\frac{\epsilon-1}{\Gamma(\epsilon)}\int_0^t (t-s)^{\epsilon-2}g(s)ds - \frac{\delta-1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-2}h(s)ds + c_1(\delta-1)t^{\delta-2}.$$

Now from the third boundary condition, we have

$$c_1 = \frac{\epsilon-1}{(\delta-1)\Gamma(\epsilon)}\int_0^1 (1-s)^{\epsilon-2}g(s)ds + \frac{1}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-2}h(s)ds.$$

Hence,

$$u(t) = -\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}h(s)ds + \frac{1}{\Gamma(\delta)}\int_0^1 t^{\delta-1}(1-s)^{\delta-2}h(s)ds$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\epsilon)}\int_0^t (t-s)^{\epsilon-1}g(s)ds+\frac{\epsilon-1}{(\delta-1)\Gamma(\epsilon)}\int_0^1 t^{\delta-1}(1-s)^{\epsilon-2}g(s)ds\\
=&\int_0^1 G_1(t,s)h(s)ds+\int_0^1 G_2(t,s)g(s)ds. \qquad \square
\end{aligned}
$$

Corollary 3.1. *Let $\Phi,\Psi\in C([0,1]\times[0,+\infty))$. Then u is a solution of problem* (1.1) *if and only if u is a solution of integral equation*

$$
u(t)=\int_0^1 G_1(t,s)\Phi(s,u(s))ds+\int_0^1 G_2(t,s)\Psi(s,u(s))ds. \tag{3.5}
$$

Lemma 3.2. *The functions $G_1(t,s)$, $G_2(t,s)$ defined by* (3.3) *and* (3.4) *have the following properties:*

(a) $t^{\delta-1}G_1(1,s)\le G_1(t,s)\le \frac{t^{\delta-1}(1-s)^{\delta-2}}{\Gamma(\delta)}$;

(b) $t^{\delta-1}G_2(1,s)\le G_2(t,s)\le \frac{t^{\delta-1}(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}$.

Proof. Statement (a) concluded from [38]. We prove the statement (b). For $s\le t$ we have

$$
\begin{aligned}
(\delta-1)\Gamma(\epsilon)G_2(t,s)=&(\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2}-(\delta-1)(t-s)^{\epsilon-1}\\
\ge&(\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2}-(\delta-1)(t-s)(t-st)^{\epsilon-2}\\
=&(\epsilon-1)t^{\delta-1}(1-s)^{\epsilon-2}-(\delta-1)(t-s)t^{\epsilon-2}(1-s)^{\epsilon-2}\\
\ge&t^{\delta-1}\left[(\epsilon-1)(1-s)^{\epsilon-2}-(\delta-1)(t-s)(1-s)^{\epsilon-2}\right]\\
\ge&t^{\delta-1}\left[(\epsilon-1)(1-s)^{\epsilon-2}-(\delta-1)(1-s)^{\epsilon-1}\right]\\
=&t^{\delta-1}(\delta-1)\Gamma(\epsilon)G_2(1,s).
\end{aligned}
$$

Thus, for $s\le t$, we have $G_2(t,s)\ge t^{\delta-1}G_2(1,s)$. On the other hand, for $s>t$, we have

$$
\frac{G_2(t,s)}{G_2(1,s)}=\frac{(\epsilon-1)t^{\delta-1}(t-s)^{\epsilon-2}}{(\epsilon-1)(1-s)^{\epsilon-2}}=t^{\delta-1}.
$$

So, for all $(t,s)\in[0,1]\times[0,1]$, we have

$$
G_2(t,s)\ge t^{\delta-1}G_2(1,s).
$$

The other side of the inequality in the statement (b) is clearly established. $\square$

4. Main Results

In this section, by using Theorem 2.1 and Theorem 2.2, we prove some existence and uniquness results for the FBVP (1.1). For convenience, we list the following hypothesis:

($H1$) $\Phi,\Psi\in C([0,1]\times[0,+\infty))$ and they are increasing functions with respect to the second variable, also $\Psi(t,0)\not\equiv 0$;

($H2$) for $0<\mu<1$, $(t,u)\in[0,1]\times[0,+\infty)$, we have $\Psi(t,\mu u)\ge\mu\Psi(t,u)$;

($H3$) for $0<\mu,\eta<1$, $(t,u)\in[0,1]\times[0,+\infty)$, we have $\Phi(t,\mu u)\ge\mu^\eta\Phi(t,u)$;

$(H4)$ there exists a constant $\rho_0 > 0$ such that $\Phi(t,u) \geq \rho_0 \Psi(t,u)$, $t \in [0,1]$, $u \geq 0$.
Now we set

$$A_1 = \int_0^1 G_1(1,s)\Phi(s,0)ds, \quad A_2 = \int_0^1 \frac{(1-s)^{\delta-2}}{\Gamma(s)}\Phi(s,1)ds,$$
$$B_1 = \int_0^1 G_2(1,s)\Psi(s,0)ds, \quad B_2 = \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(s)}\Psi(s,1)ds.$$

Theorem 4.1. *Assume that $(H1)$-$(H4)$ hold. Then, fractional boundary value problem* (1.1) *has unique positive solution. In fact, the problem has unique solution u in P_h, with $h(t) = t^{\delta-1}$, $t \in [0,1]$. Also, for any initial value $u_0 \in P_h$, the successive sequence*

$$u_{n+1}(t) = \int_0^1 G_1(t,s)\Phi(s,u_n(s))ds + \int_0^1 G_2(t,s)\Psi(s,u_n(s))ds, \quad n = 0,1,\ldots,$$

converges to the solution u^*.

Proof. Let P be the cone of all positive functions and $P_h \subset P$. From Corollary 3.1 we know that problem (1.1) has an integral formulation given by

$$u(t) = \int_0^1 G_1(t,s)\Phi(s,u(s))ds + \int_0^1 G_2(t,s)\Psi(s,u(s))ds, \tag{4.1}$$

where G_1, G_2 are defined by (3.3) and (3.4). We define two operators $T_1, T_2 : P \to E$ by

$$(T_1u)(t) = \int_0^1 G_1(t,s)\Phi(s,u(s))ds, \quad (T_2u)(t) = \int_0^1 G_2(t,s)\Psi(s,u(s))ds. \tag{4.2}$$

It is clear that u is the solution of FBVP (1.1) if and if only $u = T_1u + T_2u$. From $(H1)$-$(H2)$, we know that $T_1 : P \to P$ and $T_2 : P \to P$. In the following, we check that T_1, T_2 satisfy all assumptions of Theorem 2.1. This will be done in the following steps.

Step 1. T_1 and T_2 are increasing operators.

Let $u_1, u_2 \in P$ and $u_1 \leq u_2$, then for all $t \in [0,1]$ we have $u_1(t) \leq u_2(t)$. So, by $(H1)$,

$$(T_1u_1)(t) = \int_0^1 G_1(t,s)\Phi(s,u_1(s))ds \leq \int_0^1 G_1(t,s)\Phi(s,u_2(s))ds = (T_1u_2)(t).$$

By a similar way one can show $(T_2u_1)(t) \leq (T_2u_2)(t)$.

Step 2. T_1 is a η-concave and T_2 is a sub-homogeneous operator. Let $\mu \in (0,1)$ and $u \in P$, then from (H3), we have

$$(T_1(\mu u))(t) = \int_0^1 G_1(t,s)\Phi(s,\mu u(s))ds \geq \mu^\eta \int_0^1 G_1(t,s)\Phi(s,u(s))ds = \mu^\eta(T_1u)(t).$$

So, T_1 is a η-concave operator. Now, from $(H2)$ and the same properties for μ, we get

$$(T_2(\mu u))(t) = \int_0^1 G_2(t,s)\Psi(s,\mu u(s))ds \geq \mu \int_0^1 G_2(t,s)\Psi(s,u(s))ds = \mu(T_2u)(t).$$

Hence, we can conclude that T_2 is a sub-homogeneous operator.

Step 3. For $h(t) = t^{\delta-1}$ we have $T_1 h, T_2 h \in P_h$.

In view of $(H1)$ and Lemma 3.2, we get

$$(T_1 h)(t) = \int_0^1 G_1(t,s)\Phi(s, s^{\delta-1})ds \le \frac{t^{\delta-1}}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-2}\Phi(s,1)ds,$$
$$(T_1 h)(t) = \int_0^1 G_1(t,s)\Phi(s, s^{\delta-1})ds \ge t^{\delta-1} \int_0^1 G_1(1,s)\Phi(s,0)ds.$$

Since $A_2 \ge A_1 > 0$, we can conclude $A_1 h(t) \le T_1 h(t) \le A_2 h(t)$. Thus, $T_1 h \in P_h$. By a similar way

$$(T_2 h)(t) = \int_0^1 G_2(t,s)\Psi(s, s^{\delta-1})ds \le t^{\delta-1} \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}\Psi(s,1)ds,$$
$$(T_2 h)(t) = \int_0^1 G_2(t,s)\Psi(s, s^{\delta-1})ds \ge t^{\delta-1} \int_0^1 G_2(1,s)\Psi(s,0)ds.$$

So, $T_2 \in P_h$.

Step 4. For some $\lambda > 0$ and all $u \in P$, $T_1 u \ge \lambda T_2 u$.

Let $u \in P$. Since both G_1 and G_2 are positive constinuous and bounded functions, there exists a constant such that $G_1(t,s) \ge \kappa G_2(t,s)$. Hence, by $(H4)$, we have

$$\begin{aligned}(T_1 u)(t) = &\int_0^1 G_1(t,s)\Phi(s,u(s))ds \ge \kappa \int_0^1 G_2(t,s)\Phi(s,u(s))ds \\ \ge &\kappa\rho_0 \int_0^1 G_2(t,s)\Psi(s,u(s))ds = \lambda \int_0^1 G_2(t,s)\Psi(s,u(s))ds \\ = &\lambda(T_2 u)(t),\end{aligned}$$

where $\lambda = \kappa\rho_0$.

Thus, from Step 1–4. we conclude that all conditions of Theorem 2.1 are satisfied and the operator

$$Tu = T_1 u + T_2 u \tag{4.3}$$

has a unique fixed-point that is the unique positive solution of the FBVP (1.1). Also, from Theorem 2.1, we know for any initial value $u_0 \in P_h$, the successive sequence $u_n = T_1 u_{n-1} + T_2 u_{n-1}$, $n = 1, 2, \ldots$ converges to $u^* \in P_h$. In another words, the successive sequence

$$u_{n+1}(t) = \int_0^1 G_1(t,s)\Phi(s,u_n(s))ds + \int_0^1 G_2(t,s)\Psi(t,u_n(s))ds \to u^*, \quad n = 1, 2, \ldots,$$

as $n \to +\infty$. □

Our second result is based on Theorem 2.2. Let us add the following hypothesis to the previous hypothesis $(H1)$-$(H4)$.

$(H'1)$ $\Phi, \Psi : [0,1] \times [0,+\infty) \to [0,+\infty)$ are respectively increasing and decreasing function with respect to the second variable and $\Phi(t,0) \not\equiv 0$, $\Psi(t,1) \not\equiv 0$.

$(H5)$ For any $\mu \in (0,1)$, there exist $f(\mu), g(\mu) \in (\mu, 1)$ such that for all $t \in [0,1]$ we have

$$\Phi(t, \mu u) \geq f(\mu)\Phi(t, u), \quad \Psi(t, \mu u) \leq \frac{1}{g(\mu)}\Psi(t, u).$$

Theorem 4.2. *Assume* $(H'1)$ *and* $(H5)$ *hold, then FBVP* (1.1) *has unique solution* u^* *in* P_h *with* $h(t) = t^{\delta-1}$, $t \in [0,1]$. *Also, for any initial value problem* u_0 *and* v_0 *in* P_h *constructing successively the sequences*

$$u_{n+1}(t) = \int_0^1 G_1(t,s)\Phi(s, u_n(s))ds + \int_0^1 G_2(t,s)\Psi(s, v_n(s))ds, \quad n = 0, 1, \ldots,$$
$$v_{n+1}(t) = \int_0^1 G_1(t,s)\Phi(s, v_n(s))ds + \int_0^1 G_2(t,s)\Psi(s, u_n(s))ds, \quad n = 0, 1, \ldots,$$

we have $u_n(t) \to u^*(t)$, $v_n(t) \to u^*(t)$ *as* $n \to +\infty$, *where* $G_1(t,s)$ *and* $G_2(t,s)$ *are given in* (3.3) *and* (3.4).

Proof. Again, we consider the operators defined in (4.3), from $(H'1)$, (H5), and similar to the proof of previous theorem, one can show T_1 and T_2 satisfy the first condition of Theorem 2.2. So, we need only to verify the second condition of Theorem 2.2. Let us set

$$A_3 = \int_0^1 G_1(1,s)\Phi(s,0)ds + \int_0^1 G_2(1,s)\Psi(s,1)ds,$$
$$B_3 = \int_0^1 \frac{(1-s)^{\delta-2}}{\Gamma(\delta)}\Phi(s,1)ds + \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}\Psi(s,0)ds.$$

In view of Lemma 3.2 and $(H'1)$, $(H5)$, we have

$$\begin{aligned}
(T_1h)(t) + (T_2h)(t) =& \int_0^1 G_1(t,s)\Phi(s, s^{\delta-1})ds + \int_0^1 G_2(t,s)\Psi(s, s^{\delta-1})ds\\
\geq& t^{\delta-1}\left[\int_0^1 G_1(1,s)\Phi(s,0)ds + \int_0^1 G_2(1,s)\Psi(s,1)ds\right]\\
=& t^{\delta-1}A_3,\\
(T_1h)(t) + (T_2h)(t) =& \int_0^1 G_1(t,s)\Phi(s, s^{\delta-1})ds + \int_0^1 G_2(t,s)\Psi(s, s^{\delta-1})ds\\
\leq& t^{\delta-1}\left[\int_0^1 \frac{(1-s)^{\delta-2}}{\gamma(\delta)}\Phi(s,1)ds + \int_0^1 \frac{(\epsilon-1)(1-s)^{\epsilon-2}}{(\delta-1)\Gamma(\epsilon)}\Psi(s,0)ds\right]\\
=& t^{\delta-1}B_3.
\end{aligned}$$

Therefore, $A_3h(t) \leq (T_1h)(t) + (T_2h)(t) \leq B_3h(t)$ and $(T_1h)(t) + (T_2h)(t) \in P_h$. Thus, all conditions of Theorem 2.2 are satisfied and for any initial values v_0 and u_0 in P_h, constructing successively the sequences

$$u_{n+1}(t) = \int_0^1 G_1(t,s)\Phi(s, u_n(s))ds + \int_0^1 G_2(t,s)\Psi(s, v_n(s))ds, \quad n = 0, 1, \ldots,$$
$$v_{n+1}(t) = \int_0^1 G_1(t,s)\Phi(s, v_n(s))ds + \int_0^1 G_2(t,s)\Psi(s, u_n(s))ds, \quad n = 0, 1, \ldots,$$

we have $u_n(t) \to u^*(t)$, $v_n(t) \to u^*(t)$ as $n \to +\infty$. $\square$

5. Examples

Example 5.1. Let us consider the following FBVP

$$\begin{aligned} &D_{0+}^{\frac{5}{2}}(u(t) + I_{0+}^{\frac{8}{3}}\Psi(t,u(t)) + \Phi(t,u(t)) = 0, \\ &\lim_{t\to 0} t^{-\frac{1}{2}}u(t) = \lim_{t\to 0} t^{-\frac{1}{2}}u'(t) = u'(1) = 0, \end{aligned} \tag{5.1}$$

where $\Phi(t,u) = u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})}$, $\Psi(t,u) = \frac{u}{1+u}e^t + \frac{(b-a)t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})}$, with $b > a > 0$. Now

$$\begin{aligned} \Phi(t,\mu u) =& \mu^{\frac{1}{3}}u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})} \geq \mu^{\frac{1}{3}}\left(u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})}\right) = \mu^{\eta}\Phi(t,u), \\ \Psi(t,\mu u) =& \frac{\mu u}{1+\mu u}e^t + \frac{(b-a)t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})} \geq \mu\left[\left(\frac{u}{1+u}\right)e^t + \frac{b-a}{\Gamma(\frac{7}{2})}t^{\frac{7}{2}}\right]. \end{aligned}$$

If we set $\rho_0 \in [0, a/(e+b-a)]$, then

$$\begin{aligned} \Phi(t,u) =& u^{\frac{1}{3}} + a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})} \geq a\frac{t^{\frac{7}{2}}}{\Gamma(\frac{7}{2})(e+b-a)}(e+b-a) \\ \geq& \rho_0\left[\frac{u}{1+u}et + \frac{b-a}{\Gamma(\frac{7}{2})}t\frac{7}{2}\right] = \rho_0\Psi(t,u). \end{aligned}$$

So, all conditions of Theorem 4.1 are satisfied. Therefore, the problem (5.1) with Φ, Ψ has positive solution.

Example 5.2. Again we consider FBVP (5.1) with $\Phi(t,u) = u^{\frac{1}{3}}e^t + \alpha$, $\alpha > 0$ and $\Psi(t,u) = \frac{e^t}{1+u^{\frac{1}{4}}}$. It is clear, $\Phi : [0,1]\times[0,+\infty) \to [0,+\infty)$ is continuous and increasing with respect to the second variable and $\Phi(t,0) = \alpha > 0$, $\Psi : [0,1]\times[0,+\infty) \to [0,+\infty)$ is continuous and decreasing with respect to the second variable and $\Psi(t,1) = \frac{e^t}{2} \not\equiv 0$. Now, if we set $f_1(\mu) = \mu^{\frac{1}{3}}$ and $f_2(\mu) = \mu^{\frac{1}{4}}$, then $f_1(\mu), f_2(\mu) \in (\mu, 1)$ for all $\mu \in (0,1)$ and

$$\begin{aligned} \Phi(t,\mu u) &= \mu^{\frac{1}{3}}u^{\frac{1}{3}}e^t + \alpha \geq \mu^{\frac{1}{3}}(u^{\frac{1}{3}}e^t + \alpha) = f_1(\mu)\Phi(t,u), \\ \Psi(t,\mu u) &= \frac{e^t}{1+(\mu u)^{\frac{1}{4}}} \leq \frac{e^t}{\mu^{\frac{1}{4}}(1+u^{\frac{1}{4}})} = \frac{1}{f_2(\mu)}\Psi(t,u(t)). \end{aligned}$$

Consequently, all conditions of Theorem 4.2 are satisfied. So, problem (5.1) has unique positive solution in P_h with $h(t) = t^{\frac{3}{2}}$, $t \in [0,1]$.

Example 5.3. Consider the fractional boundary value problem

$$\begin{aligned} &D_{0+}^{2.3}(u(t) + I_{0+}^{2.2}\Psi(t,u(t)) + \Phi(t,u(t)) = 0, \\ &\lim_{t\to 0} t^{-0.7}u(t) = \lim_{t\to 0} t^{-0.8}u'(t) = u'(1) = 0, \end{aligned} \tag{5.2}$$

where $\Phi(t,u)=\sqrt[3]{3u^2(t)+t^3+3}$ and $\Psi(t,u)=\frac{3\cos^2 t}{\sqrt{5u^2(t)+\sin^2 t+1}}$.

Clearly, $\Phi:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and increasing with respect to the second variable and $\Phi(t,0)=\sqrt[3]{3}>0$, $\Psi:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and decreasing with respect to the second variable and $\Psi(t,1)\not\equiv 0$. Let $f_1(\mu)=\mu^{\frac{2}{3}}$ and $f_2(\mu)=\mu^{\frac{1}{2}}\in(\mu,1)$ for all $\mu\in(0,1)$ and

$$\begin{aligned}\Phi(t,\mu u(t)) =&\sqrt[3]{3\mu^2u^2(t)+3}\geq\sqrt[3]{\mu^2(3u^2(t)+t^3+3)}=\mu^{\frac{2}{3}}\Phi(t,u(t)),\\ \Psi(t,\mu u(t)) =&\frac{3\cos^2 t}{\sqrt{5\mu^2u^2(t)+\sin^2 t}}\leq\frac{3\cos^2 t}{\mu^{\frac{1}{2}}\sqrt{5u^2(t)+\sin^2 t}}=\frac{1}{f_2(\mu)}\Psi(t,u(t)).\end{aligned}$$

Consequently, all conditions of Theorem 4.2 are satisfied. So, problem (5.2) has unique positive solution in P_h with $h(t)=t^{1.3}$, $t\in[0,1]$.

6. Numerical Solution

Since we have already established the existence and uniqueness of a solution to (1.1), our focus here will be on its numerical solution. The method is straightforward to some degree by recalling Theorems 4.1 and 4.2, and the recurrence relation formula is the equation given in Theorem 4.1, which comes from operator (4.1). The formula for the recurrence relation can be employed without much difficulty by the initial trial solution, say, for example, $u_0(t)\equiv 0$, and then the programme iterates to find sequential $u_n(t)$ stopping when the maximum difference in two successive iterations drops below a given tolerance value. The computer algebra system Mathematica is used to execute this iterative scheme.

The passing from an iteration to the next one is done symbolically and numerically. The latter happens when, to approximate the integral appearing in the equation given in Theorem 4.1, cubic spline interpolation is used.

Firstly, we consider Example 5.1 to confirm the validity of the presented numerical method.

Using the Green's function method, we have following algorithm.

Step 1. The node points $t_0,t_1,\dots,t_M$ are considered for adequately large number of M.

Step 2. Cubic spline interpolation is used to obtain $u_n(s)$'s.

Step 3. The following approximate solution is obtained by the numerical integration:

$$\begin{aligned}u_{n+1}(t_j) =&\frac{1}{\Gamma(\delta)}\int_0^{t_j}\left[t_j^{\delta-1}(1-s)^{\delta-2}-(t_j-s)^{\delta-1}\right]\left[u_n^{1/3}(s)+a\frac{s^{7/2}}{\Gamma(7/2)}\right]ds\\ &+\frac{t_j^{\delta-1}}{\Gamma(\delta)}\int_{t_j}^1(1-s)^{\delta-2}\left[u_n^{1/3}(s)+a\frac{s^{7/2}}{\Gamma(7/2)}\right]ds\\ &+\frac{1}{(\delta-1)\Gamma(\epsilon)}\int_0^t\left[(\epsilon-1)t_j^{\delta-1}(1-s)^{\epsilon-2}-(\delta-1)(t_j-s)^{\epsilon-1}\right]\end{aligned}$$

$$\times \left[\frac{u_n(s)}{1+u_n(s)}e^s + \frac{(b-a)s^{7/2}}{\Gamma(7/2)}\right] ds$$
$$+ \frac{(\epsilon-1)t_j^{\delta-1}}{(\delta-1)\Gamma(\epsilon)} \int_{t_j}^{1} (1-s)^{\epsilon-2} \left[\frac{u_n(s)}{1+u_n(s)}e^s + \frac{(b-a)s^{7/2}}{\Gamma(7/2)}\right] ds, \quad n = 0, 1, \ldots$$

Step 4. Steps 1, 2, 3 are iterated to find consecutive $u_n(u)$ stopping when $|u_{n+1} - u_n| < TOL$.

The exact solution is unknown infact, but the iteration stopping criteria used is set $|u_{n+1} - u_n| < 10^{-10}$, and then, the numerical solution is obtained. For the step size of the node points, $h = 0.05$, the number of iterations, M=20, and $TOL = 10^{-10}$, the errors are of order 10^{-10}. The solution curve $u(t)$ is shown graphically in Figure 1 for $\delta = 2.5$ and $\epsilon = 2.66$ when $a = 1$ and $b = 2$. For other graphical simulations, (δ, ϵ)'s are taken as (2.1,2.5), (2.5,2.5), (2.5,2.9), (2.8,2.9), and (3,3). The solution curves $u(t)$'s are displayed in Figures 2-6, respectively. For $\delta = 2.5$ and $\epsilon = 2.66$ when $a = 1$ and $b = 2$, the convergence is plotted in Figure 7, and the error is plotted in Figure 8.

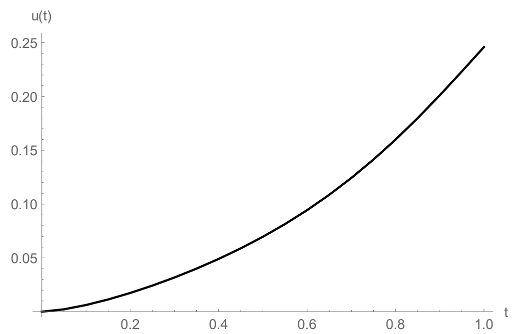

FIGURE 1. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.66$.

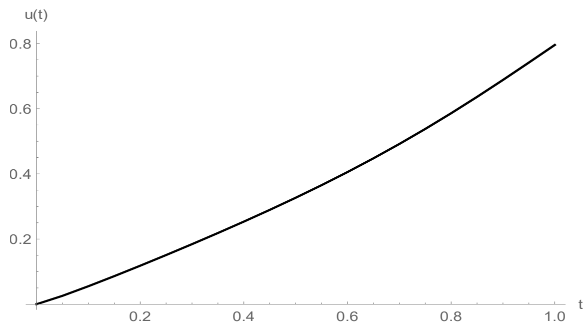

FIGURE 2. Solution curve $u(t)$ for $\delta = 2.1$ and $\epsilon = 2.5$.

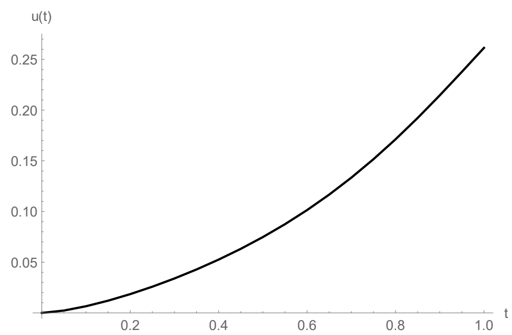

FIGURE 3. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.5$.

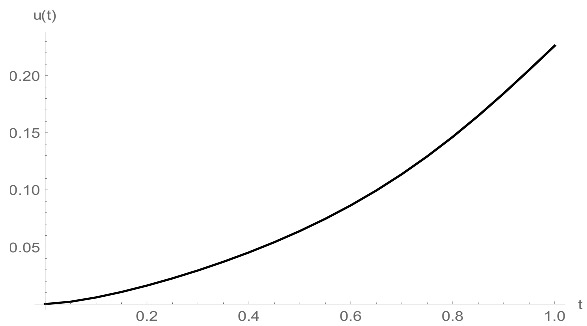

FIGURE 4. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.9$.

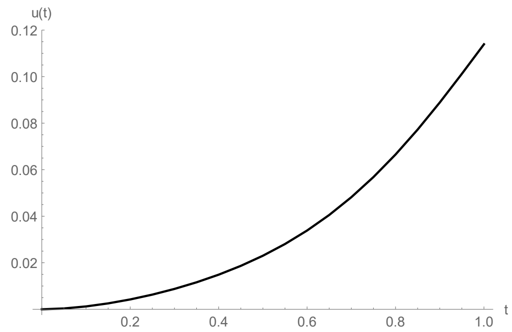

Figure 5. Solution curve $u(t)$ for $\delta = 2.8$ and $\epsilon = 2.9$.

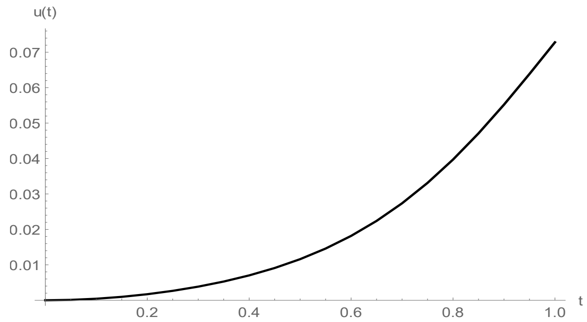

Figure 6. Solution curve $u(t)$ for $\delta = 3$ and $\epsilon = 3$.

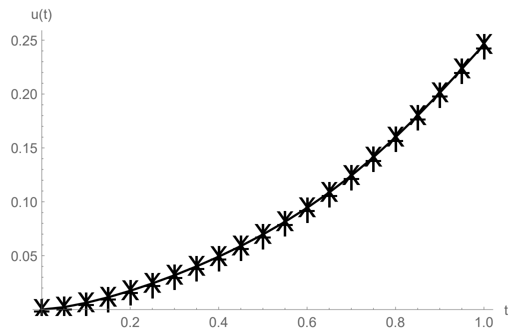

Figure 7. Convergence curve $n = 5$ (horizontal bar), $n = 10$ (vertical bar), $n = 15$ (x) and $n = 20$ (solid) for $\delta = 2.5$ and $\epsilon = 2.66$.

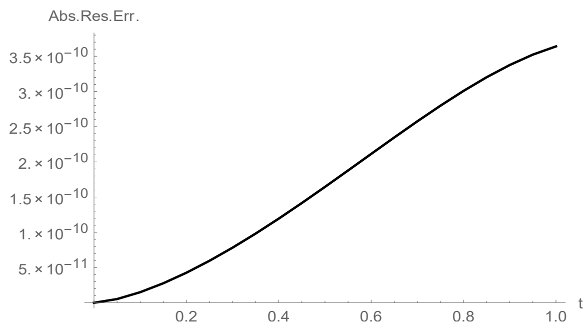

Figure 8. Error curve for $\delta = 2.5$ and $\epsilon = 2.66$.

Now, let us consider Example 5.2 to confirm the validity of the presented numerical method. Similar to the previous algorithm, the following solution is obtained:

$$\begin{aligned} u_{(n+1)}(t_j) =& \frac{1}{\Gamma(\delta)} \int_0^{t_j} \left[t_j^{\delta-1}(1-s)^{\delta-2} - (t_j - s)^{\delta-1} \right] \left[u_n^{1/3}(s) e^s + \alpha \right] ds \\ &+ \frac{t_j^{\delta-1}}{\Gamma(\delta)} \int_{t_j}^1 (1-s)^{\delta-2} \left[u_n^{1/3}(s) e^s + \alpha \right] ds + \frac{1}{(\delta-1)\Gamma(\epsilon)} \\ &\times \int_0^t \left[(\epsilon-1) t_j^{\delta-1} (1-s)^{\epsilon-2} - (\delta-1)(t_j - s)^{\epsilon-1} \right] \left[\frac{e^s}{1 + u_n^{1/4}(s)} \right] ds \\ &+ \frac{(\epsilon-1) t_j^{\delta-1}}{(\delta-1)\Gamma(\epsilon)} \int_{t_j}^1 (1-s)^{\epsilon-2} \left[\frac{e^s}{1 + u_n^{1/4}(s)} \right] ds, \quad n = 0, 1, \dots \end{aligned}$$

For the step size of the node points, $h = 0.05$, the number of iterations, M=15, and $TOL = 10^{-10}$, the errors is of order 10^{-16}. The solution curve $u(t)$ is shown graphically in Figure 9 for $\delta = 2.5$ and $\epsilon = 2.66$ when $\alpha = 1$. For other graphical simulations, (δ, ϵ)'s are taken as (2.1,2.5), (2.5,2.5), (2.5,2.9), (2.8,2.9), and (3,3). The solution curves $u(t)$'s are displayed in Figures 10–14, respectively. For $\delta = 2.5$ and $\epsilon = 2.66$ when $\alpha = 1$, the convergence is plotted in Figure 15, and the error is plotted in Figure 16.

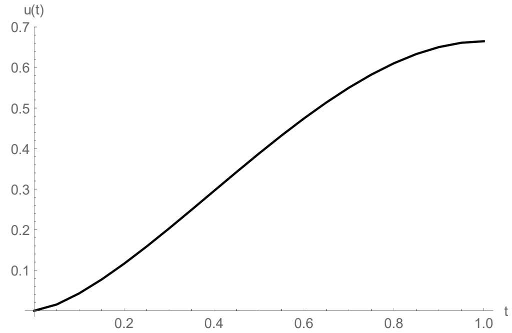

FIGURE 9. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.66$.

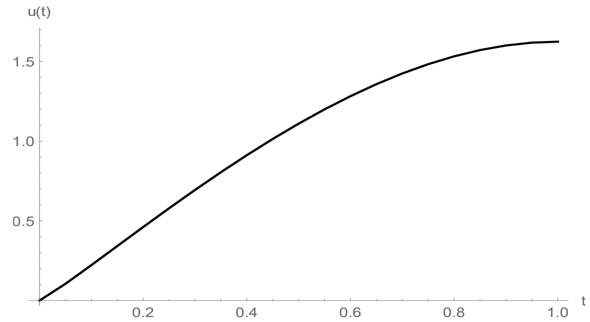

FIGURE 10. Solution curve $u(t)$ for $\delta = 2.1$ and $\epsilon = 2.5$.

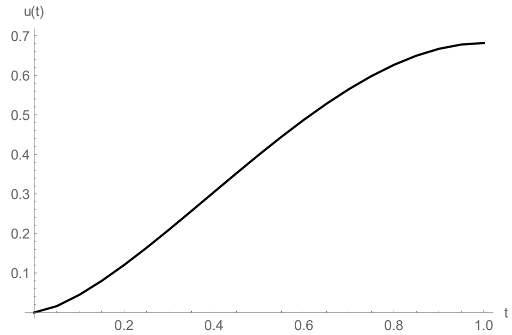

FIGURE 11. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.5$.

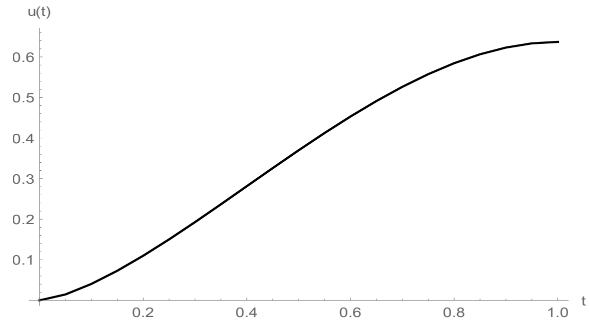

FIGURE 12. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.9$.

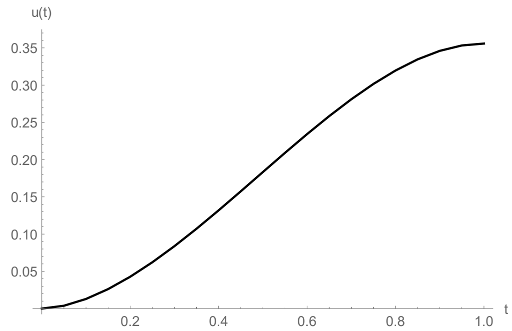

FIGURE 13. Solution curve $u(t)$ for $\delta = 2.8$ and $\epsilon = 2.9$.

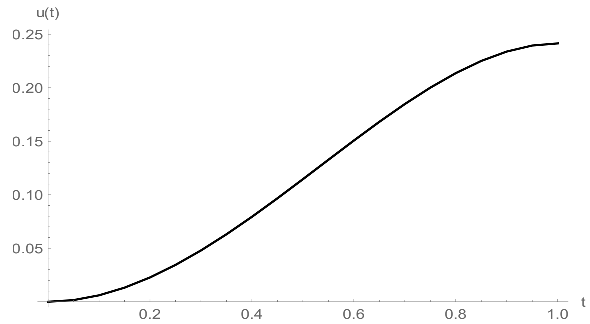

FIGURE 14. Solution curve $u(t)$ for $\delta = 3$ and $\epsilon = 3$.

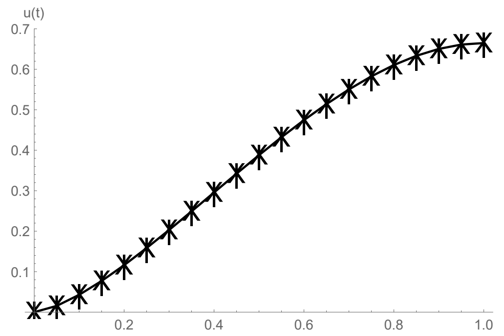

FIGURE 15. Convergence curve $n = 5$ (horizontal bar), $n = 10$ (vertical bar), $n = 15$ (x) and $n = 20$ (solid) for $\delta = 2.5$ and $\epsilon = 2.66$.

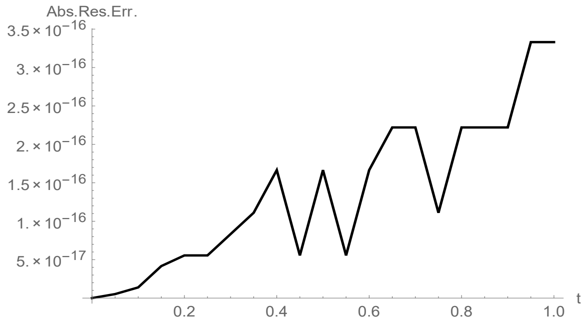

FIGURE 16. Error curve for $\delta = 2.5$ and $\epsilon = 2.66$.

Finally, let us consider Example 5.3 to confirm the validity of the presented numerical method. Similar to the current algorithm, the following solution is obtained:

$$\begin{aligned}
u_{n+1}(t_j) =& \frac{1}{\Gamma(\delta)} \int_0^{t_j} \left[t_j^{\delta-1}(1-s)^{\delta-2} - (t_j - s)^{\delta-1} \right] \sqrt[3]{3u_n^2(s) + s^3 + 3}\,\mathrm{ds} \\
&+ \frac{t_j^{\delta-1}}{\Gamma(\delta)} \int_{t_j}^1 (1-s)^{\delta-2} \sqrt[3]{3u_n^2(s) + s^3 + 3}\,\mathrm{ds} \\
&+ \frac{1}{(\delta-1)\Gamma(\epsilon)} \int_0^t \left[(\epsilon-1) t_j^{\delta-1}(1-s)^{\epsilon-2} - (\delta-1)(t_j - s)^{\epsilon-1} \right] \\
&\times \left(\frac{3\cos^2 s}{\sqrt{5u_n^2(s) + \sin^2 s + 1}} \right) \mathrm{ds} \\
&+ \frac{(\epsilon-1) t_j^{\delta-1}}{(\delta-1)\Gamma(\epsilon)} \int_{t_j}^1 (1-s)^{\epsilon-2} \left(\frac{3\cos^2 s}{\sqrt{5u_n^2(s) + \sin^2 s + 1}} \right) \mathrm{ds}, \quad n = 0, 1, \ldots
\end{aligned}$$

For the step size of the node points, $h = 0.05$, the number of iterations, $M = 10$, and $TOL = 10^{-10}$, the order of errors is of around 10^{-10}. The solution curve $u(t)$ is shown graphically in Figure 17 for $\delta = 2.3$ and $\epsilon = 2.2$. For other graphical simulations, $(\delta,\ \epsilon)$'s are taken as $(2.1, 2.5)$, $(2.5, 2.5)$, $(2.5, 2.9)$, $(2.8, 2.9)$, and $(3, 3)$. The solution curves $u(t)$'s are displayed in Figures 18-22, respectively. For $\delta = 2.3$ and $\epsilon = 2.2$, the convergence is plotted in Figure 23, and the error is plotted in Figure 24.

Table 1 shows the numerical results and absolute residual errors of the present method for $M = 10$, $\delta = 2.3$ and $\epsilon = 2.2$.

Table 1. Numerical solution and absolute residual error of Example 5.3 for $M = 10,\ \delta = 2.3$ and $\epsilon = 2.2$.

t	Numerical solution	Absolute residual error
0.0	0	
0.1	0.1145872311	4.57641×10^{-11}
0.2	0.2543086295	1.02926×10^{-10}
0.3	0.3838093159	1.47172×10^{-10}
0.4	0.4917811004	1.67472×10^{-10}
0.5	0.5736563020	1.63345×10^{-10}
0.6	0.6285251227	1.40516×10^{-10}
0.7	0.6578210989	1.06733×10^{-10}
0.8	0.6644649486	6.91701×10^{-11}
0.9	0.6521485611	3.31993×10^{-11}
1.0	0.6246825709	2.08899×10^{-12}

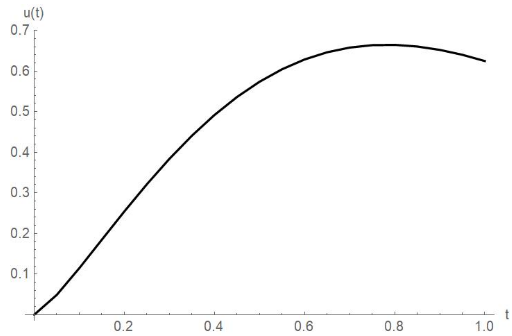

FIGURE 17. Solution curve $u(t)$ for $\delta = 2.3$ and $\epsilon = 2.2$.

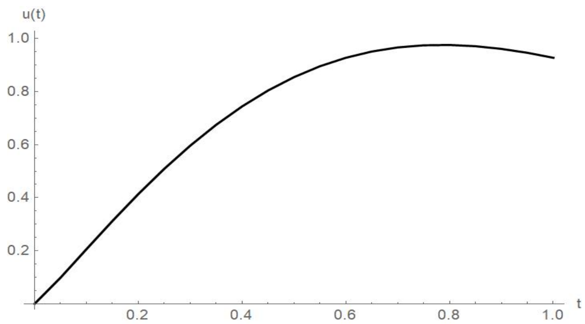

FIGURE 18. Solution curve $u(t)$ for $\delta = 2.1$ and $\epsilon = 2.5$.

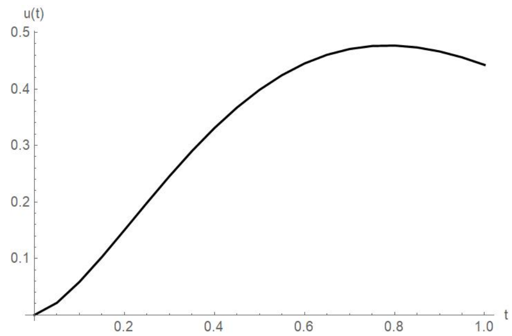

FIGURE 19. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.5$.

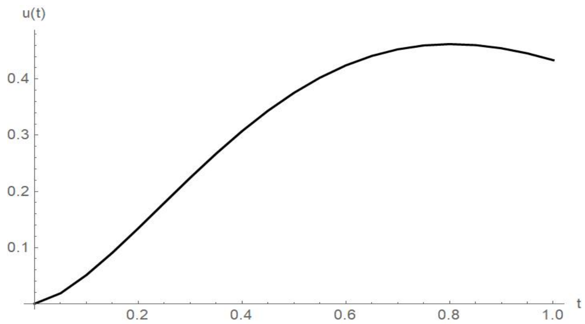

FIGURE 20. Solution curve $u(t)$ for $\delta = 2.5$ and $\epsilon = 2.9$.

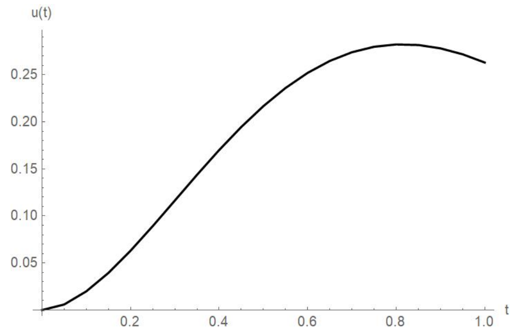

FIGURE 21. Solution curve $u(t)$ for $\delta = 2.8$ and $\epsilon = 2.9$.

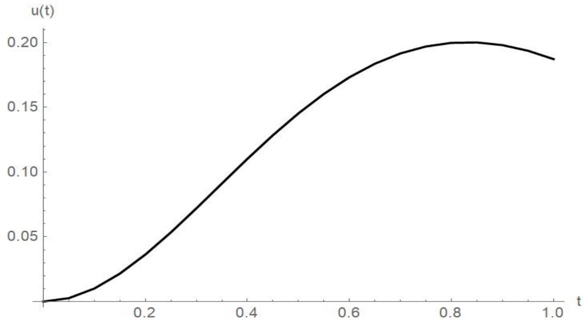

FIGURE 22. Solution curve $u(t)$ for $\delta = 3$ and $\epsilon = 3$.

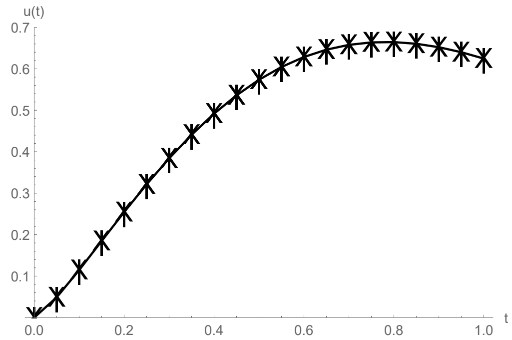

FIGURE 23. Convergence curve $n = 5$ (vertical bar), $n = 10$ (x) and $n = 15$ (solid) for $\delta = 2.3$ and $\epsilon = 2.2$.

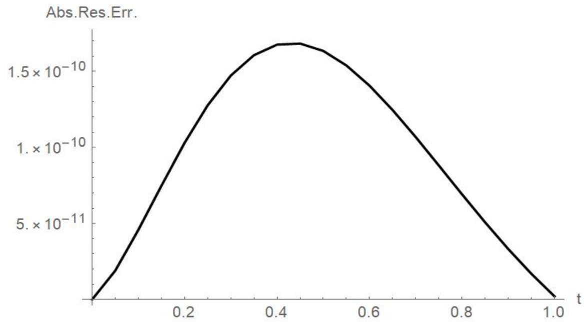

FIGURE 24. Error curve for $\delta = 2.3$ and $\epsilon = 2.2$.

7. CONCLUSION

In this article, we have considered a class of FBVPs with a Riemann-Liouville derivative and integral for deriving some novel, necessary, and sufficient conditions for the existence and uniqueness of the positive solution. We have utilised some fixed-point theorems on cones. A convergent successive sequence for finding the solution of the proposed FBVP has been derived. We have verified the validity of the proposed results by implementing some problems with the derivation of numerical methodology. The obtained results will be beneficial in proving the existence and uniqueness of positive solutions while dealing with the proposed FBVPs. In the future, the researchers can try to model real-life problems using the given fractional boundary value problem along with its qualitative and quantitative analyses.

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