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THE $\overline{\partial}$-CAUCHY PROBLEM ON WEAKLY q-CONVEX DOMAINS IN $\mathbb{C}P^n$

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Abstract. Let D be a weakly q-convex domain in the complex projective space $\mathbb{C}P^n$. In this paper, the (weighted) $\overline{\partial}$-Cauchy problem with support conditions in D is studied. Specifically, the modified weight function method is used to study the L^2 existence theorem for the $\overline{\partial}$-Neumann problem on D. The solutions are used to study function theory on weakly q-convex domains via the $\overline{\partial}$-Cauchy problem.

1. Introduction and Main Results

The $\overline{\partial}$-problem is one of the important central problems of complex variables. A classical result due to Hörmander tells us that the $\overline{\partial}$-problem is solvable in pseudoconvex domains, and hence, pseudoconvex domains has been widely accepted as the standard domain which we can solve the $\overline{\partial}$-problem. In [16], Ho extend this problem to weakly q-convex domains. In fact, Ho is the first person to study the $\overline{\partial}$-problem in q-convex domains in $\mathbb{C}^n$. This paper is devoted to studying the L^2 $\overline{\partial}$ Cauchy problem and the $\overline{\partial}$-closed extension problem for forms on a weakly q-convex domain D in the complex projective space $\mathbb{C}P^n$. These problems were first studied by Kohn and Rossi [20] (see also [12]). They proved the holomorphic extension of smooth CR functions and the $\overline{\partial}$-closed extension of smooth forms from the boundary bD of a strongly pseudoconvex domain to the whole domain D. The L^2 theory of these problems has been obtained for pseudoconvex domains in $\mathbb{C}^n$ or, more generally, for domains in complex manifolds with strongly plurisubharmonic weight functions (see Chapter 9 in [6] and the references therein). The L^2 $\overline{\partial}$ Cauchy problem was considered by Derridj [8,9]. In [30,31] Shaw has obtained a solution to this problem on a pseudoconvex domain with C^1 boundary in $\mathbb{C}^n$. Also, in the setting of strictly

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q-convex (or q-concave) domains, this problem has been studied by Sambou in his thesis (see [29]). In [1], Abdelkader-Saber studied this problem on pseudoconvex manifolds satisfing property B. In [26, 27], Saber studied this problem on a weakly q-convex domain with C^1-smooth boundary and on a q-pseudoconvex domain D in $\mathbb{C}^n$, $1 \leqslant q \leqslant n$, with Lipschitz boundary. Recently, Saber [28] studied this result to a q-pseudoconvex domain D in a Stein manifold. On a pseudoconvex domain in $\mathbb{C}P^n$, Cao-Shaw-Wang [4] (cf. also [5]) obtained the L^2 existence theorem for the $\overline{\partial}$-Neumann operator N and obtained the (weighted) L^2 $\overline{\partial}$ Cauchy-problem on such domains. The aim of this paper is to extend this result to the situation in which the boundaries are assumed weakly q-convex domain D in $\mathbb{C}P^n$. Moreover, the solutions are used to study function theory on such domains via the $\overline{\partial}$-Cauchy problem.

2. Notation and Preliminaries

Let $(x_0, x_1, \ldots, x_n)$ be a (fixed) homogeneous coordinates of $\mathbb{C}P^n$. If U_0 is the open set in $\mathbb{C}P^n$ defined by $x_0 \neq 0$ and if $(z_1, z_2, \ldots, z_n)$, where $z_i = x_i/x_0$, is the homogeneous coordinates of U_0, we assume that

$$\omega = \frac{\sum_{i=1}^n |\, dz_i|^2}{1+\sum_{i=1}^n |z_i|^2} - \frac{|\sum_{i=1}^n z_i\, d\overline{z}_i|^2}{(1+\sum_{i=1}^n |z_i|^2)^2} \quad \text{on } U_0.$$

The Fubini-Study metric of $\mathbb{C}P^n$ determined by $(x_0, x_1, \ldots, x_n)$. This is well-known standard Kähler metric of $\mathbb{C}P^n$.

Let D be a bounded domain in $\mathbb{C}P^n$ and let $C^\infty_{p,q}(D)$ be the space of complex-valued differential forms of class C^∞ and of type (p,q) on D. Denote by $L^2(D)$ the space of square integrable functions on D with respect to the Lebesgue measure in $\mathbb{C}P^n$, $L^2_{p,q}(D)$ the space of (p,q)-forms with coefficients in $L^2(D)$ and $L^2_{p,q}(D,\phi)$ the space of (p,q)-forms with coefficients in $L^2(D)$ with respect to the weighted function $e^{-\phi}$. For $u, v \in L^2_{p,q}(D)$, the inner product $\langle u, v\rangle$ and the norm $\| u \|$ are denoted by:

$$\langle u, v\rangle = \int_D u \wedge \star \overline{v} \quad \text{and} \quad \| u \|^2 = \langle u, u\rangle,$$

where $\star$ is the Hodge star operator. Let $\operatorname{dist}(z, bD)$ be the Fubini distance from $z \in D$ to the boundary bD and let δ be a C^2 defining function for D normalized by $|d\delta| = 1$ on bD such that

$$\delta = \delta(z) = \begin{cases} -\operatorname{dist}(z, bD), & \text{if } z \in D, \\ \operatorname{dist}(z, bD), & \text{if } z \in \mathbb{C}P^n \backslash D. \end{cases}$$

Let $\phi_t = -t \log|\delta|$, $t \geqslant 0$, for $u, v \in L^2_{p,q}(D, \phi_t)$, the inner product $\langle u, v\rangle_{\phi_t}$ and the norm $\| u \|_{\phi_t}$ are denoted by:

$$\langle u, v\rangle_{\phi_t} = \langle u, v\rangle_t = \int_D u \wedge \star_{(t)} \overline{v},$$
$$\|u\|^2_{\phi_t} = \| u \|^2_t = \langle u, u\rangle_t,$$

where $\star_{(t)} = \delta^t \star = \star\, \delta^t$. Since ϕ_t is bounded on $\overline{D}$, the two norms $\|\cdot\|$ and $\|\cdot\|_t$ are equivalent. Let $\overline{\partial} : \mathrm{dom}\, \overline{\partial} \subset L^2_{p,q}(D, \phi_t) \to L^2_{p,q+1}(D, \phi_t)$ be the maximal closure of the Cauchy-Riemann operator and $\overline{\partial}^*_\phi$ be its Hilbert space adjoint. Let $\Box_t = \overline{\partial}\,\overline{\partial}^*_t + \overline{\partial}^*_t \overline{\partial}$ be the Laplace-Beltrami operator, where $\overline{\partial}^*_t = \overline{\partial}^*_{\phi_t}$.

Denote by ∇ the Levi-Civita connection of $\mathbb{C}P^n$ with the standard Fubini-Study metric ω. Let $\{e_i\}$ be an orthonormal basis of vector fields. For any two vector fields f, g, the curvature operator of the connection ∇ is denoted by

$$\mathcal{R}(f, g) = \nabla_f \nabla_g - \nabla_g \nabla_f - \nabla_{[f,g]}.$$

By setting $\mathcal{R}_{ijkl} = \omega(\mathcal{R}(e_i, e_j)e_k, e_l)$, the Ricci tensor $\mathcal{R}_{ij}$ is denoted by

$$\mathcal{R}_{ij} = \sum_k \varepsilon_k \mathcal{R}_{ikkj},$$

which turns out to be self-adjoint with respect to ω and the scalar curvature

$$\Theta = \sum_i \mathcal{R}_{ii} = \sum_{i,j} \varepsilon_i \varepsilon_j \mathcal{R}_{jiij} \tag{2.1}$$

as the trace of the Ricci tensor.

Definition 2.1. Let D be an open set in an n-dimensional complex manifold X, let k be an integer with $1 \le k \le n-1$ and put $E = X \backslash D$. The set D is said to be pseudoconvex of order k in X if, for every $b \in E$ and for every coordinate neighborhood $(U, (z_1, \dots, z_n))$ which contains b as the origin, the set

$$\left\{ (z_1, \dots, z_n) \in U : \, z_i = 0, \, 1 \le i \le k, \, 0 < \sum_{i=k+1}^n |z_i|^2 < t \right\}$$

contains no points of E for some $t > 0$, then there exists $\ell > 0$ such that for each $(z'_1, \dots, z'_k)$ with $|z'_i| < \ell$, $1 \le i \le k$, the set

$$\left\{ (z_1, \dots, z_n) \in U : \, z_i = z'_i, \, 1 \le i \le k, \, \sum_{i=k+1}^n |z_i|^2 < t \right\}$$

contains at least one point of E.

Definition 2.2. Let D be an n-dimensional complex manifold and let q be an integer, $1 \le q \le n$. By Fujita ([13], Proposition 8) a C^2 function $\phi : D \to \mathbb{R}$ is pseudoconvex of order $n - q$, if and only if its Levi form $\partial\overline{\partial}\phi$ has at least $n - q + 1$ non negative eigenvalues at each point of D.

Definition 2.3. Let D be an open subset of an n-dimensional complex manifold X. D is said to have C^2 boundary in X if for all $z \in bD$ there exist an open neighborhood U of z and a C^2 function $\delta : U \to \mathbb{R}$, called a defining function of D at z such that $d\delta(z) \ne 0$ and $D \cap U = \{z \in U : \delta(z) < 0\}$. Following Ho [16], D is said to be a weakly q-convex ($q \geqslant 1$) if at every point $x_0 \in bD$ we have

$$\sum_{|K|}^{\prime}\sum_{j,k} \frac{\partial^2\delta}{\partial z_j \partial \overline{z}_k} u_{jK}\, \overline{u_{kK}} \geqslant 0, \quad \text{for every } (0,q)\text{-form},$$

where

$$u = \sum_{|J|=q} u_J\, d\overline{z}^J \text{ such that } \sum_{j=1}^{n} \frac{\partial \delta}{\partial z_j} u_{jK} = 0, \quad \text{for all } |K| = q-1.$$

Moreover, D is weakly q-convex if and only if for any $z \in bD$ the sum of any q eigenvalues $\delta_{i_1}, \ldots, \delta_{i_q}$, with distinct subscripts, of the Levi-form at z satisfies $\sum_{j=1}^{q} \delta_{i_j} \geqslant 0$ (cf. [15] and Lemma 4.7 in [34]).

Definition 2.4. Let D be a smooth domain in $\mathbb{C}^n$, D is said to be a weakly q-concave if $\overline{D}^c$ is weakly q-convex.

Lemma 2.1 ([16])**.** *Let D be a smooth domain in $\mathbb{C}^n$ and ρ be its defining function. The following two conditions are equivalent.*

(1) *D is weakly q-convex.*
(2) *For any $z \in bD$ the sum of any q eigenvalues $\rho_{i_1}, \ldots, \rho_{i_q}$, with distinct subscripts, of the Levi-form at z satisfies $\sum_{j=1}^{q} \rho_{i_j} \geqslant 0$.*

It follows from Lemma 2.1 that D is weakly q-concave if and only if for any q eigenvalues $\rho_{i_1}, \ldots, \rho_{i_q}$ of the Levi-form at $z \in bD$ with distinct subscripts we have $\sum_{j=1}^{q} \rho_{i_j} \leqslant 0$.

Example 2.1. Let D be an open subset of an n-dimensional complex manifold X and suppose that the boundary bD is a real hypersurface of class C^2 in X, that is, there exist, for each $z \in bD$, a neighborhood U of z and a C^2 function $\rho : U \to \mathbb{R}$ such that $d\rho(z) \neq 0$ and $D \cap U = \{z \in U : \rho(z) < 0\}$. Then D is pseudoconvex of order $n-q$ in X, if and only if the Levi form $\partial\overline{\partial}\rho$ has at least $n-q$ non-negative eigenvalues on $T'_z(bD)$ for each defining function ρ of D near z, where $T'_z(bD)(\subset T_z(bD))$ is the holomorphic tangent space of the real hypersurface bD at z (cf. [10, 35] called such a subset D a $(q-1)$-pseudoconvex open subset with C^2 boundary).

Theorem 2.1 ([23])**.** *Let $D \Subset \mathbb{C}P^n$ be a pseudoconvex domain of order $n-q$, $1 \leq q \leq n$. Let $d(z, bD)$ be the Fubini distance from $z \in D$ to the boundary bD. Then the function $-\log d(z, bD)$ is $(q-1)$-pluirsubharmonic in D.*

Lemma 2.2 ([17], Lemma 2.6)**.** *Let ϕ be a real valued function of class C^2 defined in an n-dimensional complex manifold D. Then ϕ is $(q-1)$-plurisubharmonic, $1 \leq q \leq n$, in D if and only if ϕ is weakly q-convex in D.*

Remark 2.1. Pseudoconvex open sets in the original sense are pseudoconvex of order $n-1$.

Remark 2.2. The pseudoconvexity of order $n-q$ of an open subset D in X is a local property of the boundary $bD \subset X$ of D. More precisely, D is pseudoconvex of order

$n-q$ in X if, for each $p \in bD$, there exists a neighborhood $U \subset X$ of p such that $D \cap U$ is pseudoconvex of order $n-q$ in U.

Remark 2.3. If an open set D in an n-dimensional complex manifold X is weakly q-convex, $1 \leq q \leq n$, then D is pseudoconvex of order $n-q$ in X. However, the converse is not valid even if $X = \mathbb{C}^n$ (see [10] and [22]). By Fujita [13], an open subset D of $\mathbb{C}^n$ is pseudoconvex of order $n-q$ in $\mathbb{C}^n$, if and only if D has an exhaustion function which is pseudoconvex of order $n-q$ on D. Thus, by the approximation theorem of Bungart [3], an open subset D of X is pseudoconvex of order $n-q$ in X, if and only if D is locally q-complete with corners in X in the sense of Peternell [24].

Proposition 2.1 (Bochner-Hörmander-Kohn-Morrey formula)**.** *Let D be a compact domain with C^2-smooth boundary bD and $\delta(x) = -d(x, bD)$. Suppose that Θ is the curvature term defined in* (2.1) *with respect to the Fubini-Study metric ω. Then, for any $u \in C^\infty_{p,q}(\overline{D}) \cap \mathrm{dom}\overline{\partial}^*_\phi$ with $1 \leqslant q \leqslant n-1$, and $\phi \in C^2(\overline{D})$, we have*

$$\overline{\partial} u\|^2_\phi + \| \,\overline{\partial}^*_\phi u\|^2_\phi = \langle \Theta u, \overline{u}\rangle_\phi + \left\| \frac{\partial u_{IJ}}{\partial \overline{z}^k} \right\|^2_\phi + \langle (i\partial\overline{\partial}\phi)u, \,\overline{u}\rangle_\phi + \int_{bD} ((i\partial\overline{\partial}\delta)u, \,\overline{u})\, e^{-\phi} ds. \tag{2.2}$$

This formula is known (*cf.* [2, 7, 15, 18, 19, 32, 36]) *for some special cases, although it has not been stated in the literature in the form* (2.2)*. If u has compact support in the interior of D, the* (2.2) *was proved in* [2]*, Chapter* 8 *of* [7] *and* (2.12) *of* [36]*. The boundary term had been computed in* [14]*, Chapter* 3 *by combining the Morrey-Kohn technique on the boundary with non-trivial weight function. If one combines the results of* [15] *and* [37] *with the interior formulae discussed above, one can prove that* (2.2) *holds for the general case with a weight function $e^{-\phi}$ and the curvature term. Specially, for $\phi = 0$,* (2.2) *was proved in* [32].

Proposition 2.2. *For any (p, q)-form u of $D \Subset \mathbb{C}P^n$ with $q \geqslant 1$,*

$$(\Theta u, \overline{u}) = q(2n+1)|u|^2, \quad \textit{when}\, u\, \textit{is a}\, (0,q)\textit{-form},$$
$$(\Theta u, \overline{u}) = 0, \quad \textit{for any}\ (n,q)\textit{-form}\ u,$$
$$(\Theta u, \overline{u}) \geq 0, \quad \textit{when}\ p \geq 1\ \textit{and}\ u\ \textit{is a}\,(p,q)\textit{-form}.$$

The statement for $(0, q)$*-forms and* (n, q)*-forms was computed in* [32] *and* [36]*. Also, following Lemma* 3.3 *of Henkin-Iordan* [14] *and its proof showed that the curvature operator Θ acting on $L^2_{p,q}(D)$ is a non-negative operator.*

3. The $\overline{\partial}$-Cauchy Problem on Weakly q-Convex Domains

This section is devoted to showing the existence of the $\overline{\partial}$-Neumann operator on a weakly q-convex domain D in $\mathbb{C}P^n$, $1 \leqslant q \leqslant n$, and by applying these existence to solve the $\overline{\partial}$ problem with support conditions on D. The boundary integral in (2.2) is non-negative for $q \geqslant 1$ by the assumption on D. Also, by taking $\phi \equiv 0$ in (2.2) and using Proposition 2.2, we find the fundamental estimate

$$\|u\|^2 \leqslant c\left(\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2\right).$$

This means that $\Box$ has closed range and $\ker \Box = \{0\}$. Thus, one can establish the L^2-existence theorem of the $\overline{\partial}$-Neumann operator N.

Theorem 3.1. *Let $D \Subset \mathbb{C}P^n$ be a weakly q-convex domain with C^2 smooth boundary. Then, for each $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, there exists a bounded linear operator $N : L^2_{p,q}(D) \to L^2_{p,q}(D)$ with the following properties:*

(i) $\mathrm{Range}N \subset \mathrm{dom}\,\Box$, $\Box\, N = N\, \Box = Id$ *on* $\mathrm{dom}\Box$*;*

(ii) *for* $f \in L^2_{p,q}(D)$,

$$f = \overline{\partial}\,\overline{\partial}^* Nf \,\oplus\, \overline{\partial}^*\overline{\partial}Nf;$$

(iii) $N\,\overline{\partial} = \overline{\partial}\,N$ *on* $\mathrm{dom}\overline{\partial}$, $1 \leqslant q \leqslant n-1$*;*

(iv) $\overline{\partial}^* N = N\,\overline{\partial}^*$ *on* $\mathrm{dom}\overline{\partial}^*$, $2 \leqslant q \leqslant n$*;*

(v) N, $\overline{\partial}N$ *and* $\overline{\partial}^* N$ *are bounded linear operators on* $L^2_{p,q}(D)$.

Using the duality relations pertaining to the $\overline{\partial}$-Neumann problem, one solve the L^2 $\overline{\partial}$ Cauchy problem on weakly q-convex domains in $\mathbb{C}P^n$, $1 \leqslant q \leqslant n$. This method was first used by Kohn-Rossi [20] for smooth forms on strongly pseudoconvex domains. More precisely, we prove the following L^2 Cauchy problem for $\overline{\partial}$ in $\mathbb{C}P^n$:

Theorem 3.2. *Let $D \Subset \mathbb{C}P^n$ be a weakly q-convex domain, $1 \leqslant q \leqslant n$ with C^2 smooth boundary. Then, for $f \in L^2_{p,q}(\mathbb{C}P^n)$, $\mathrm{supp}\, f \subset \overline{D}$, $1 \leqslant q \leqslant n-1$, satisfying $\overline{\partial} f = 0$ in the distribution sense in $\mathbb{C}P^n$, there exists $u \in L^2_{p,q-1}(\mathbb{C}P^n)$, $\mathrm{supp}\, u \subset \overline{D}$ such that $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$.*

Proof. Let $f \in L^2_{p,q}(\mathbb{C}P^n)$, $\mathrm{supp}\, f \subset \overline{D}$, then $f \in L^2_{p,q}(D)$. From Theorem 3.1, $N_{n-p,n-q}$ exists for $n-q \geqslant 1$. Since $N_{n-p,n-q} = \Box^{-1}_{n-p,n-q}$ on $\mathrm{Range}\,\Box_{n-p,n-q}$ and $\mathrm{Range}\, N_{n-p,n-q} \subset \mathrm{dom}\,\Box_{n-p,n-q}$, then $N_{n-p,n-q} \star \overline{f} \in \mathrm{dom}\,\Box_{n-p,n-q} \subset L^2_{n-p,n-q}(D)$, for $q \leqslant n-1$. Thus, we can define $u \in L^2_{p,q-1}(D)$ by

$$u = -\star\ \overline{\overline{\partial}\, N_{n-p,n-q}\ \star \overline{f}}.$$

Thus $\mathrm{supp}\, u \subset \overline{D}$ and u vanishes on bD. Now, we extend u to $\mathbb{C}P^n$ by defining $u = 0$ in $\mathbb{C}P^n \setminus D$. It follows from the same arguments of Theorem 9.1.2 in [6] and Theorem 2.2 in [1] that the form u satisfies the equation $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$. Thus the proof follows. $\Box$

4. The Weighted $\overline{\partial}$-Cauchy Problem

In this section, we assume that D is a weakly q-convex domain, $1 \leqslant q \leqslant n$, with C^2 smooth boundary in $\mathbb{C}P^n$. Also, we will choose $\phi_t = -t \log|\delta|$, $t > 0$ in (2.2), and using Remark 2.3 and by using Proposition 2.2, the inequality (2.2) implies the weighted L^2-existence for the $\overline{\partial}$. Also, for $u \in \mathrm{Dom}(\Box_t)$ of degree $q \geqslant 1$ and for $t > 0$, we have

$$\begin{aligned} t\|u\|_t^2 &\leqslant (\|\overline{\partial}u\|_t^2 + \|\overline{\partial}_t^* u\|_t^2) \\ &= \langle \Box_t u, u\rangle_t \\ &\leqslant \|\Box_t f\|_t \|u\|_t, \end{aligned}$$

i.e.,

$$t\|u\|_t \leqslant \|\Box_t u\|_t.$$

Since $\Box_t$ is a linear closed densely defined operator, then, from [15, Theorem 1.1.1], $\mathrm{Range}(\Box_t)$ is closed. Thus, from (1.1.1) in [15] and the fact that $\Box_t$ is self adjoint, we have the Hodge decomposition

$$L^2_{p,q}(D,\phi_t) = \overline{\partial}\,\overline{\partial}_t^* \mathrm{dom}(\Box_t) \oplus \overline{\partial}_t^*\overline{\partial}\,\mathrm{dom}(\Box_t).$$

Since $\Box_t$ is one to one on $\mathrm{dom}(\Box_t)$ from (1.5.3) in [15], then there exists a unique bounded inverse operator

$$N_t : \mathrm{Ran}(\Box_t) \to \mathrm{dom}(\Box_t) \cap (\ker(\Box_t))^{\perp}$$

such that $N_t\Box_t f = f$ on $\mathrm{dom}(\Box_t)$. Therefore, we can establish the existence theorem of the inverse of $\Box_t$ the so called weighted $\overline{\partial}$-Neumann operator N_t.

Theorem 4.1. *For any $1 \leqslant q \leqslant n$ and $t > 0$, there exists a bounded linear operator $N_t : L^2_{p,q}(D,\phi_t) \to L^2_{p,q}(D,\phi_t)$ satisfies the following properties:*

(i) $\mathrm{Range}\,(N_t) \subset \mathrm{dom}(\Box_t)$, $N_t\Box_t = I$ *on* $\mathrm{dom}(\Box_t)$;
(ii) *for* $f \in L^2_{p,q}(D,\phi_t)$, *we have* $u = \overline{\partial}\,\overline{\partial}_t^* N_t f \;\oplus\; \overline{\partial}_t^*\overline{\partial} N_t f$;
(iii) $\overline{\partial} N_t = N_t\overline{\partial}$, $1 \leqslant q \leqslant n-1$;
(iv) $\overline{\partial}^* N_t = N_t\,\overline{\partial}^*$, $2 \leqslant q \leqslant n$;
(v) *for all* $f \in L^2_{p,q}(D,\phi_t)$, *we have the estimates*

$$\begin{aligned} t\|N_t f\|_t &\leqslant \|f\|_t, \\ \sqrt{t}\|\overline{\partial} N_t f\|_t + \sqrt{t}\|\overline{\partial}_t^* N_t f\|_t &\leqslant \|f\|_t; \end{aligned}$$

(vi) *if* $\overline{\partial} f = 0$, *then* $u_t = \overline{\partial}_t^* N_t f$ *solves the equation* $\overline{\partial} u_t = f$.

Theorem 4.2. *For* $f \in L^2_{p,q}(D,\phi_t)$, $1 \leqslant q \leqslant n-1$, $\mathrm{supp}\, f \subset \overline{D}$, *satisfying* $\overline{\partial} f = 0$ *in the distribution sense in* $\mathbb{C}P^n$, *there exists* $u \in L^2_{p,q-1}(D,\phi_t)$, $\mathrm{supp}\, u \subset \overline{D}$ *such that* $\overline{\partial} u = f$ *in the distribution sense in* $\mathbb{C}P^n$.

Proof. Following Theorem 4.1, N_t exists for forms in $L^2_{n-p,n-q}(D,\phi_t)$. Thus, one can defines $u_t \in L^2_{p,q-1}(D,\phi_t)$ by

$$u_{(t)} = -\star_{(t)}\ \overline{\overline{\partial}\, N_{n-p,n-q}\ \star_{(-t)}\ \overline{f}}. \tag{4.1}$$

Thus $\mathrm{supp}\, u_t \subset \overline{D}$ and u_t vanishes on bD. Now, we extend u_t to $\mathbb{C}P^n$ by defining $u_t = 0$ in $\mathbb{C}P^n \setminus D$. We want to prove that the extended form u_t satisfies the equation

$\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. For $\eta \in L^2_{n-p,n-q-1}(D, -\phi_t) \cap \operatorname{dom} \overline{\partial}$, we have

$$
\begin{aligned}
\langle \overline{\partial}\, \eta, \star_{(t)} f \rangle_D &= \int_D \overline{\partial}\, \eta \wedge \star_{(-t)} (\star_{(t)} f) \\
&= \int_D \overline{\partial}\, \eta \wedge \star_{(-t)} \star_{(t)}\; f \\
&= (-1)^{p+q} \int_D \overline{\partial}\, \eta \wedge\; f \\
&= (-1)^{p+q} \langle f, \star_{(-t)} \overline{\partial}\, \eta \rangle_D \\
&= (-1)^{p+q} \langle f, \star_{(-t)} \overline{\partial}\, \eta \rangle_{\mathbb{C}P^n},
\end{aligned}
$$

because $\operatorname{supp} f \subset \overline{D}$. Since $\vartheta|_D = \overline{\partial}^*|_D$, when ϑ acts in the distribution sense (see [15]), then we obtain

$$
\begin{aligned}
\langle \overline{\partial} \eta, \star_{(t)} f \rangle_D &= \langle f, \vartheta\; \star_{(-t)}\; \eta \rangle_{\mathbb{C}P^n} \\
&= \langle \overline{\partial}\, f, \star_{(-t)} \eta \rangle_{\mathbb{C}P^n} \\
&= 0.
\end{aligned}
$$

It follows that $\overline{\partial}^*_t (\star_{(t)} f) = 0$ on D. Using Theorem 4.1 (iv), we have

$$
\overline{\partial}^*_t N_t (\star_{(t)} f) = N_t \overline{\partial}^*_t (\star_{(t)} f) = 0. \tag{4.2}
$$

Thus, from (4.1) and (4.2), one obtains

$$
\begin{aligned}
\overline{\partial} u_t &= - \overline{\partial\; \star_{-t}\;\; \overline{\partial} N_{n-p,n-q}\;\; \star_t\, \overline{f}} \\
&= (-1)^{p+q+1} \overline{\star \star\; \partial\; \star\;\; \overline{\partial} N_{n-p,n-q}\;\; \star\, \overline{f}} \\
&= (-1)^{p+q} \overline{\star \overline{\partial}^* \overline{\partial} N_{n-p,n-q} \star \overline{f}} \\
&= (-1)^{p+q} \overline{\star\, (\overline{\partial}^* \overline{\partial} + \overline{\partial}\, \overline{\partial}^*)\, N_{n-p,n-q} \star \overline{f}} \\
&= (-1)^{p+q} \overline{\star \star \overline{f}} \\
&= f,
\end{aligned}
$$

in the distribution sense in D. Since $u = 0$ in $\mathbb{C}P^n \backslash D$, then for $u \in L^2_{p,q}(\mathbb{C}P^n) \cap \operatorname{dom} \overline{\partial}^*$, one obtains

$$
\begin{aligned}
< u, \overline{\partial}^* u >_{\mathbb{C}P^n} &= < u, \overline{\partial}^* u >_D \\
&= < \star \overline{\partial}^* u, \star_{(-t)} u >_{(t)D} \\
&= (-1)^{p+q} < \overline{\partial} \star u, \star_{(-t)} u >_{(t)D} \\
&= (-1)^{p+q} < \star u, \overline{\partial}^*\, \star_{(-t)}\; u >_{(t)D} \\
&= < \star u, \star_{(-t)} \overline{\partial} u >_{(t)D} \\
&= < f, u >_D \\
&= < f, u >_{\mathbb{C}P^n},
\end{aligned}
$$

where the third equality holds since $\star u = (-1)^{q+1}\, \overline{\partial} N_{n-p,n-q} \star\, f \in \operatorname{dom} \overline{\partial}^*$. Thus $\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. $\square$

As in [5], we prove the following results.

Proposition 4.1. *Let D be the same as in Theorem 3.1. Put $\Omega = \mathbb{C}P^n \setminus \overline{D}$. Then, for any $f \in W^{1+\varepsilon}_{p,q}(\Omega)$, $\overline{\partial} f = 0$, $0 \le \varepsilon < \frac{1}{2}$, there exists $F \in W^{\varepsilon}_{p,q}(\mathbb{C}P^n)$ such that $F|_\Omega = f$ and $\overline{\partial} F = 0$ in $\mathbb{C}P^n$.*

Proof. Since D has C^2 smooth boundary, there exists a bounded extension operator from $W^s_{p,q}(\Omega)$ to $W^s_{p,q}(\mathbb{C}P^n)$ for all $s \geqslant 0$ (cf. e.g. [33]). Let $\tilde{f} \in W^{1+\varepsilon}_{p,q}(\mathbb{C}P^n)$ be the extension of f so that $\tilde{f}|_\Omega = f$ with

$$\|\tilde{f}\|_{W^{1+\varepsilon}(\mathbb{C}P^n)} \leqslant C\|f\|_{W^{1+\varepsilon}(\Omega)}.$$

Furthermore, we can choose an extension such that $\overline{\partial}\tilde{f} \in W^{\varepsilon}(D) \cap L^2(D, \phi_{2\varepsilon})$.

One defines $T\tilde{f}$ by $T\tilde{f} = -\star_{2\varepsilon}\overline{\partial} N_{2\varepsilon}(\star_{-2\varepsilon}\overline{\partial}\tilde{f})$ in Ω. As in Theorem 4.2, $T\tilde{f} \in L^2(D, \phi_{2\varepsilon})$. But for a C^2-smooth domain, we have that $T\tilde{f} \in L^2(D, \phi_{2\varepsilon})$ is comparable to $W^{\varepsilon}(\Omega)$ for $0 \leqslant \varepsilon < \frac{1}{2}$. This gives that $T\tilde{f} \in W^{\varepsilon}_{p,q}(\Omega)$ and $T\tilde{f}$ satisfies $\overline{\partial} T\tilde{f} = \overline{\partial}\tilde{f}$ in $\mathbb{C}P^n$ in the distribution sense if we extend $T\tilde{f}$ to be zero outside Ω.

Since $0 \leqslant \varepsilon < \frac{1}{2}$, the extension by 0 outside Ω is a continuous operator from $W^{\varepsilon}(\Omega)$ to $W^{\varepsilon}(\mathbb{C}P^n)$ (cf. e.g. [21]). Thus we have $T\tilde{f} \in W^{\varepsilon}(\mathbb{C}P^n)$.

Define

$$F = \begin{cases} f, & \text{if } z \in \overline{D}, \\ \tilde{f} - T\tilde{f}, & \text{if } z \in \Omega. \end{cases}$$

Then $F \in W^{\varepsilon}_{p,q}(\mathbb{C}P^n)$ and F is $\overline{\partial}$-closed extension of f to $\mathbb{C}P^n$. □

Corollary 4.1. *Let $D \Subset \mathbb{C}P^n$ be a weakly q-concave domain, $n \geqslant 2$ with C^2 smooth boundary. Then $W^{1+\varepsilon}_{p,0}(D) \cap \ker \overline{\partial} = \{0\}$, $1 \le p \le n$ and $W^{1+\varepsilon}_{0,0}(D) \cap \ker \overline{\partial} = \mathbb{C}$.*

Proof. Using Proposition 4.1 for $q = 0$, we have that any holomorphic $(p, 0)$-form on D extends to be a holomorphic $(p, 0)$ in $\mathbb{C}P^n$, which are zero (when $p > 0$) or constants (when $p = 0$). □

Corollary 4.2. *Let $D \Subset \mathbb{C}P^n$ be a weakly q-concave domain, $n \geqslant 2$ with C^2 smooth boundary. Then, for any $f \in W^{1+\varepsilon}_{p,q}(D)$, where $0 \le p \le n$, $1 \leqslant q \leqslant n-2$, $p \neq q$, and $0 \leqslant \varepsilon < \frac{1}{2}$, such that $\overline{\partial} f = 0$ in D, there exists $u \in W^{1+\varepsilon}_{p,q-1}(D)$ such that $\overline{\partial} u = f$ in D.*

Proof. If $p \neq q$, we have that $F = \overline{\partial} u$ for some $U \in W^1_{p,q-1}(\mathbb{C}P^n)$. Let $u = U$ on D, we have $u \in W^1_{p,q-1}(D)$ satisfying $\overline{\partial} u = f$ in D. □

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