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ON TWO PEXIDERIZED FUNCTIONAL EQUATIONS OF DAVISON TYPE

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Abstract. In this paper, we present the general solution of two Pexiderized functional equations of Davison type without assuming any regularity assumption on the unknown functions.

1. Introduction

In 1979, during the 17^{th} International Symposium on Functional Equations (ISFE), Davison [2] introduced the following functional equation

$$f(xy) + f(x + y) = f(xy + x) + f(y), \tag{1.1}$$

where the domain and range of f is a (commutative) field. At ISFE 17^{th} Benz [1] determined the continuous solution of Davison functional equation. Indeed, he proved that if $f : \mathbb{R} \to \mathbb{R}$, then every continuous solution of the equation (1.1) is of the form $f(x) = ax + b$, where a and b are real constants. In 2000, Girgensohn and Lajkó [3] obtained the general solution of the Davison equation without any regularity assumption. They showed that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1.1) for all $x, y \in \mathbb{R}$ if and only if f is of the form $f(x) = A(x) + b$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and b is an arbitrary real constant. For more on Davison functional equation (1.1) and its stability interested readers should referred to the book [5] and references therein. In [4] we studied the following functional

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equations

$$f(x+y) = f(xy) + f(y) + f(x - xy), \tag{1.2}$$

$$f(x+y) = f(x - xy) + f(y + xy), \tag{1.3}$$

$$f(x+y) = f(x + y - xy) + f(xy), \tag{1.4}$$

$$f(x+y) = f(x - xy) + f(y) - f(-xy), \tag{1.5}$$

$$2f(x) + 2f(y) = f(x + y + xy) + f(x + y - xy), \tag{1.6}$$

without any regularity assumption on the unknown function f.

Let $\mathbb{X}$ be a nonempty set. The list $(\mathbb{X}, +, \cdot)$ is called a linear (or vector) space if $(\mathbb{X}, +)$ is an abelian group, and $\cdot$ is a mapping that assigns to each $(\lambda, x) \in \mathbb{R} \times \mathbb{X}$ an element $\lambda \cdot x$ of $\mathbb{X}$ (which will be denoted simply as λx) such that for all $\alpha, \lambda \in \mathbb{R}$ and $x, y \in \mathbb{X}$, we have (i) $\alpha(\lambda x) = (\alpha\lambda)x$; (ii) $(\alpha + \lambda)x = \alpha x + \lambda x$ and $\lambda(x + y) = \lambda x + \lambda y$; (iii) $1x = x$. A function $f : \mathbb{R} \to \mathbb{X}$, where $\mathbb{X}$ is a linear space, is said to be additive if and only if f satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $\mathbb{X} = \mathbb{R}$, it is well known that every regular (measurable, continuous, integrable, or locally integrable) additive function is of the form $f(x) = ax$, where a is an arbitrary constant in $\mathbb{R}$.

The aim of the present paper is to present the general solutions (f, g, h, k) on the pexiderized functional equations

$$f(x+y) + g(-xy) = h(x - xy) + k(y) \tag{1.7}$$

and

$$2f(x) + 2g(y) = h(x + y + xy) + k(x + y - xy), \tag{1.8}$$

for all $x, y \in \mathbb{R}$ without assuming any regularity assumption of the unknown functions. This paper ends with two open problems related to the above functional equations.

2. General Solutions of (1.7) and (1.8) on $\mathbb{R}$

In this section $\mathbb{X}$ denotes a linear space.

Theorem 2.1. *The functions $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy the functional equation* (1.7) *for all $x, y \in \mathbb{R}$ if and only if they have the form $f(x) = A(x) + b_1$, $g(x) = A(x) + b_2$, $h(x) = A(x) + b_3$ and $k(x) = A(x) + b_4$, where $A : \mathbb{R} \to \mathbb{X}$ is additive and $b_1, b_2, b_3, b_4 \in \mathbb{X}$ are constants with $b_1 + b_2 = b_3 + b_4$.*

Proof. Sufficiency is obvious. Let $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy (1.7). Substituting $x = 0$, $y = 0$ and $y = 1$, respectively, in (1.7), we get

$$f(y) + g(0) = h(0) + k(y), \tag{2.1}$$

$$f(x) + g(0) = h(x) + k(0), \tag{2.2}$$

$$f(x+1) + g(-x) = h(0) + k(1). \tag{2.3}$$

If we use these equations in (1.7), we obtain

$$f(x+y) - f(1 + xy) = f(x - xy) + f(y) + 2g(0) - 2h(0) - k(0) - k(1), \tag{2.4}$$

for all $x, y \in \mathbb{R}$. Letting $x = 1$ in (2.4), we obtain

$$f(1-y) = -f(y) - 2g(0) + 2h(0) + k(0) + k(1) \quad (y \in \mathbb{R}).$$

Hence,

$$f(1+xy) = -f(-xy) - 2g(0) + 2h(0) + k(0) + k(1) \quad (x, y \in \mathbb{R}). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$f(x+y) + f(-xy) = f(x-xy) + f(y) \quad (x, y \in \mathbb{R}).$$

Therefore f is of the form $f(x) = A(x) + b_1$, where $A : \mathbb{R} \to \mathbb{X}$ is additive and $b_1 \in \mathbb{X}$ is a constant (see [4, Theorem 3.1]). Now we obtain the asserted form of g, h and k by using (2.1), (2.2) and (2.3). The proof of the theorem is now complete. ☐

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{X}$ be an odd function. Then f satisfies*

$$f(x) + f(y) + f(y+1) = f(x+y+xy) + f(y-xy+1) \quad (x, y \in \mathbb{R}) \tag{2.6}$$

if and only if f is additive.

Proof. Sufficiency is clear. Let f satisfy (2.6). Replacing y by $y+1$ and $y-1$, respectively, we get

$$f(x) + f(y+1) + f(y+2) = f(2x+y+xy+1) + f(y-x-xy+2), \tag{2.7}$$

$$f(x) + f(y-1) + f(y) = f(y+xy-1) + f(x+y-xy), \tag{2.8}$$

for all $x, y \in \mathbb{R}$. Interchanging x and y in (2.8), we see that

$$f(y) + f(x-1) + f(x) = f(x+xy-1) + f(x+y-xy) \quad (x, y \in \mathbb{R}). \tag{2.9}$$

Subtracting (2.9) from (2.8), we get

$$f(y-1) - f(x-1) = f(y+xy-1) - f(x+xy-1) \quad (x, y \in \mathbb{R}). \tag{2.10}$$

Replacing x by $x+1$ and y by $y+1$, respectively, in (2.10), we have

$$f(y) - f(x) = f(2y+x+xy+1) - f(2x+y+xy+1) \quad (x, y \in \mathbb{R}). \tag{2.11}$$

Adding the equations (2.7) and (2.11), we have

$$f(y) + f(y+1) + f(y+2) = f(2y+x+xy+1) + f(y-x-xy+2), \tag{2.12}$$

for all $x, y \in \mathbb{R}$. Let $u, v \in \mathbb{R}$ with $u+v \neq -2$. Setting $x = \frac{v-u}{2+u+v}$ and $y = \frac{u+v}{2}$ in (2.12), we get

$$f\Big(\frac{u+v}{2}\Big) + f\Big(\frac{u+v}{2}+1\Big) + f\Big(\frac{u+v}{2}+2\Big) = f\Big(\frac{u+v}{2}+v+1\Big) + f(u+2). \tag{2.13}$$

If $u+v=-2$, then (2.13) reduces to $f(-1)+f(0)+f(1) = f(v)+f(-v)$, which holds automatically, since f is odd. Thus, (2.13) is true for all $u, v \in \mathbb{R}$. Replacing v by $v-u$ in (2.13), we have

$$f\Big(\frac{v}{2}\Big) + f\Big(\frac{v}{2}+1\Big) + f\Big(\frac{v}{2}+2\Big) = f\Big(\frac{3v-2u}{2}+1\Big) + f(u+2). \tag{2.14}$$

Replacing u by $u-2$ and v by $-\frac{2}{3}v$ in (2.14), we have

$$f\Big(-\frac{v}{3}\Big)+f\Big(-\frac{v}{3}+1\Big)+f\Big(-\frac{v}{3}+2\Big)=f\left(3-(u+v)\right)+f(u).$$

This functional equation is a Pexider functional equation of the form

$$F(x)=G(x+y)+H(y) \quad (x,y\in\mathbb{R}), \tag{2.15}$$

where

$$\begin{aligned}F(t)&:=f\Big(-\frac{t}{3}\Big)+f\Big(-\frac{t}{3}+1\Big)+f\Big(-\frac{t}{3}+2\Big),\\ G(t)&:=f(3-t),\\ H(t)&:=f(t).\end{aligned}$$

It is easy to show that (2.15) implies $H(x+y)=H(x)+H(y)$ for all $x,y\in\mathbb{R}$ since $G(x)=F(0)-H(x)$, $F(x)=F(0)-H(x)$ and $H(0)=0$. Hence, H is additive and thus f is additive. The proof of the lemma is now complete. □

Theorem 2.2. *The functions $f,g,h,k:\mathbb{R}\to\mathbb{X}$ satisfy* (1.8) *for all $x,y\in\mathbb{R}$ if and only if they have the form $f(x)=A(x)+b_1$, $g(x)=A(x)+b_2$, $h(x)=A(x)+b_3$, $k(x)=A(x)+b_4$, where $A:\mathbb{R}\to\mathbb{X}$ is additive and $b_1,b_2,b_3,b_4\in\mathbb{X}$ are constants with $2b_1+2b_2=b_3+b_4$.*

Proof. Sufficiency is clear. Let $f,g,h,k:\mathbb{R}\to\mathbb{X}$ satisfy (1.8). Setting $x=0$, $y=0$ and $y=1$, respectively, in (1.8), we get

$$2f(0)+2g(y)=h(y)+k(y), \tag{2.16}$$

$$2f(x)+2g(0)=h(x)+k(x), \tag{2.17}$$

$$2f(x)+2g(1)=h(2x+1)+k(1). \tag{2.18}$$

Therefore from (2.16) and (2.17), we obtain

$$f(x)-f(0)=g(x)-g(0) \quad (x\in\mathbb{R}). \tag{2.19}$$

Replacing y by $2y+1$ in (1.8), we have

$$2f(x)+2g(2y+1)=h(2x+2y+2xy+1)+k(2y-2xy+1) \quad (x,y\in\mathbb{R}). \tag{2.20}$$

Using (2.18) and (2.19) in (2.20), we get

$$\begin{aligned}2f(x)+2f(2y+1)=&2f(x+y+xy)+k(2y-2xy+1)+2f(0)-2g(0)\\ &+2g(1)-k(1),\end{aligned} \tag{2.21}$$

for all $x,y\in\mathbb{R}$. It follows from (2.17) that

$$2f(2y-2xy+1)+2g(0)=h(2y-2xy+1)+k(2y-2xy+1) \quad (x,y\in\mathbb{R}).$$

Using (2.18) in this equation, we have

$$\begin{aligned}k(2y-2xy+1)=&2f(2y-2xy+1)-2f(y-xy)\\ &+2g(0)-2g(1)+k(1) \quad (x,y\in\mathbb{R}).\end{aligned}$$

Using this equation in (2.21), we get

$$f(x) + f(2y+1) - f(0) = f(x+y+xy) + f(2y-2xy+1) - f(y-xy), \tag{2.22}$$

for all $x, y \in \mathbb{R}$. Letting $x = -1$ and replacing y by $\frac{1}{2}y$ in (2.22), we have

$$f(2y+1) = f(y+1) + f(y) - f(0) \quad (y \in \mathbb{R}). \tag{2.23}$$

Replacing y by $y - xy$ in (2.23), we obtain

$$f(2y-2xy+1) = f(y-xy+1) + f(y-xy) - f(0) \quad (y \in \mathbb{R}).$$

Using this equation directly in the right-hand side of (2.22) and using (2.23) in the left-hand side of (2.22), we get

$$f(x) + f(y) - f(0) = f(x+y+xy) + f(y-xy+1) - f(y+1), \tag{2.24}$$

for all $x, y \in \mathbb{R}$. Since the left-hand side of (2.24) is symmetric in x and y, we get

$$f(y-xy+1) - f(y+1) = f(x-xy+1) - f(x+1), \tag{2.25}$$

for all $x, y \in \mathbb{R}$. Replacing y by $2y-1$ in (2.25), we get

$$f(x+2y-2xy) - f(2y) = f(2(x-xy)+1) - f(x+1).$$

Using (2.23) in this equation, we have

$$f(x+2y-2xy) - f(2y) = f(x-xy+1) + f(x-xy) - f(x+1) - f(0), \tag{2.26}$$

for all $x, y \in \mathbb{R}$. Using (2.25) in (2.26), we have

$$f(x+2y-2xy) - f(2y) = f(y-xy+1) + f(x-xy) - f(y+1) - f(0), \tag{2.27}$$

for all $x, y \in \mathbb{R}$. Setting $x = 1$ in (2.27), we get

$$f(2y) = f(1+y) - f(1-y) + f(0) \quad (y \in \mathbb{R}). \tag{2.28}$$

Replacing y by $-y$ in (2.28) and adding the obtained equation to (2.28), we get

$$f(2y) + f(-2y) = 2f(0) \quad (y \in \mathbb{R}).$$

Hence, $f - f(0)$ is odd. Since f satisfies (2.24), $f - f(0)$ satisfies (2.6). Therefore, $f - f(0)$ is additive by Lemma 2.1. Thus, $f(x) = A(x) + b_1$, where $A : \mathbb{R} \to \mathbb{X}$ is an additive function and $b_1 \in \mathbb{X}$ is a constant. Now, using (2.19), (2.18) and (2.17), we obtain the asserted form of g, h and k. This finishes the proof of the theorem. □

3. Open problems

In this section, we pose two open problems. Determine the general solution (f, g, h, k) of the functional equations (1.7) and (1.8), respectively, where the domain and range of the unknown functions f, g, h, k are (commutative) fields. It should be noted that our arguments are not valid in Theorems 2.1 and 2.2 if the field characteristic (in domain) is equal to 2 or 3.

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