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# SYMMETRIES, NOETHER'S THEOREM, CONSERVATION LAWS AND NUMERICAL SIMULATION FOR SPACE-SPACE-FRACTIONAL GENERALIZED POISSON EQUATION

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Abstract. In the present paper Lie theory of differential equations is expanded for finding symmetry geometric vector fields of Poisson equation. Similarity variables extracted from symmetries are applied in order to find reduced forms of the considered equation by using Erdélyi-Kober operator. Conservation laws of the space-space-fractional generalized Poisson equation with the Riemann-Liouville derivative are investigated via Noether's method. By means of the concept of non-linear self-adjointness, Noether's operators, formal Lagrangians and conserved vectors are computed. A collocation technique is also applied to give a numerical simulation of the problem.

## 1. Introduction

Theory of fractional order differential equations (FDEs) due to the non-local property of fractional derivatives are used to describe many phenomena and various fields of physics and other sciences for example fluid mechanics, physics, chemistry, biology, engineering, control, signal and image processing, dynamic systems, biology, environmental science, materials, economic, etc. Also the use of fractional differentiation for the mathematical modeling of real world has been widespread at the recent years. Sun et al. are given a comprehensive package for the tangible examples of fractional calculus in the nature [24].

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Notwithstanding several definitions for fractional calculus, such as Sonin–Letnikov derivative, Liouville derivative, Caputo derivative, Hadamard derivative, Riesz–Miller derivative, Che–Machado derivative, Caputo–Katugampola derivative, Hilfer–Katugampola derivative, Pichaghchi derivative, etc., have been presented by different researchers [5,15,19,25]. Also Lin et al. established another important integral identity for once differentiable function involving Riemann–Liouville fractional integrals which will be used to derive some new Riemann–Liouville fractional Hermite–Hadamard inequalities via $r$-convex function and geometric-arithmetically $s$-convex function respectively [13].

The concept of symmetry of FDEs is similar to symmetry of PDEs. Symmetries of a fractional equation are transformations that map any solution to another solution of the equation but unlike PDEs that are considered in many references, symmetries of time-fractional differential equations (TFDEs) have been investigated somewhat and symmetries of space-time-fractional differential equations (STFDE) and space-space-fractional differential equations (SSFDE) have been attended seldom [8–10, 17, 23].

One of the applications of symmetries is to calculate conservations laws of a given system [12, 20]. Conservation laws are fundamental laws of science especially physics that keep a certain quantity that are not variable in time during of processes and they can be used to reduce dimension of equations. To calculate conservation laws by symmetries we should utilize Noether's theorem, Euler-Lagrange operator and formal Lagrangian [1, 7, 11, 14, 16].

In mathematics, Poisson's equation is a PDE of elliptic type with broad utility in mechanical engineering and theoretical physics. It arises, for instance, to describe the potential field caused by a given charge or mass density distribution; with the potential field known, one can then calculate gravitational or electrostatic field. It is a generalization of Laplace's equation, which is also frequently seen in physics. The equation is named after the French mathematician, geometer, and physicist Simeon Denis Poisson.

In this paper the generalized fractional order of the Poisson equation, the space-space-fractional equation of the form,

$$\mathcal{D}_x^{\beta}(u) + \mathcal{D}_y^{\alpha}(u) = F(u) \tag{1.1}$$

is considered where $1 < \alpha, \beta < 1$ [22].

If $\alpha = \beta = 2$, (1.1) becomes to the second order elliptic PDE $u_{xx}+u_{yy} = F(u)$ which is named Poisson equation. The generalized form of (1.1) by considering $1 < \alpha, \beta < 2$ can be written as

$$\mathcal{D}_x^{\beta}(u) + \mathcal{D}_y^{\alpha}(u) = F(u, u_x). \tag{1.2}$$

(1.2) has some special cases. For example if $y$ replace by $t$ and $u_{xx}$ replaces by $-u_{xx}$ it converts to Klein-Gordon equation. Also, if we replace $F(u)$ by $\sin u$ the equation reduces to sine-Gordon equation [22].

The rest of the paper is organized as follow. In Section 2, the method of finding symmetry operators is introduced and this section is concluded by reduction of the equation via the similarity variables obtaining from symmetries. Section 3 establishes the Noether's operators and the associated conservation laws for the fractional Poisson equation. Finally, in section 4 a collocation method based on Jacobi polynomials is suggested to obtain the numerical solution of the problem. It in noteworthy that Pintarelli et al. give a useful method for finding the numerical solutions of the both linear and non-linear Poisson equation [18].

## 2. Symmetry Analysis of Space-Time-Fractional Poisson Equation

This section is devoted to Lie group analysis of the (1.2).

**2.1. Lie symmetry method for** (1.2)**.** Consider an SSFDE of the form

$$\mathcal{M} = \left\{\partial_y^\alpha u + \partial_x^\beta u - F(x, y, u, u_x, u_{xx}, \dots) = 0\right\}, \tag{2.1}$$

for an arbitrary function $F(x, y, u, u_x, u_{xx}, \dots)$. To obtain symmetries of (2.1) two different cases are considered.

Case 1. Let us substitute $F(x, y, u, u_x, u_{xx}, \dots) = -uu_x$ into (1.2). Then (2.2) is obtained:

$$\Delta := \mathcal{D}_y^\alpha(u) + \mathcal{D}_x^\beta(u) + uu_x = 0. \tag{2.2}$$

First, we should investigate one-parameter Lie group of infinitesimal transformation with a small group parameter $\varepsilon \ll 1$ such as:

$$x \mapsto x + \varepsilon\xi(x, y, u), \quad y \mapsto y + \varepsilon\rho(x, y, u), \quad u \mapsto u + \varepsilon\eta(x, y, u). \tag{2.3}$$

The transformation (2.3) takes

$$X = \xi\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} \tag{2.4}$$

as an associated infinitesimal generator. If $\Delta$ admitted $X$ as a symmetry, the invariance condition implies that

$$\mathrm{Pr}^{(\alpha,\beta,1)}X(\Delta)|_{\Delta=0} = 0, \tag{2.5}$$

where $\mathrm{Pr}^{(\alpha,\beta,1)}X$ denotes the prolongation of (2.4). The extended form of Eq. (2.5) yields the following appearance:

$$\mathrm{Pr}^{(\alpha,\beta,1)}X = \xi\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta_y^\alpha\frac{\partial}{\partial u_y^\alpha} + \eta_x^\beta\frac{\partial}{\partial u_x^\beta} + \eta^x\frac{\partial}{\partial u_x},$$

which determines the point symmetries of (2.2). Expanding the invariance condition (2.5) yields:

$$\mathrm{Pr}^{(\alpha,\beta,1)}X(F)|_{(F=0)} = \eta^{(\alpha,y)} + \eta^{(\beta,x)} + \eta^x u + \eta u_x = 0, \tag{2.6}$$

where

$$\begin{aligned}
\eta^{(\beta,x)} &= \mathfrak{D}_x^{\beta}(\eta) + \rho\mathfrak{D}_x^{\beta}(u_y) - \mathfrak{D}_x^{\beta}(\rho u_y) + \mathfrak{D}_x^{\beta}(uD_x(\xi)) - \mathfrak{D}_x^{\beta+1}(\xi u) + \xi\mathfrak{D}_x^{\beta+1}(u),\\
\eta^{(\alpha,y)} &= \mathfrak{D}_y^{\alpha}(\eta) + \xi\mathfrak{D}_y^{\alpha}(u_x) - \mathfrak{D}_y^{\alpha}(\xi u_x) + \mathfrak{D}_y^{\alpha}(uD_y(\rho)) - \mathfrak{D}_y^{\alpha+1}(\rho u) + \rho\mathfrak{D}_y^{\alpha+1}(u),\\
\eta^{x} &= D_x(\eta) - u_y D_x(\rho) - u_x D_x(\xi),
\end{aligned} \tag{2.7}$$

are the prolongation's coefficients.

The operators $D_y$ and $D_x$ mention the total derivatives dependent to $y$ and $x$, respectively. Also operator $\mathfrak{D}_x^{\beta}, \mathfrak{D}_y^{\alpha}$ are the total space fractional derivative with respect to $x, y$. These operators are not similar to ordinary operators. Because there are differents between PDEs and FDEs for using Leibniz rule, non-commutation and Laplace transform, see [21] for more details. By inserting (2.7) to (2.6), the following solution is obtained:

$$\xi = -C_2\alpha x + C_1, \quad \rho = -C_2\beta y + C_3, \quad \eta = C_2\beta\alpha u - C_2\alpha u,$$

where $C_i$ $(i = 1, 2, 3)$ are arbitrary constants.

Because of the maintaining the structure of SSFDE in the lower limit of the integral in Riemann-Liouville derivative respect to $x$ or $y$, we should work under the assumption that $\rho(x, y, u)\Big|_{y=0} = 0$ and $\xi(x, y, u)\Big|_{x=0} = 0$. According to Lie symmetry theory we have the following Lie algebra for (2.2), with arbitrary $\alpha, \beta \in (1, 2)$,

$$X_1 = -\alpha x\frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} + (u\beta\alpha - u\alpha)\frac{\partial}{\partial u}. \tag{2.8}$$

The condition $\rho(x, y, u)\Big|_{y=0} = \xi(x, y, u)\Big|_{x=0} = 0$ does not involve any assumption about PDEs then $X_2 = \frac{\partial}{\partial x}$ and $X_3 = \frac{\partial}{\partial y}$ are more symmetries for (2.2) by $\alpha = \beta = 2$.

Case 2. Similarly if $F(x, y, u, u_x, u_{xx}, \dots) = u$, with $\alpha, \beta \in (1, 2)$, the equation

$$F = \mathfrak{D}_y^{\alpha}(u) + \mathfrak{D}_x^{\beta}(u) - u = 0,$$

has the following symmetries:

$$X_1 = 2\alpha x\frac{\partial}{\partial x} + 2\beta y\frac{\partial}{\partial y} + \alpha(\beta - 1)u\frac{\partial}{\partial u}, \quad X_2 = u\frac{\partial}{\partial u}.$$

2.2. **Reductions.** In this section we will acquire reduction of (2.2) by the obtained symmetries of section 2 and then we will obtain a space-fractional order differential equation (SFODE). For Case 1, pursuant to the infinitesimals (2.8), we can write the similarity variables those are found by solving the corresponding characteristic equations in the form

$$\frac{dy}{-\beta y} = \frac{du}{\alpha(\beta - 1)u} = \frac{dx}{-\alpha x}.$$

Solving the above differential equation, one can get the similarity variables $xy^{-\frac{\alpha}{\beta}}$ and $uy^{\frac{\alpha(\beta-1)}{\beta}}$. As regards we arrive an answer that has the form $u(x, y) = y^{-\frac{\alpha(\beta-1)}{\beta}}H(z)$

where the function $z$ is given by $xy^{-\frac{\alpha}{\beta}}$. Then we can write left-hand-sided of the Riemann-Liouville fractional derivative $\mathcal{D}_y^{\alpha}(u)$ in the form

(2.9)

$$\mathcal{D}_y^{\alpha}u(x,y) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial y^n}\int_0^y \frac{H\left(xs^{-\frac{\alpha}{\beta}}\right)s^{-\frac{\alpha(\beta-1)}{\beta}}ds}{(y-s)^{\alpha-n+1}}, \quad n-1<\alpha<n,\, n=2,3,\ldots$$

The assumptions $\omega=\frac{y}{s}$ ($s=\frac{y}{\omega}$ and $ds=\frac{-y}{\omega^2}d\omega$) is considered in the sequel. Then (2.9) can be written as:

$$\mathcal{D}_y^{\alpha}u(y,x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial y^n}\int_1^{\infty}\frac{y^{n-\alpha-\frac{\alpha(\beta-1)}{\beta}}}{\omega^{n-\alpha-\frac{\alpha(\beta-1)}{\beta}+1}}(\omega-1)^{n-\alpha-1}H(z\omega^{\frac{\alpha}{\beta}})d\omega.$$

Thus, we obtain

$$\mathcal{D}_y^{\alpha}u(y,x) = \frac{\partial^n}{\partial y^n}\left[y^{n-\alpha-\frac{\alpha(\beta-1)}{\beta}}\left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1-\frac{\alpha(\beta-1)}{\beta},n-\alpha}H\right)(z)\right], \tag{2.10}$$

where

$$\left(K_{\delta}^{\zeta,\alpha}g\right)(z) = \begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\zeta+\alpha)}g(zu^{\frac{1}{\delta}})du, & \alpha>0,\\ g(z), & \alpha=0,\end{cases}$$

$$n = \begin{cases}[n]+1, & \alpha\notin\mathbb{N},\\ \alpha, & \alpha\in\mathbb{N}.\end{cases}$$

By inserting $z=xy^{-\frac{\alpha}{\beta}}$ into (2.10) and by considerinmg $y\frac{d}{dy}\Psi(z)=-\frac{\alpha}{\beta}z\frac{d}{dz}\Psi(z)$, it concludes that

$$\mathcal{D}_y^{\alpha}u(y,x) = y^{-\alpha-\frac{\alpha(\beta-1)}{\beta}}\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha-\frac{\alpha(\beta-1)}{\beta},\alpha}H\right)(z),$$

where $\mathcal{P}_{\beta}^{\zeta,\alpha}$ is the left-hand-sided of Erdélyi-Kober fractional differential operator [5].

The left-hand-side of the Riemann-Liouville fractional derivative $\mathcal{D}_x^{\beta}(u)$ can be written the same as the $\mathcal{D}_y^{\alpha}(u)$, but with a few differences. First of all, we will need the assumptions $\omega=\frac{x}{s}$,$x=\omega s$ and $ds=\frac{-x}{\omega^2}d\omega$. So we get

$$\mathcal{D}_x^{\beta}u(x,y) = y^{-\frac{\alpha(\beta-1)}{\beta}}x^{-\beta}\left(\mathcal{P}_{-1}^{1-\beta,\beta}H\right)(z).$$

Thus, (2.2) can be reduced to the following FPDE where it is written in terms of Erdelyi-Kober fractional differential operator,

$$\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha-\frac{\alpha(\beta-1)}{\beta},\alpha}H\right)(z) + z^{-\beta}\left(\mathcal{P}_{-1}^{1-\beta,\beta}H\right)(z) = -H(z)H'(z)y^{\frac{\alpha(\beta-1)}{\beta}}.$$

## 3. Conservation Laws

In this section conservation laws are computed via the modified version of Noether's theorem based on non-linear self-adjointness concept [9].

3.1. **Non-linear self-adjointness of the fractional Poisson equation.** There are two methods during the conservation laws calculations, first method is stablished by a usual Lagrangian and the second method is constructed by a formal Lagrangian.

The formal Lagrangian for the (1.2) can be written as follows:

$$\mathcal{L} = vF(x, y, u, \mathcal{D}_y^\alpha u, \mathcal{D}_x^\beta u, u_x) = v\mathcal{D}_y^\alpha(u) + v\mathcal{D}_x^\beta(u) + vuu_x, \quad v = v(x, y),$$

where $v$ is new dependent variable. Indeed, by multiplying a new dependent variable in the equation that is equaled to zero we can find the formal Lagrangian equation.

The Euler-Lagrange operator with respect to $u$ for a finite space interval $x \in [x, 0]$ and $y \in [0, y]$ is defined by:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (\mathcal{D}_y^\alpha)^* \frac{\partial}{\partial \mathcal{D}_y^\alpha u} + (\mathcal{D}_x^\beta)^* \frac{\partial}{\partial \mathcal{D}_x^\alpha u} + \sum_{m=1}^{\infty} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1,\ldots,i_m}},$$

where $(\mathcal{D}_j^\mu)^*$ is adjoint operator for Riemman-Liouville derivative $(\mathcal{D}_j^\mu)$ such that $j = x, y$ and $\mu = \alpha, \beta$, that is defined by (see [14]):

$$({}_0\mathcal{D}_x^\beta)^* = (-1)^n {}_xJ_X^{n-\beta}(\mathcal{D}_x^n) \equiv {}_x^C\mathcal{D}_X^\beta, \tag{3.1}$$

where ${}_xJ_X^{n-\beta}$ is the right-sided fractional integral in Riemann-Liouville derivative, and ${}_x\mathcal{D}_X^\beta$ and ${}_x^C\mathcal{D}_X^\beta$ are the right-sided Riemann-Liouville and Caputo fractional derivative of order $\beta$. By applying operator $\frac{\delta}{\delta u}$ and formal Lagrangian we obtain the adjoint equation for (2.2) as

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = (\mathcal{D}_y^\alpha)^* v + (\mathcal{D}_x^\beta)^* v - v_x u. \tag{3.2}$$

The (2.2) is non-linearly self-adjoint if the adjoint (3.2) holds for all solution $u$ of the initial (2.2) upon the substitution $v = \Phi(x, y, u)$, where $v = \Phi(x, y, u)$ satisfies the condition $\Phi(x, y, u) \neq 0$. It means that the following equation holds:

$$F^*\Big|_{v=\Phi(x,y,u)} = \lambda F, \tag{3.3}$$

where the coefficients $\lambda$ is indefinite function, which is obtained during calculations.

For the first position we consider $v$ in general term $v = \Phi(x, y, u)$ and its necessary derivative is $v_x = \Phi_x + \Phi_u u_x$. By inserting elements $v$ and $v_x$, we shall write the expression (3.3) as

$$-\Phi_x u - \Phi_u u_x u + (\mathcal{D}_y^\alpha)^* v + (\mathcal{D}_x^\beta)^* v = \lambda((\mathcal{D}_y^\alpha u) + (\mathcal{D}_x^\beta u) + u_x u). \tag{3.4}$$

Then expansion of (3.4) and comparing of the coefficients for $1, u_x$ one can verify that $\lambda = \Phi_u$ and $\Phi_x = 0$. Hence, (2.2) is non-linearly self-adjoint by considering $v = a_1$, where $a_1$ is a constant.

For the second position, without loss of generality we can certainly assume $v = \varphi(y)\chi(x)$ then $v_x = \varphi(y)\chi'(x)$. Inserting $v$ and $v_x$ into (3.3) the (2.4) yields

$$-\varphi\chi' + (\mathcal{D}_y^\alpha)^* \varphi\chi + (\mathcal{D}_x^\beta)^* \varphi\chi = \lambda((\mathcal{D}_y^\alpha u) + (\mathcal{D}_x^\beta u) + u_x u)\Big|_{(2.2))} = 0.$$

According to (3.1), the above expression is written as:

$$-\varphi\chi' + \chi(x)^{c}_{y}\mathcal{D}^{\alpha}_{Y}(X)\Big(\varphi(y)\Big) + \varphi(y)^{c}_{x}\mathcal{D}^{\beta}_{X}\Big(\chi(x)\Big) = 0. \tag{3.5}$$

The expanded form of (3.5) concludes that $\chi(x) = a_2$ and $\varphi(y) = y$.

3.2. **Basic definitions for constructing conservation laws.** The main difficulty in carrying out conservation laws is that we can't define usual Lagrangian for many equations. The generalized Poisson (2.2) hasn't got the usual Lagrangian. Accordingly under the above results, Noether's operators for the Riemann-Liouville based on the formal Lagrangian are given by

$$C^y = \mathcal{N}^y = \sum_{k=0}^{n-1}(-1)^k\mathcal{D}_y^{\alpha-1-k}(W)\mathcal{D}_y^k\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_y^{\alpha}u)}\right) \times (-1)^n J\left(W, \mathcal{D}_y^n\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_y^{\alpha}u)}\right)\right), \tag{3.6}$$

$$C^x = \mathcal{N}^x = \sum_{k=0}^{m-1}(-1)^k\mathcal{D}_x^{\beta-1-k}(W)\mathcal{D}_x^k\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_x^{\beta}u)}\right) \times (-1)^m J_1\left(W, \mathcal{D}_x^m\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_x^{\beta}u)}\right)\right), \tag{3.7}$$

where $J$ and $J_1$ are defined as:

$$J(f,g) = \frac{\displaystyle\int_0^y\int_y^Y \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}}d\mu d\tau}{\Gamma(n-\alpha)},$$

$$J_1(f,g) = \frac{\displaystyle\int_0^x\int_x^X \frac{f(\tau,y)g(\mu,y)}{(\mu-\tau)^{\beta+1-m}}d\mu d\tau}{\Gamma(m-\beta)},$$

and $W$ is the characteristic of Lie's symmetry generator defined by [4, 16]:

$$W = \eta - \xi u_x - \rho u_y.$$

The formal Lagrangian for (2.2) after substitution acceptable $v = a_1$ is defined as $\mathcal{L} = a_1\Big(\mathcal{D}_y^{\alpha}(u) + \mathcal{D}_x^{\beta}(u) + uu_x\Big)$. In this case, using (3.6), (3.7) and considering $W = \alpha(\beta-1)u + \alpha x u_x + \beta y u_y$, one can get components of conserved vectors:

$$\begin{aligned} C^y &= a_1\alpha(\beta-1)\mathcal{D}_y^{\alpha-1}(u) + a_1\alpha x\mathcal{D}_y^{\alpha-1}(u_x) + a_1\beta\mathcal{D}_y^{\alpha-1}(yu_y),\\ C^x &= a_1\alpha(\beta-1)\mathcal{D}_x^{\beta-1}(u) + a_1\alpha\mathcal{D}_x^{\beta-1}(xu_x) + a_1\beta y\mathcal{D}_x^{\beta-1}(u_y). \end{aligned}$$

The formal Lagrangian for (2.2) after replacement $v = a_2 y$ is given by $\mathcal{L} = a_2 y\Big(\mathcal{D}_y^{\alpha}(u) + \mathcal{D}_x^{\beta}(u) + uu_x\Big)$ and then we can formulate main results as following:

$$\begin{aligned} C^y =& a_2\alpha(\beta-1)y\mathcal{D}_y^{\alpha-1}(u) + a_2\alpha yx\mathcal{D}_y^{\alpha-1}(u_x) + a_2\beta y\mathcal{D}_y^{\alpha-1}(yu_y)\\ &- a_2\alpha(\beta-1)\mathcal{D}_y^{\alpha-2}(u) - a_2\alpha x\mathcal{D}_y^{\alpha-2}(u_x) - a_2\beta\mathcal{D}_y^{\alpha-2}(yu_y),\\ C^x =& a_2\alpha(\beta-1)\mathcal{D}_x^{\beta-1}(u) + a_2\alpha y\mathcal{D}_x^{\beta-1}(xu_x) + a_2\beta y^2\mathcal{D}_x^{\beta-1}(u_y). \end{aligned}$$

## 4. A Numerical Simulation

In this section, we will propose a numerical solution for the equation

$$D_x^\beta(u) + D_y^\alpha(u) = f(u, u_x). \tag{4.1}$$

To do this, we consider a specific example with $f(u, u_x) = u(x, y)$ and the boundary conditions as follows

$$u(x, 0) = \frac{\sin x}{100}, \quad x \in [0, 1], \tag{4.2}$$

$$u(0, y) = \frac{\sin y}{100}, \quad y \in [0, 1]. \tag{4.3}$$

Since that is not the focus of this work, we briefly describe an efficient numerical method to perform it here. It should be noted that in some research works [2, 3, 6], the authors used an operational matrix of fractional differentiation to solve the problems of this type. However, in this part, we use a collocation method based on Jacobi polynomials and compute them and their fractional derivative by some suitable commands in `MAPLE` software. The well-known Jacobi polynomials are defined on the interval $[-1, 1]$ and can be generated with the aid of the following recurrence formula [2]

$$\begin{aligned} J_i^{(a,b)}(t) = &\frac{(a+b+2i-1)\{a^2-b^2+t(a+b+2i)(a+b+2i-2)\}}{2i(a+b+i)(a+b+2i-2)} J_{i-1}^{(a,b)} \\ &- \frac{(a+i-1)(b+i-1)(a+b+2i)}{i(a+b+i)(a+b+2i-2)} J_{i-2}^{(a,b)}, \quad i = 2, 3, \dots, \end{aligned}$$

where

$$J_0^{(a,b)}(t) = 1 \quad \text{and} \quad J_1^{(a,b)}(t) = \frac{a+b+2}{2} t + \frac{a-b}{2}.$$

In order to use these polynomials on the interval $[0, 1]$ we defined the so-called shifted Jacobi polynomials by introducing the change of variable $t = 2x-1$. Let the shifted Jacobi polynomials $J_{1,i}^{(a,b)}(2x-1)$ be denoted by $J_{1,i}^{(a,b)}(x)$. Then $J_{1,i}^{(a,b)}(x)$ can be generated from

$$\begin{aligned} J_{1,i}^{(a,b)}(x) = &\frac{(a+b+2i-1)\{a^2-b^2+(2x-1)(a+b+2i)(a+b+2i-2)\}}{2i(a+b+i)(a+b+2i-2)} J_{1,i-1}^{(a,b)} \\ &- \frac{(a+i-1)(b+i-1)(a+b+2i)}{i(a+b+i)(a+b+2i-2)} J_{1,i-2}^{(a,b)}, \quad i = 2, 3, \dots, \end{aligned}$$

where

$$J_{1,0}^{(a,b)}(x) = 1 \quad \text{and} \quad J_{1,1}^{(a,b)}(x) = \frac{a+b+2}{2}(2x-1) + \frac{a-b}{2}.$$

The analytic form of the shifted Jacobi polynomials $J_{1,i}^{(a,b)}(x)$ of degree $i$ is given by

$$J_{1,i}^{(a,b)}(x) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(i+b+1)\Gamma(i+k+a+b+1)}{\Gamma(k+b+1+1)\Gamma(i+a+b+1)(i-k)!k!} x^k,$$

where
$$J_{1,i}^{(a,b)}(0) = (-1)^i \frac{\Gamma(i+b+1)}{\Gamma(b+1)i!} \quad \text{and} \quad J_{1,i}^{(a,b)}(1) = \frac{\Gamma(i+a+1)}{\Gamma(a+1)i!}.$$
The choice $a = b = 0$ yields the Legendre polynomials, while choosing $a = b = -\frac{1}{2}$ gives Chebyshev polynomials. We now assume that, the solutions of (4.1)–(4.3) can be approximated by the shifted Jacobi polynomials as follows
$$u(x,y) \simeq u_N(x,y) = \sum_{m=0}^{N}\sum_{n=0}^{N} u_{mn} J_{1,m}^{(a,b)}(x) J_{1,n}^{(a,b)}(y),$$
where $u_{mn}$, $m = 0,1,\dots,N$, and $n = 0,1,\dots,N$, are unknown coefficients to be determined and $J_{1,m}^{(a,b)}(x)$ and $J_{1,n}^{(a,b)}(y)$ are the shifted Jacobi polynomials. In our numerical simulation we consider $a = b = 0$.

The shifted Jacobi polynomials $J_{1,i}^{(a,b)}(x)$ can be written in the `MAPLE` software in the form
$$J_{1,i}^{(a,b)}(x) = \mathrm{Jacobi}P(i,a,b,2x-1).$$
The fractional integral of order $\alpha > 0$ of function $f$ can also be determined using the following command
$${}_0I_x^{\alpha} f(x) = \mathrm{fracdiff}(f(x), x, -\alpha).$$
Then the Rieman–Liouville fractional derivative of order $1 \le \alpha \le 2$ of function $f$ is normally written as
$$D_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^2 I_x^{2-\alpha} f(x) = \mathrm{diff}(\mathrm{fracdiff}(f(x), x, \alpha-2), x, x).$$
Since there exist $(N+1)^2$ unknown coefficients $u_{mn}$, $m = 0,1,\dots,N$, and $n = 0,1,\dots,N$, we should construct system of $(N+1)^2$ algebraic equations. For this reason, we define the well-known Chebyshev Gauss Lobatto (CGL) collocation points as
$$\eta_i = \frac{1}{2}\left(1 - \cos\left(\frac{(i-1)\pi}{N}\right)\right), \quad 1 \le i \le N+1,$$
$$\tau_j = \frac{1}{2}\left(1 - \cos\left(\frac{(j-1)\pi}{N}\right)\right), \quad 1 \le j \le N+1.$$
We discretize now equation (4.1) using the CGL points as
$$D_x^{\beta} u_N(\eta_i,\tau_j) + D_y^{\alpha} u_N(\eta_i,\tau_j) = u_N(\eta_i,\tau_j), \quad 2 \le i \le N+1,\ 2 \le j \le N+1,$$
and the boundary conditions (4.2)–(4.3) as follows
$$u_N(\eta_i, 0) = \frac{\sin\eta_i}{100}, \quad 1 \le i \le N+1,$$
$$u_N(0, \tau_j) = \frac{\sin\tau_j}{100}, \quad 2 \le j \le N+1.$$
In this case, the considered equations are collocated and then transformed into the associated systems of $(N+1)^2$ algebraic equations and $(N+1)^2$ unknowns which can

be solved through an iterative method in Maple software by *fsolve* command. We can also check the accuracy of our proposed numerical approach. To do this, we replace the $u_N(x,y), D_x^\beta u_N(x,y)$ and $D_y^\alpha u_N(x,y)$ in equations (4.1)–(4.3). Then (4.1)–(4.3) can be satisfied approximately. In other words we define the absolute error as

$$E = \left| D_x^\beta u_N(x,y) + D_y^\alpha u_N(x,y) - u_N(x,y) \right| \simeq 0.$$

Diagrams of the solutions of the system (4.1)–(4.3) using the suggested numerical method are shown in Figure 1. Figure 2, gives some numerical results obtained by this method for $u(x,y)$. The absolute error $E$ is also depicted in Figure 3.

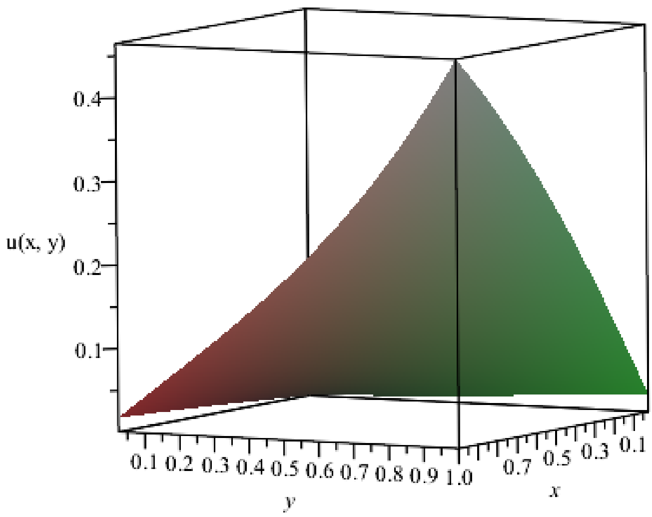


FIGURE 1. Numerical solution $u_N(x,y)$ for $\beta = 1.9$ and $\alpha = 2$ with $N = 5$.

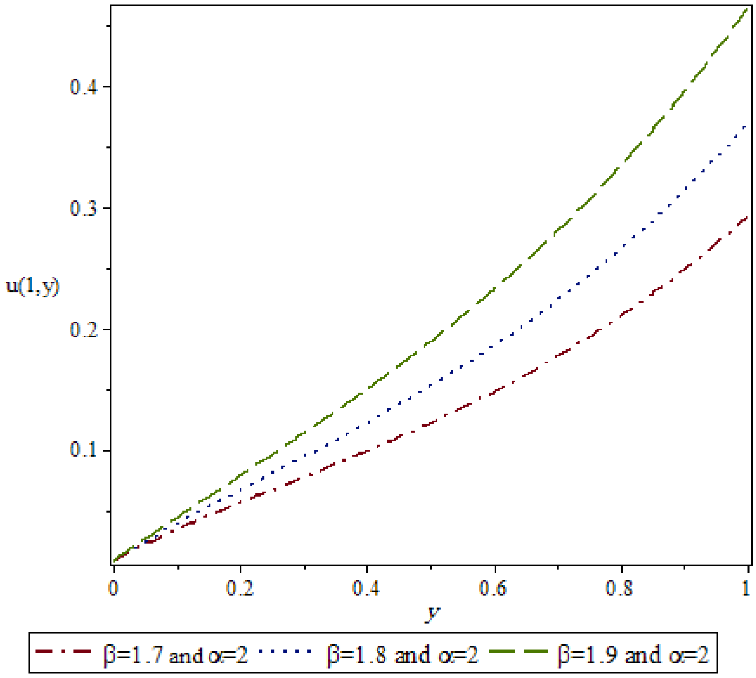


FIGURE 2. Numerical solutions of the $u_N(x, y)$ for $\beta = 1.9, 1.8, 1.7$ and $\alpha = 2$ with $x = 1$ and $N = 5$.

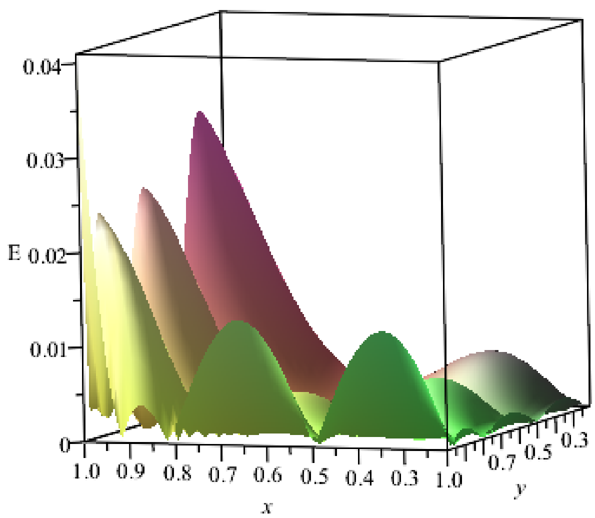


FIGURE 3. Absolute error $E$ of the presented method for $N = 5$.

TABLE 1. Some numerical results of $u_N(x, y)$ with $N = 5$ and different values of $\alpha$ and $\beta$.

| $(x, y)$ | $\alpha = 2,\ \beta = 1.9$ | $\alpha = 2,\ \beta = 1.8$ | $\alpha = 2,\ \beta = 1.7$ |
|---|---|---|---|
| (1, 0.1) | 0.04400193 | 0.03886688 | 0.03410741 |
| (1, 0.2) | 0.07856733 | 0.06686172 | 0.05642837 |
| (1, 0.3) | 0.11351537 | 0.09422739 | 0.07751767 |
| (1, 0.4) | 0.15008387 | 0.12244699 | 0.09901679 |
| (1, 0.5) | 0.18939548 | 0.15273855 | 0.12217625 |
| (1, 0.6) | 0.23250915 | 0.18613527 | 0.14796318 |
| (1, 0.7) | 0.28047157 | 0.22356557 | 0.17716893 |
| (1, 0.8) | 0.33436854 | 0.26593330 | 0.21051668 |
| (1, 0.9) | 0.39537657 | 0.31419779 | 0.24876897 |
| (1, 1) | 0.46481412 | 0.36945403 | 0.29283537 |

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