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INTEGRAL OPERATORS THAT DEFINE THE SOLUTION OF HIGHER-ORDER EQUATIONS

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Abstract. Integral operators that transform arbitrary analytic functions into regular solutions of equations in partial derivatives of the elliptic type of the second and higher orders are constructed. The application of the constructed operators is illustrated by the solutions of the Cauchy problem, the special Cauchy problem, and the Riquier problem for the axisymmetric Helmholtz fourth order equation. The transitions to equations of parabolic and hyperbolic types of higher orders are proposed. An integral representation of the solution of these equations is obtained.

1. Introduction

Elliptic-type differential equations of mathematical physics play one of the central roles in mathematical modeling of various processes in physics and technology. Established processes of physical nature are described by equations of the elliptic type of the fourth and higher orders. Differential equations in partial derivatives containing differential operators of the form

$$L_{\mu,k,s} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\mu}{x} \cdot \frac{\partial}{\partial x} + \frac{k}{y} \cdot \frac{\partial}{\partial y} + S$$

and their iterations are widely used in modeling liquid and gas diffusion processes, as well as biological and environmental phenomena. These equations are found in problems of statics of the theory of elasticity, in problems of the theory of combustion, theory of filtration, in spectrography problems.

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The methods of solving such equations are the creation of integral and differential operators that determine the solutions of equations and systems of elliptic and hyperbolic types [1–4, 10].

In the theory of general complex representations of solutions of elliptic equations, is important the discovery made by I. N. Vekua about the possibility of an equivalent reduction of any boundary value problem for the equation

$$\Delta^n U + \sum_{k=1}^{n} \left(\Delta^{n-k} U\right) = 0$$

to the corresponding boundary value problem for the system of analytical functions [9].

The subject of research in this paper is the construction of integral operators that translate arbitrary analytic functions into regular solutions of differential equations (in partial derivatives) of higher orders ($n \geq 2$), that is, a method of finding solutions of the corresponding iterated differential equations in analytical form has been developed.

The essence of the method is to obtain an integral representation, which is the solution of the Cauchy problem.

As is known [6, 7], there exists and only one solution to the Cauchy problem for an equation of elliptic type with analytical coefficients.

The Riemann integral operator is constructed on the basis of the biaxially symmetric Helmholtz equation [2].

As an example of the application of the constructed operators, the Riquier problem has been solved.

2. Main Results

Let G be an arbitrary stellar region relative to $z = 0$, $z^* \in G^* = \{x - iy \mid x + iy \in G\}$, τ - real variable (or complex), $\tau \in T$.

Let us consider a differential equation of the form

$$(L_{\mu,k,S}) \qquad L_{\mu,k,S}\Phi = \Phi_{xx} + \Phi_{yy} + \frac{2\mu}{x}\Phi_x + \frac{k}{y}\Phi_y + S\Phi = 0, \quad \mu, k > 0,$$

where $\Phi = \Phi(x, y, \tau)$, S is the linear operator depends only on τ, $\tau \in T$.

The integral representation of the solutions of the equation $(L_{\mu,k,S})$, as established in the work [2], has the form

$$\begin{aligned}\Phi(x,y,\tau) = & \frac{x^{-\mu}}{B\left(\frac{k}{2},\frac{1}{2}\right)} \int_0^{\pi} \omega(x + iy\cos t, \tau)(x + iy\cos t)^{\mu} \sin^{k-1} t \\ & \otimes \sum_2 \left(\mu, 1-\mu, \frac{k}{2}, \frac{-y^2 \sin^2 t}{4x(x + iy\ \cos t)}, -\frac{S}{4} y^2 \sin^2 t\right) dt,\end{aligned} \tag{2.1}$$

where $\omega(z, \tau)$ is a function, analytic in G and continuous in $\bar{G}$ function, B is the beta function, $\sum_2$ is a degenerate hypergeometric Gorn function of two variables [5] $\sum_2(\alpha, \beta, \gamma, x, y) = \sum_{m,n=0}^{+\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n$, $|x| < 1$, $(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}$.

At $\mu = 0$ the equation $(L_{\mu,k,S})$ and formula (2.1) will have the following form

$$(L_{k,S}) \qquad L_{k,S}\Phi = \Phi_{xx} + \Phi_{yy} + \frac{k}{y}\Phi_y + S\Phi = 0,$$

$$(2.2) \qquad \Phi(x,y,\tau) = C_k \int_0^{\pi} \omega(x + iy\cos t, \tau)\, {}_0F_1\left[\frac{k}{2}; -\frac{S}{4}y^2 \sin^2 t\right] \sin^{k-1} t\, dt,$$

where $C_k = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{1}{2}\right)}$, ${}_0F_1\left[\frac{k}{2}; z\right]$ is a partial case of the generalized hypergeometric function, which is related to the Bessel function by equation

$${}_0F_1\left[\frac{k}{2}; z\right] = \Gamma\left(\frac{k}{2}\right)\left(i\sqrt{z}\right)^{-\frac{k}{2}+1} J_{\frac{k}{2}-1}\left(2i\sqrt{z}\right).$$

And also

a) with $S = \frac{\alpha^2}{4}$, $k = 1$, the equation $(L_{k,S})$ becomes the axisymmetric Helmholtz equation

$$(2.3) \qquad \Phi_{xx} + \Phi_{yy} + \frac{1}{y}\Phi_y + \frac{\alpha^2}{4}\Phi = 0,$$

and formula (2.2) is the general integral representation of its solutions

$$(2.4) \qquad \Phi(x,y) = -\frac{i}{\pi}\int_{\bar z}^{z} \omega(\sigma) \frac{\cos\left(\frac{\alpha}{2}\sqrt{(z-\sigma)(\bar z - \sigma)}\right)}{\sqrt{(z-\sigma)(\bar z-\sigma)}} d\sigma,$$

where the integration from $\bar z$ to z is carried out along any rectified contour placed in G and satisfies the condition $\Phi(x, 0) = \omega(x)$,

b) with $S = -\left(a + b\frac{\partial}{\partial\tau}\right)$, $a, b = \text{const.}$, $k = 1$, equation $(L_{k,S})$ becomes a parabolic type equation

$$\Phi_{xx} + \Phi_{yy} + \frac{1}{y}\Phi_y - b\Phi_\tau - a\Phi = 0,$$

with a solution

$$(2.5) \qquad \Phi(x,y,\tau) = \frac{1}{2\pi i}\oint_K \frac{1}{\pi}\frac{d\xi}{\xi - \tau}\int_0^{\pi} \Phi_3\left(1; \frac{1}{2}; \frac{b\left(\frac{y}{2}\right)^2 \sin^2 t}{\xi - \tau}, a\left(\frac{y}{2}\right)^2 \sin^2 t\right) \omega(x + iy\cos t, \xi) dt.$$

Here K is a circle in $T_0 \subset T$ with the center in $\xi = \tau$.

c) With $S = -\left(b\frac{\partial}{\partial\tau}\right)^2$, $b \in \mathbb{R}$, $b = \text{const.}$, $k = 1$, equation $(L_{k,S})$ becomes a hyperbolic type equation

$$\Phi_{xx} + \Phi_{yy} + \frac{1}{y}\Phi_y - b^2\Phi_{\tau\tau} = 0,$$

with a solution

$$\Phi(x,y,\tau)=\frac{1}{2\pi i}\oint\limits_K \frac{d\xi}{\xi-\tau}\int\limits_0^\pi \omega(x+iy\cos t,\xi)_1F_1\left(\frac{1}{2},\frac{1}{2},\frac{by^2\sin^2 t}{(\xi-\tau)^2}\right)dt. \tag{2.6}$$

Consider a differential equation

$$L^n_{k,S}\Phi=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{k}{y}\cdot\frac{\partial}{\partial y}+S\right)^n\Phi=0, \tag{$L^n_{k,S}$}$$

where $\Phi=\Phi(x,y,\tau)$ and S - linear operator of $\tau\in T$.

Lemma 2.1. *If $\Phi_r(z,\bar z,\tau)$, $r=\overline{0,n-1}$ are $2(r+1)$ times continuously differentiable solutions of the equation $(L_{k,S})$, then function defined by equality*

$$\Phi(x,y,\tau)=\sum_{r=0}^{n-1}\Phi_r(x,y,\tau)x^r \tag{2.7}$$

satisfies the equation $(L^n_{k,S})$.

Proof. Let us prove it by the method of mathematical induction.

1) Let us show the validity of the statement for $n=2$, that is

$$L^2_{k,S}\Phi=0. \tag{$L^2_{k,S}$}$$

By direct verification, we make sure that the function

$$\Phi(x,y,\tau)=\Phi_0(x,y,\tau)+x\Phi_1(x,y,\tau),$$

where $\Phi_r(x,y,\tau)$, $r=\overline{0,1}$, satisfies equation $(L_{k,S})$, is a solution of the equation $(L^2_{k,S})$

$$L^2_{k,S}\Phi=L_{k,S}(L_{k,S}\Phi)=L_{k,S}\left(L_{k,S}\Phi_0+L_{k,S}\left(x\Phi_1\right)\right)=2\frac{\partial}{\partial x}\left(L_{k,S}\Phi_1\right)=0.$$

2) Let Lemma 2.1 hold for some natural number $n-1$, that is

$$\Phi(x,y,\tau)\sum_{r=0}^{n-2}\Phi_r(x,y,\tau)x^r,$$

where $L_{k,S}\left(\Phi_r(x,y,\tau)\right)=0$, $r=\overline{0,n-2}$, satisfies equation $(L^{n-1}_{k,S})$.

3) Based on the assumption, we will prove the validity of Lemma 2.1 for the following natural number n. We make sure that the function is defined by equality (2.2) and is a solution of the equation $(L_{k,S})$, satisfies equation $(L^n_{k,S})$. We have

$$L^n_{k,S}(\Phi)=L_{k,S}\left(L^{n-1}_{k,S}\Phi\right)=L_{k,S}\left(L^{n-1}_{k,S}\left(\sum_{r=0}^{n-2}\Phi_r x^r+\Phi_{n-1}x^{n-1}\right)\right)=L^n_{k,S}\left(x^{n-1}\Phi_{n-1}\right),$$

because $L^{n-1}_{k,S}\left(\sum_{r=0}^{n-2}\Phi_r x^r\right)=0$ by assumption. So, let us prove that

$$L^n_{k,S}\left(x^{n-1}\Phi_{n-1}\right)=0. \tag{2.8}$$

Equality (2.8) is proved again by the method of mathematical induction.

With $n=1$, $L_{k,S}(\Phi_0)=0$ is true under the condition of Lemma 2.1.

Let $\Phi_{n-1} = \varphi$. Then, $L^n_{k,S}\left(x^{n-1}\varphi\right) = L^{n-1}_{k,S}\left(L_{k,S}\left(x^{n-1}\varphi\right)\right)$.
Let us consider

$$\begin{aligned} L_{k,S}\left(x^{n-1}\varphi\right) =& x^{n-1}\left(\frac{\partial^2\varphi}{\partial y^2} + \frac{k}{y}\cdot\frac{\partial\varphi}{\partial y} + S\varphi\right) + \frac{\partial^2}{\partial x^2}\left(x^{n-1}\varphi\right) \\ =& x^{n-1}L_{k,S}\,\varphi + (n-1)(n-2)x^{n-3}\varphi + 2(n-1)x^{n-2}\frac{\partial\varphi}{\partial x}. \end{aligned}$$

We assume the validity of (2.8) at $r < n$, that is

$$L^r_{k,S}\left(x^{r-1}\varphi\right) = 0, \quad r < n.$$

From here we have $L^n_{k,S}\left(x^{r-1}\varphi\right) = 0$, $r < n$. Actually,

$$L^n_{k,S}\left(x^{r-1}\varphi\right) = L^{n-r}_{k,S}\left(L^r_{k,S}\left(x^{r-1}\varphi\right)\right) = 0.$$

We prove the validity of (2.8) for n.

$$\begin{aligned} L^n_{k,S}\left(x^{n-1}\varphi\right) =& L^{n-1}_{k,S}\left(2(n-1)x^{n-2}\frac{\partial\varphi}{\partial x} + (n-1)(n-2)x^{n-3}\varphi\right) \\ =& 2(n-1)L^{n-1}_{k,S}\left(x^{n-2}\frac{\partial\varphi}{\partial x}\right) + (n-1)(n-2)L^{n-1}_{k,S}\left(x^{n-3}\varphi\right) = 0. \end{aligned}$$

So, Lemma 2.1 is proved. □

Therefore, the following theorem holds.

Theorem 2.1. *For all functions $\omega_r(z,\tau)$, $r = \overline{0, n-1}$, analytic in G and continuous in $\bar{G}$,*

$$\Phi\left(z,\bar{z},\tau\right) = \sum_{r=0}^{n-1} x^r \int_0^{\pi} \omega_r(x + sy\cos t, \tau)\,{}_0F_1\left[\frac{k}{2}; -\frac{S}{4}y^2\sin^2 t\right]\sin^{k-1} t dt \tag{2.9}$$

is a solution of the equation

$$L^n_{k,S}\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{k}{y}\cdot\frac{\partial}{\partial y} + S\right)^n \Phi = 0,$$

for arbitrary $\tau \in T_0$, $T_0 \subset T$, and $z, \bar{z}$ from neighborhood of $z = 0$, $\bar{z} = 0$.

Let us apply integral operators to solving problems of mathematical physics.

1. Let's take $S = \frac{\alpha^2}{4}$, $n = 2$, $k = 1$.

Cauchy problem. In the region $0 < x, y < +\infty$ find a regular solution of the axisymmetric Helmholtz equation of the 4th order

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y}\cdot\frac{\partial}{\partial y} + \frac{\alpha^2}{4}\right)^2 \Phi = 0, \tag{2.10}$$

which satisfies the conditions

$$\left.\frac{\partial^m\Phi}{\partial x^m}\right|_{x=0} = f_m(y),\ m = \overline{0,3}, \quad \left.\frac{\partial\Phi}{\partial y}\right|_{y=0} = 0, \tag{2.11}$$

where $f_m(y)$ is a function given sufficiently many times continuously differentiable.

We look for a solution to the problem in the form (2.7), with $n = 2$

$$\Phi(x, y) = \Phi_0(x, y) + x\Phi_1(x, y).$$

Since, according to Lemma 2.1, Φ_0 and Φ_1 satisfy (2.3), then satisfying the boundary conditions (2.11) for finding $\Phi_0(x, y)$, $\Phi_1(x, y)$ we obtain the following boundary value problems:

$$\frac{\partial^2\Phi_0}{\partial x^2} + \frac{\partial^2\Phi_0}{\partial y^2} + \frac{1}{y}\cdot\frac{\partial\Phi_0}{\partial y} + \frac{\alpha^2}{4}\Phi_0 = 0, \qquad \frac{\partial\Phi_0}{\partial x}\Big|_{x=0} = \varphi(y), \quad \frac{\partial\Phi_0}{\partial y}\Big|_{y=0} = 0, \tag{2.12}$$

$$\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_1}{\partial y^2} + \frac{1}{y}\cdot\frac{\partial\Phi_1}{\partial y} + \frac{\alpha^2}{4}\Phi_1 = 0, \qquad \frac{\partial\Phi_1}{\partial x}\Big|_{x=0} = \psi(y), \quad \frac{\partial\Phi_1}{\partial y}\Big|_{y=0} = 0, \tag{2.13}$$

where (see [5])

$$\varphi(y) = \frac{\pi}{2}\left(N_0\left(\frac{\alpha}{2}y\right)\int_0^y J_0\left(\frac{\alpha}{2}y_1\right)y_1F(y_1)dy_1 - J_0\left(\frac{\alpha}{2}y\right)\int_0^y N_0\left(\frac{\alpha}{2}y_1\right)y_1F(y_1)dy_1\right),$$

($J_0(z), N_0(z)$ are cylindrical functions of the 1st and 2nd kind),

$$F(y) = \frac{1}{2}\left(3f_1''(y) + \frac{1}{y}f_1'(y) + \frac{\alpha^2}{4}f_1(y)\right) + f_3(y),$$

$$\psi(y) = \frac{1}{2}\left(f_2(y) + f_0''(y) + \frac{1}{y}f_0'(y) + \frac{\alpha^2}{4}f_0(y)\right).$$

Let us find a regular solution of problem (2.12) in the region $x > 0$, $y > 0$.

We look for a solution in the form (2.4), where $\omega_0(z) = u_0(x, y) + iv_0(x, y)$ is an analytical function in the given region, $v_0(x, 0) = 0$. Using the inversion formula [8], we obtain

$$\frac{\partial u_0}{\partial x}(0, y) = \frac{\partial}{\partial y}\int_0^y \varphi(\xi)\frac{\operatorname{ch}\left(\frac{\alpha}{2}\sqrt{y^2-\xi^2}\right)}{\sqrt{y^2-\xi^2}}\xi d\xi,$$

or, using that $\frac{\partial u_0}{\partial x}(x, y) = \frac{\partial v_0}{\partial y}(x, y)$, we obtain

$$v_0(0, y) = \int_0^y \varphi(\xi)\frac{\operatorname{ch}\left(\frac{\alpha}{2}\sqrt{y^2-\xi^2}\right)}{\sqrt{y^2-\xi^2}}\xi d\xi.$$

Considering that $v_0(0, -y) = -v_0(0, y)$, and when approaching infinity

$$\left|v_0(0, y)\right| \le \frac{M}{|y|^\varepsilon} \quad (M, \varepsilon > 0),$$

we find $\omega_0(z)$ in form

$$\omega_0(z) = u_0(x,y) + iv_0(x,y) = -\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{v_0(0,t)}{t+iz}dt.$$

Therefore, the solution of the problem (2.12) will be:

$$\Phi_0(x,y) = -\frac{2}{\pi^2}\int\limits_0^y \frac{\cos\left(\frac{\alpha}{2}\sqrt{y^2-\xi^2}\right)}{\sqrt{y^2-\xi^2}}d\xi$$
$$\otimes\int\limits_{-\infty}^{+\infty}\frac{(t-\xi)dt}{(t-\xi)^2+x^2}\int\limits_0^t \varphi(\tau)\frac{\operatorname{ch}\left(\frac{\alpha}{2}\sqrt{t^2-\tau^2}\right)}{\sqrt{t^2-\tau^2}}\tau d\tau.$$

Similarly, we obtain the solution of problem (2.13). Finally, the solution of the Cauchy problem (2.10), (2.11) will be

$$\Phi(x,y) = -\frac{2}{\pi^2}\int\limits_0^y \frac{\cos\left(\frac{\alpha}{2}\sqrt{y^2-\xi^2}\right)}{\sqrt{y^2-\xi^2}}d\xi$$
$$\otimes\int\limits_{-\infty}^{+\infty}\frac{(t-\xi)dt}{(t-\xi)^2+x^2}\int\limits_0^t (\varphi(\tau)+x\psi(\tau))\frac{\operatorname{ch}\left(\frac{\alpha}{2}\sqrt{t^2-\tau^2}\right)}{\sqrt{t^2-\tau^2}}\tau d\tau.$$

Cauchy special problem. Let G be a domain symmetric with respect to segment d of the real axis and let $\omega_m(z)$ (m is even) are arbitrary analytic functions in this domain. Find in the domain G a regular solution of the axisymmetric Helmholtz equation of the 4th order (2.10) that satisfies the conditions

$$\Phi(x,0) = \omega_0(x), \quad \frac{\partial\Phi}{\partial y}\Big|_{y=0} = 0, \quad \frac{\partial^2\Phi}{\partial y^2}\Big|_{y=0} = \omega_2(x), \quad \frac{\partial^3\Phi}{\partial y^3}\Big|_{y=0} = 0. \tag{2.14}$$

We will call the problem (2.10), (2.14) a special Cauchy problem, which is used to establish the general integral representation of all regular solutions of the equation (2.10).

In accordance with Lemma 2.1, we look for a solution to the problem (2.10), (2.14) in the form

$$\Phi(x,y) = \Phi_0(x,y) + x\Phi_1(x,y),$$

where $\Phi_0(x,y)$, $\Phi_1(x,y)$ satisfy (2.3). Checking the conditions (2.14), to find $\Phi_0(x,y)$, $\Phi_1(x,y)$ we will get the following Cauchy problems:

$$\frac{\partial^2\Phi_0}{\partial x^2} + \frac{\partial^2\Phi_0}{\partial y^2} + \frac{1}{y}\cdot\frac{\partial\Phi_0}{\partial y} + \frac{\alpha^2}{4}\Phi_0 = 0,$$
$$\Phi_0\Big|_{y=0} = a(x), \quad \frac{\partial\Phi_0}{\partial y}\Big|_{y=0} = 0,$$
$$\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_1}{\partial y^2} + \frac{1}{y}\cdot\frac{\partial\Phi_1}{\partial y} + \frac{\alpha^2}{4}\Phi_1 = 0,$$

$$\Phi_1\Big|_{y=0} = b(x), \quad \frac{\partial \Phi_1}{\partial y}\Big|_{y=0} = 0,$$

where

$$\begin{aligned} a(x) =& \omega_0(x) - x\int_0^x \left(\omega_2(t) + \frac{1}{2}\omega_0''(t) + \frac{\alpha^2}{8}\omega_0(t)\right) dt, \\ b(x) =& \int_0^x \left(\omega_2(t) + \frac{1}{2}\omega_0''(t) + \frac{\alpha^2}{8}\omega_0(t)\right) dt. \end{aligned}$$

According to the integral representation (2.4), the desired solution of the special Cauchy problem will be

$$\begin{aligned} \Phi(x,y) =& -\frac{i}{\pi}\int_{\bar z}^{z} \omega_0(\sigma)\frac{\cos\left(\frac{\alpha}{2}\sqrt{(z-\sigma)(\bar z-\sigma)}\right)}{\sqrt{(z-\sigma)(\bar z-\sigma)}} d\sigma \\ & -\frac{i}{\pi}\int_{\bar z}^{z} \left(\omega_2(\sigma) + \frac{1}{2}\omega_0''(\sigma) + \frac{\alpha^2}{8}\omega_0(\sigma)\right) \sin\left(\frac{\alpha}{2}\sqrt{(z-\sigma)(\bar z-\sigma)}\right) d\sigma. \end{aligned}$$

Riquier problem. In the region $0 < x, y < +\infty$ find a four-times irreverently differentiable solution of the axisymmetric Helmholtz equation (2.10)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y}\cdot\frac{\partial}{\partial y} + \frac{\alpha^2}{4}\right)^2 \Phi = 0,$$

which satisfies the conditions

$$\Phi(0,y) = a(y), \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y}\cdot\frac{\partial}{\partial y} + \frac{\alpha^2}{4}\right)\Phi\Big|_{x=0} = b(y), \quad \frac{\partial \Phi}{\partial y}\Big|_{y=0} = 0, \tag{2.15}$$

where $a(y)$, $b(y)$ are given sufficiently smooth functions. In accordance with Lemma 2.1, we look for a solution to the problem (2.10), (2.15) in the form $\Phi(x,y) = \Phi_0(x,y) + x\Phi_1(x,y)$.

Since Φ_0 and Φ_1 satisfy the equation (2.3), satisfying the condition (2.15), the problem (2.10), (2.15) reduces to problems relative to Φ_0 and Φ_1 with the conditions:

$$\Phi_0(0,y) = a(y), \quad \frac{\partial \Phi_0}{\partial y}\Big|_{y=0} = 0, \tag{2.16}$$

$$\frac{\partial \Phi_1}{\partial x}\Big|_{x=0} = \frac{1}{2}b(y), \quad \frac{\partial \Phi_1}{\partial y}\Big|_{y=0} = 0. \tag{2.17}$$

We are looking for a solution to the problem (2.3), (2.16) in the form

$$\Phi_0(x,y) = \frac{2}{\pi}\int_0^y u_0(x,\tau)\cos\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)\left(y^2-\tau^2\right)^{-\frac{1}{2}} d\tau,$$

where $u_0(x,y) = \operatorname{Re}\omega_0(z)$. Satisfying the condition (2.16), by the inversion formula [8] we obtain

$$u_0(0,y) = \frac{\partial}{\partial y}\int\limits_0^y a(\tau)\operatorname{ch}\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)\left(y^2-\tau^2\right)^{-\frac{1}{2}}\tau d\tau.$$

Let $u_0(0,-y) = u_0(0,y)$ and when approaching infinity $|u_0(0,y)| \le \frac{M}{|y|^\varepsilon}$, $M, \varepsilon \to 0$. Then, the analytic function $\omega_0(z)$ is determined by equality

$$\omega_0(z) = u_0(x,y) + iv_0(x,y) = -\frac{1}{\pi i}\int\limits_{-\infty}^{+\infty}\frac{u_0(0,t)}{t+iz}dt,$$

and the solution of the problem (2.3), (2.16) will look like

$$\Phi_0(x,y) = \left(\frac{2}{\pi}\right)^2 x\int\limits_0^y \frac{\cos\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}d\tau \tag{2.18}$$
$$\otimes\int\limits_0^{+\infty}\frac{(t^2+\tau^2+x^2)\,dt}{\left(t^2-\tau^2+x^2\right)^2+4x^2\tau^2}\cdot\frac{\partial}{\partial t}\int\limits_0^t a(\xi)\frac{\operatorname{ch}\left(\frac{\alpha}{2}\sqrt{t^2-\xi^2}\right)}{\sqrt{t^2-\xi^2}}\xi d\xi.$$

We are looking for a solution to the problem (2.3), (2.17) in the form

$$\Phi_1(x,y) = \frac{2}{\pi}\int\limits_0^y u_1(x,\tau)\cos\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)\left(y^2-\tau^2\right)^{-\frac{1}{2}}d\tau,$$

$u_1(x,y) = \operatorname{Re}\omega_1(x+iy)$.

By similar reasoning, we obtain the solution to the problem (2.3), (2.17)

$$\Phi_1(x,y) = -\left(\frac{2}{\pi}\right)^2\frac{1}{2}\int\limits_0^y\frac{\cos\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}d\tau \tag{2.19}$$
$$\otimes\int\limits_0^{+\infty}\frac{t\,(t^2-\tau^2+x^2)\,dt}{\left(t^2-\tau^2+x^2\right)^2+4x^2\tau^2}\int\limits_0^t b(\xi)\frac{\operatorname{ch}\left(\frac{\alpha}{2}\sqrt{t^2-\xi^2}\right)}{\sqrt{t^2-\xi^2}}\xi d\xi.$$

Finally, the solution of Riquier problem is

$$\Phi(x,y) = \Phi_0(x,y) + x\Phi_1(x,y),$$

where $\Phi_0(x,y)$ is given by (2.18), and $\Phi_1(x,y)$ by (2.19).

2. When $S = -\left(a + b\frac{\partial}{\partial\tau}\right)$, relying on Lemma 2.1 and using the representation (2.5), we come to the following theorem.

Theorem 2.2. *For all functions $\omega_r(z,\tau)$, $r=\overline{0,n-1}$, holomorphic in G and continuous in $\bar{G}$*

$$\Phi\left(z,\bar{z},\tau\right) = \frac{1}{2\pi i}\sum_{r=0}^{n-1}x^r\oint\limits_K\frac{d\xi}{\xi-\tau}\int\limits_0^\pi\omega_r(x+iy\cos t,\xi)$$

$$\otimes \Phi_3 \left(1, \frac{1}{2}, a\left(\frac{y}{2}\right)^2 \sin^2 t, \frac{b\left(\frac{y}{2}\right)^2 \sin^2 t}{\xi - \tau}\right) dt$$

is a solution of a parabolic equation of the n-th order

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \cdot \frac{\partial}{\partial y} - b\frac{\partial}{\partial \tau} - a\right)^n \Phi = 0,$$

for arbitrary $\tau \in T_0$, *and* $z, \bar{z}$ *from neighborhood of the* $z = 0$, $\bar{z} = 0$.

3. When $S = -\left(b\frac{\partial}{\partial \tau}\right)^2$. Relying on Lemma 2.1 and using representation (2.6), we will have the following theorem.

Theorem 2.3. *For all functions* $\omega_r(z, \tau)$, $r = \overline{0, n-1}$, *that are analytic in* G *and continuous in* $\bar{G}$

$$\Phi\left(z, \bar{z}, \tau\right) = \frac{1}{2\pi i} \sum_{r=0}^{n-1} x^r \oint_K \frac{d\xi}{\xi - \tau} \int_0^\pi \omega_r(x + iy\cos t, \xi)_1 F_1\left(\frac{1}{2}; \frac{1}{2}; \frac{by^2 \sin^2 t}{(\xi - \tau)^2}\right) dt$$

is a solution of an equation of hyperbolic type of the n-th order

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \cdot \frac{\partial}{\partial y} - b^2\frac{\partial^2}{\partial \tau^2}\right)^n \Phi = 0,$$

for arbitrary $\tau \in T_0$, *and* $z, \bar{z}$ *from neighborhood of the* $z = 0$, $\bar{z} = 0$.

3. Conclusions

Riemann operator method allows studying iterative generalized axisymmetric equations from a single position. The involvement of the apparatus of special functions contributes to the successful solution of boundary value problems for iterative equations.

New integral representations of solutions of some iterative equations of elliptic, parabolic and hyperbolic types are obtained.

References

[1] I. M. Alexandrovich, O. S. Bondar, N. I. Lyashko, S. I. Lyashko and M.V.-S. Sydorov, *Integral operators that determine the solution of an iterated Hyperbolic-type equation*, Cybernet. Systems Anal. **56** (2020), 401–409. `https://dio.org/10.1007/s10559-020-00256-3`

[2] I. M. Alexandrovich, S. I. Lyashko, M. V.-S. Sydorov, N. I. Lyashko and O. S. Bondar, *Riemann integral operator for stationary and non-stationary processes*, Cybernet. Systems Anal. **57**(6) (2021), 918–926. `https://dio.org/10.1007/s10559-021-00418-x`

[3] I. M. Alexandrovich and M. V. Sydorov, *Differential operators specifying the solution of an elliptic iterated equation*, Ukrainian Math. J. **71** (2019), 495–504. `https://dio.org/10.1007/s11253-019-01659-y`

[4] I. M. Alexandrovich, M. V.-S. Sydorov and S. A. Salnikova, *Some generalizations of the Riemann operator method*, Azerb. J. Math. **11**(2) (2021), 163–175.

[5] H. Bateman, *Higher Transcendental Functions*, Volumes I-III, McGraw-Hill Book Co., New York-Toronto-London, 1953.

[6] R. P. Gilbert, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York-London, 1969.
[7] P. Henrici, *A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients*, Journal of Applied Mathematics and Physics (ZAMP) **8**(3) (1957), 169–203. https://dio.org/10.1007/BF01600500
[8] I. I. Lyashko, M. V. Sydorov and I. M. Alexandrovich, *Inversion of some integral equations*, Journal of Numerical and Applied Mathematics **2** (2004), 25–30.
[9] I. N. Vekua, *New Methods for Solving Elliptic Equations*, North-Holland Publishing Company, 1967.
[10] N. O. Virchenko, S. L. Kalla and I. M. Alexandrovich, *Inverse problems involving generalized axial-symmetric Helmholtz equation*, Mathematica Balkanica **26**(1–2) (2012), 113–122.

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