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TOPOLOGICAL HOCHSCHILD (σ,τ)-COHOMOLOGY GROUPS AND (σ,τ)-SUPER WEAK AMENABILITY OF BANACH ALGEBRAS

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Abstract. In this work, we introduce the new cohomology groups depended on homomorphisms which are extensions of the topological Hochschild cohomology groups and investigate some of their properties that are analogue to the Hochschild cohomology groups. In addition, we use some homomorphisms on Banach algebras to define a new concept of amenability, namely, (σ,τ)-super weak amenability which is a generalization of the cyclic amenability. Finally, we show that this new notion on a commutative Banach algebra $\mathcal{A}$ is equivalent to the (σ,τ)-weak amenability, where σ and τ are some continuous homomorphisms on $\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{X}$ be a Banach $\mathcal{A}$-bimodule. Let the products of $a \in \mathcal{A}$ and $x \in \mathfrak{X}$ be denoted by $a \cdot x$ and $x \cdot a$ which are both actions $\mathcal{A}$ over $\mathfrak{X}$. A derivation $D : \mathcal{A} \to \mathfrak{X}$ is a linear map which satisfies $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in \mathcal{A}$. The derivation δ is said to be *inner* if there exists $x \in \mathfrak{X}$ such that $\delta(a) = \delta_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. The linear space of all bounded (continuous) derivations and the linear subspace of inner derivations from $\mathcal{A}$ into $\mathfrak{X}$ are denoted by $Z^1(\mathcal{A}, \mathfrak{X})$ and $N^1(\mathcal{A}, \mathfrak{X})$, respectively. We consider the quotient space $H^1(\mathcal{A}, \mathfrak{X}) = Z^1(\mathcal{A}, \mathfrak{X})/N^1(\mathcal{A}, \mathfrak{X})$ which is called the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in $\mathfrak{X}$. A Banach algebra $\mathcal{A}$ is called *amenable* if every continuous derivation from $\mathcal{A}$ into every dual Banach $\mathcal{A}$-module is inner or equivalently $H^1(\mathcal{A}, \mathfrak{X}^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule $\mathfrak{X}$ [8]. Also, $\mathcal{A}$ is said to be weakly

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amenable if $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$. Recall that a bounded derivation $D:\mathcal{A}\to\mathcal{A}^*$ is called *cyclic* if $\langle D(a),b\rangle+\langle D(b),a\rangle=0$ for all $a,b\in\mathcal{A}$. A Banach algebra $\mathcal{A}$ is called *cyclically amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner [9].

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. We denote by $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ the metric space of all bounded homomorphisms from $\mathcal{A}$ into $\mathcal{B}$, with the metric derived from the bounded linear operators from $\mathcal{A}$ into $\mathcal{B}$, and denote $\mathrm{Hom}(\mathcal{A},\mathcal{A})$ by $\mathrm{Hom}(\mathcal{A})$. Let $\mathcal{X}$ be an $\mathcal{A}$-bimodule, and let $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$. A bounded linear mapping $d:\mathcal{A}\to\mathcal{X}$ is called a (σ,τ)-*derivation* if

$$d(ab)=d(a)\cdot\sigma(b)+\tau(a)\cdot d(b)\quad(a,b\in\mathcal{A}).$$

Also, a bounded linear mapping $d:\mathcal{A}\to\mathcal{X}$ is called a (σ,τ)-*inner derivation* if there exists $x\in\mathcal{X}$ such that

$$d(a)=x\cdot\sigma(a)-\tau(a)\cdot x\quad(a\in\mathcal{A}).$$

Then, $\mathcal{A}$ is called (σ,τ)-*amenable* if every (σ,τ)-derivation $d:\mathcal{A}\to\mathcal{X}$ is (σ,τ)-inner. We denote the space of continuous (σ,τ)-derivations from $\mathcal{A}$ into $\mathcal{X}$ by $Z^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ and the space of inner (σ,τ)-derivations by $B^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$. Consider the space $H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ as the quotient space $Z^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})/B^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$. The space $H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ is called the first (σ,τ)-cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$.

Let $\sigma,\tau\in\mathrm{Hom}(\mathcal{A},\mathcal{B})$. Then, $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule by the following module actions:

$$a\cdot b=\tau(a)b,\quad b\cdot a=b\sigma(a)\quad(a\in\mathcal{A},b\in\mathcal{B}).$$

We denote the above $\mathcal{A}$-bimodule by $\mathcal{B}_{\sigma,\tau}$ and denote $\mathcal{B}_{\sigma,\tau}$ by $\mathcal{B}_\sigma$ if $\sigma=\tau$. A Banach algebra $\mathcal{A}$ is called (σ,τ)-*weakly amenable* if $H^1(\mathcal{A},(\mathcal{A}_{(\sigma,\tau)})^*)=\{0\}$. These concepts are introduced and investigated in [3, 10, 11] and [12] (for the generalization of n-weak amenability refer to [4]). The (σ,τ)-weak amenability on the measure algebra $M(G)$, the group algebra $L^1(G)$ and the segal algebra $S^1(G)$, where G is a locally compact group are studied in [7]. For the module versions of these notions refer to [1] and [2].

In this work, we define the new cohomology groups which are the extensions of topological Hochschild cohomology groups and study some of their properties. In other words, we show that under which conditions $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)$ can be vanishes, where $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)$ is the n-th (σ,τ)-cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}^*$. In last section, we define a notion of amenability related to homomorphisms and find some equivalent results to the (σ,τ)-weak amenability for Banach algebras. Finally, we bring a concrete example for this new notion on a special semigroup algebra.

2. (σ,τ)-Cohomology of Banach Algebras

Throughout this paper, all mapping are assumed to be bounded. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$. From now on, we denote $\overbrace{\mathcal{A}\times\mathcal{A}\times\cdots\times\mathcal{A}}^{n-\text{times}}$ by $\mathcal{A}^n$. Suppose that $C^0(\mathcal{A},\mathcal{X})=\mathcal{X}$ and for $n\in\mathbb{N}$, define $C^n(\mathcal{A},\mathcal{X})$

by the Banach space of all bounded n-linear mappings from $\mathcal{A}^n$ into $\mathfrak{X}$ together with the multi-linear operator norm $\|f\| = \sup\{\|f(a_1,\dots,a_n)\| : a_i \in \mathcal{A}, \|a_i\| \le 1\}$. Consider the sequence of linear maps as follows:

$$0 \to C^0(\mathcal{A},\mathfrak{X}) \xrightarrow{\delta^0} C^1(\mathcal{A},\mathfrak{X}) \xrightarrow{\delta^1} C^2(\mathcal{A},\mathfrak{X}) \xrightarrow{\delta^2} \cdots,$$

where $(\delta^0 x)(a) = \tau(a)\cdot x - x\cdot\sigma(a)$ and for $n \in \mathbb{N}$,

$$\begin{aligned}(\delta^n T)(a_1,\dots,a_{n+1}) =& \tau(a_1)\cdot T(a_2,\dots,a_{n+1})\\ &+ \sum_{k=1}^{n}(-1)^k T(a_1,\dots,a_k a_{k+1},\dots,a_{n+1})\\ &+ (-1)^{n+1} T(a_1,\dots,a_n)\cdot\sigma(a_{n+1}),\end{aligned}$$

in which $T \in C^n(\mathcal{A},\mathfrak{X})$. The following result indicates the relation between δ^n's.

Lemma 2.1. *Suppose that $\mathcal{A}$ is a Banach algebra and $\mathfrak{X}$ is a Banach $\mathcal{A}$-bimodule. Then, for each n, $\delta^{n+1}\circ\delta^n = 0$.*

Proof. Considering $b^i_j = \begin{cases} a_j, & j<i,\\ a_j a_{j+1}, & j=i,\\ a_{j+1}, & j>i,\end{cases}$ we have

$$\begin{aligned}&\delta^{n+1}\circ\delta^n T(a_1,\dots,a_{n+2})\\
=&\tau(a_1)\cdot\delta^n T(a_2,\dots,a_{n+2}) + \sum_{i=1}^{n}(-1)^i \delta^n T(b^i_1,\dots,b^i_{n+1})\\
&+(-1)^{n+2}\delta^n T(a_1,\dots,a_{n+1})\cdot\sigma(a_{n+2})\\
=&\tau(a_1)\cdot[\tau(a_2)\cdot T(a_3,\dots,a_{n+2}) + \sum_{i=2}^{n+1}(-1)^{i-1}T(b^i_2,\dots,b^i_{n+1})\\
&+(-1)^{n+1}T(a_2,\dots,a_{n+1})\cdot\sigma(a_{n+2})] + \sum_{i=1}^{n+1}(-1)^i[\tau(b^i_1)T(b^i_2,\dots,b^i_{n+1})\\
&+\sum_{j=1}^{n}(-1)^j T(b^i_1,\dots,b^i_j b^i_{j+1},\dots,b^i_{n+1}) + (-1)^{n+1}T(b^i_1,\dots,b^i_n)\cdot\sigma(b^i_{n+1})]\\
&+(-1)^{n+2}[\tau(a_1)\cdot T(a_2,\dots,a_{n+1}) + \sum_{j=1}^{n}(-1)^j T(b^j_1,\dots,b^j_n)\\
&+(-1)^{n+1}T(a_1,\dots,a_n)\cdot\sigma(a_{n+1})]\cdot\sigma(a_{n+2})\\
=&-\sum_{i=1}^{n+1}(-1)^i\tau(b^i_1)\cdot T(b^i_2,\dots,b^i_{n+1}) + (-1)^{n+1}\tau(a_1)\cdot T(a_2,\dots,a_{n+1})\cdot\sigma(a_{n+2})\\
&+\sum_{i=1}^{n+1}(-1)^i\tau(b^i_1)\cdot T(b^i_2,\dots,b^i_{n+1}) + \sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j}T(b^i_1,\dots,b^i_j b^i_{j+1},\dots,b^i_{n+1})\\
&+\sum_{i=1}^{n+1}[(-1)^{i+n+1}T(b^i_1,\dots,b^i_n)\cdot\sigma(b^i_{n+1}) + (-1)^{n+2}\tau(a_1)\cdot T(a_2,\dots,a_{n+1})\cdot\sigma(a_{n+2})]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n+1}(-1)^{j+n+2}T(b_1^j,\dots,b_n^j)\cdot\sigma(b_{n+1}^j)\\
=&\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j}T(b_1^i,\dots,b_j^ib_{j+1}^i,\dots,b_{n+1}^i).
\end{aligned}
$$

With a combinatorial discussion, it can be concluded that the last summation is zero. □

We denote the kernel of δ^n and the image of δ^{n-1} by $Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ and $B^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ respectively. It follows from Lemma 2.1 that $B^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ is a subspace of $Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$. In other words, we can introduce the space $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ as the quotient space

$$Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})/B^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}).$$

Note that the elements of $Z^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ are continuous (σ,τ)-derivations and the elements of $B^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ are inner (σ,τ)-derivations. In the upcoming result we show that under some conditions $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)$ can be zero.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra with a left bounded approximate identity, $\mathcal{X}$ be a right annihilating Banach $\mathcal{A}$-bimodule. Then, for all $n>0$, $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)=\{0\}$.*

Proof. Let $(e_\nu)_{\nu\in\Lambda}$ be a left bounded approximate identity for $\mathcal{A}$. Assume that $f\in Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)$. Define $g_\nu\in C^{n-1}(\mathcal{A},\mathcal{X}^*)$ via

$$g_\nu(a_1,\dots,a_{n-1})=f(e_\nu,a_1,\dots,a_{n-1})\quad(a_1,\dots,a_{n-1}\in\mathcal{A},\ \nu\in\Lambda).$$

Note that $C^{n-1}(\mathcal{A},\mathcal{X}^*)$ is the dual of $C_{n-1}(\mathcal{A},\mathcal{X})=\underbrace{\mathcal{A}\otimes_p\cdots\otimes_p\mathcal{A}}_{n-1}\otimes_p\mathcal{X}$. Indeed, the mapping $C^{n-1}(\mathcal{A},\mathcal{X}^*)\to(C_{n-1}(\mathcal{A},\mathcal{X}))^*$ defined through

$$\phi\mapsto\tilde{\phi},\quad \tilde{\phi}(a_1\otimes\cdots\otimes a_{n-1}\otimes x)=\langle\phi(a_1,\dots,a_{n-1}),x\rangle$$

is an isometrical isomorphism of Banach spaces. Since $\|g_\nu\|\le\|f\|\|e_\nu\|$, the net $(g_\nu)_{\nu\in\Lambda}$ is bounded and so by the Banach-Alaoglu theorem, it has a subnet $(g_\nu)_{\nu\in\Omega}$ which is weak*-converging to a cochain g. Hence, for every $a_1,\dots,a_n\in\mathcal{A}$ and $x\in\mathcal{X}$, we obtain

$$\lim_\nu\langle g_\nu(a_1,\dots,a_{n-1}),x\rangle=\langle g(a_1,\dots,a_{n-1}),x\rangle.$$

Since $\mathcal{X}^*$ is a left annihilating $\mathcal{A}$-bimodule, we have

$$
\begin{aligned}
\delta^{n-1}g_\nu(a_1,\dots,a_n)&=\sum_{k=1}^{n-1}(-1)^k g_\nu(a_1,\dots,a_ka_{k+1},\dots,a_n)\\
&\quad+(-1)^n g_\nu(a_1,\dots,a_{n-1})\cdot\sigma(a_n)\\
&=\sum_{k=1}^{n-1}(-1)^k f(e_\nu,a_1,\dots,a_ka_{k+1},\dots,a_n)\\
&\quad+(-1)^n f(e_\nu,a_1,\dots,a_{n-1})\cdot\sigma(a_n)
\end{aligned}
$$

$$\begin{aligned}&+ f(e_\nu a_1, a_2, \ldots, a_n) - f(e_\nu a_1, a_2, \ldots, a_n)\\ &= - \delta^n f(e_\nu, a_1, \ldots, a_n) - f(e_\nu a_1, a_2, \ldots, a_n)\\ &= - f(e_\nu a_1, a_2, \ldots, a_n).\end{aligned}$$

Thus,

$$\begin{aligned}\lim_\nu \langle \delta^{n-1} g_\nu(a_1, \ldots, a_n), x\rangle =& \lim_\nu \langle \tau(a_1)\cdot g_\nu(a_2, \ldots, a_n)\\ &+ \sum_{k=1}^{n-1}(-1)^k g_\nu(a_1, \ldots, a_k a_{k+1}, \ldots, a_n)\\ &+ (-1)^n g_\nu(a_1, \ldots, a_{n-1})\cdot\sigma(a_n), x\rangle\\ =& \langle \tau\cdot(a_1) g_\nu(a_2, \ldots, a_n), x\rangle\\ =& \langle \delta^{n-1} g(a_1, \ldots, a_n), x\rangle,\end{aligned}$$

for all $x \in \mathcal{X}$. That is, $\lim_\nu \delta^{n-1} g_\nu = \delta^{n-1} g$ in the weak*-topology on the space $C^n(\mathcal{A}, \mathcal{X}^*)$. Also, $\lim_\nu e_\nu a_1 = a_1$ and hence

$$\begin{aligned}\langle f(a_1, \cdots, a_n), x\rangle &= \lim_\nu \langle f(e_\nu a_1, \ldots, a_n), x\rangle\\ &= -\lim_\nu \langle \delta^{n-1} g_\nu(a_1, \ldots, a_n), x)\rangle\\ &= \langle -\delta^{n-1} g(a_1, \ldots, a_n), x\rangle.\end{aligned}$$

Therefore, $f = \delta^{n-1}(-g) \in B^n_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*)$. This completes the proof. □

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then, the Banach space $C^k(\mathcal{A}, \mathcal{X})$ is an $\mathcal{A}$-bimodule with the following actions.

$$(a\cdot\phi)(a_1, \ldots, a_k) = a\cdot\phi(a_1, \ldots, a_k) \tag{2.1}$$

and

$$\begin{aligned}(\phi\cdot a)(a_1, \ldots, a_k) =& \phi(\tau^{-1}(a)a_1, a_2, \ldots, a_k)\\ &+ \sum_{i=1}^{k-1}(-1)^i \phi(\tau^{-1}(a), a_1, \ldots, a_i a_{i+1}, \ldots, a_k)\\ &+ (-1)^k \phi(\tau^{-1}(a), a_1, \ldots, a_{k-1})\tau(a_k) \quad (a \in \mathcal{A}\ \phi \in C^k(\mathcal{A}, \mathcal{X})).\end{aligned} \tag{2.2}$$

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{X}$ be as the above, and $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. Then, for all $n \geq 1$, $H^{n+k}_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = H^n_{(\sigma,\tau)}(\mathcal{A}, C^k(\mathcal{A}, \mathcal{X}))$, where the module actions $\mathcal{A}$ over $C^k(\mathcal{A}, \mathcal{X})$ are defined in* (2.1) *and* (2.2).

Proof. Let T^n be the canonical mapping from $C^{n+k}(\mathcal{A}, \mathcal{X})$ into $C^n(\mathcal{A}, C^k(\mathcal{A}, \mathcal{X}))$ defined by

$$((T^n\phi)(a_1, \ldots, a_n))(a_{n+1}, \ldots, a_{n+k}) = \phi(a_1, \ldots, a_{n+k}),$$

where $\phi \in C^k(\mathcal{A}, \mathcal{X})$. It is easy to check that T^n is a linear isometry. Let Δ^n be the multilinear mapping corresponding to δ^n when the $\mathcal{A}$-bimodule $\mathcal{X}$ is replaced by the

$\mathcal{A}$-bimodule $C^k(\mathcal{A}, \mathcal{X})$. Consider the following commutative diagram:

$$\begin{array}{ccccc} C^{n-1}(\mathcal{A}, C^k(\mathcal{A}, \mathcal{X})) & \overset{\Delta^{n-1}}{\Longrightarrow} & C^{n}(\mathcal{A}, C^k(\mathcal{A}, \mathcal{X})) & \overset{\Delta^{n}}{\Longrightarrow} & C^{n+1}(\mathcal{A}, C^k(\mathcal{A}, \mathcal{X})) \\ \Downarrow & & \Downarrow & & \Downarrow \\ C^{n+k-1}(\mathcal{A}, \mathcal{X}) & \overset{\delta^{n+k-1}}{\Longrightarrow} & C^{n+k}(\mathcal{A}, \mathcal{X}) & \overset{\delta^{n+k}}{\Longrightarrow} & C^{n+k+1}(\mathcal{A}, \mathcal{X}). \end{array}$$

The above diagram necessitates that $H^{n+k}_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = H^{n}_{(\sigma,\tau)}(\mathcal{A}, C^k(\mathcal{A}, \mathcal{X}))$. □

In analogy with Theorem 2.1, we have the next consequence, shows that (σ, τ)-amenability of a Banach algebra $\mathcal{A}$ implies that $H^n_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*) = \{0\}$.

Theorem 2.3. *Let $\mathcal{A}$ be (σ, τ)-amenable Banach algebra, where $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. Then, $H^n_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*) = \{0\}$, for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$ and for every $n \in \mathbb{N}$.*

Proof. Set $Y = \underbrace{\mathcal{A} \otimes_p \cdots \otimes_p \mathcal{A}}_{(n-1)-\text{times}} \otimes_p \mathcal{X}$. Then, Y is a Banach $\mathcal{A}$-bimodule under the following module multiplications:

$$\begin{aligned} (a_1 \otimes \cdots \otimes a_n \otimes x) \cdot a = {} & a_1 \otimes \cdots \otimes a_n \otimes x \cdot a, \\ a \cdot (a_1 \otimes \cdots \otimes a_n \otimes x) = {} & (\tau^{-1}(a) a_1) \otimes \cdots \otimes a_n \otimes x \\ & + \sum_{j=1}^{n-1} (-1)^j \tau^{-1}(a) \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes x \\ & + (-1)^n \tau^{-1}(a) \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes \tau(a_n) x. \end{aligned}$$

Also, there exists an isometric $\mathcal{A}$-bimodule isomorphism from Y^* onto $C^n(\mathcal{A}, \mathcal{X}^*)$. Therefore, $H^{n+1}_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*) \approx H^{1}_{(\sigma,\tau)}(\mathcal{A}, C^n(\mathcal{A}, \mathcal{X}^*)) \approx H^{1}_{(\sigma,\tau)}(\mathcal{A}, Y^*) = \{0\}$. □

Here and subsequently, for a Banach algebra $\mathcal{A}$ we set $\mathcal{A}^2 = \{ab : a, b \in \mathcal{A}\}$. Suppose that $\mathcal{J}$ is a closed ideal of a Banach algebra $\mathcal{A}$. The quotient $\frac{\mathcal{A}}{\mathcal{J}}$ is again a Banach algebra under the usual product and quotient norm. We also suppose that $\mathcal{J}^2 = \mathcal{J}$. Let $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$ such that $\sigma(\mathcal{J}) \subseteq \mathcal{J}$ and $\tau(\mathcal{J}) \subseteq \mathcal{J}$ and d be a (σ, τ)-derivation on $\mathcal{A}$. It is easy to check that $d(\mathcal{J}) \subseteq \mathcal{J}$. Assume that $\sigma^*, \tau^* : \frac{\mathcal{A}}{\mathcal{J}} \to \frac{\mathcal{A}}{\mathcal{J}}$ are the natural homomorphisms correspond to σ and τ, respectively. Then, the mapping

$$\begin{aligned} d_0 : \frac{\mathcal{A}}{\mathcal{J}} & \to \frac{\mathcal{A}}{\mathcal{J}}, \\ a + \mathcal{J} & \mapsto d(a) + \mathcal{J} \end{aligned}$$

is a well-defined and a (σ^*, τ^*)-derivation. We have the following diagram.

$$\begin{array}{ccc} \mathcal{A} & \overset{d}{\to} & \mathcal{A} \\ p \downarrow & & \downarrow p \\ \frac{\mathcal{A}}{\mathcal{J}} & \overset{d_0}{\to} & \frac{\mathcal{A}}{\mathcal{J}} \end{array}$$

where p is the natural projection from A onto $\frac{\mathcal{A}}{\mathcal{J}}$. The preceding discussion leads us to this challenge: With the above notations in which d_0 is an arbitrary (σ^*, τ^*)-derivation of $\frac{\mathcal{A}}{\mathfrak{J}}$, is there any (σ,τ)-derivation d of $\mathcal{A}$ which makes the above diagram commute? In other words, does d_0 *lift* to a derivation of $\mathcal{A}$? Considering $\mathcal{A}$ as a linear space (and ignoring topology), the subspace $\mathfrak{J}$ has a complementary subspace $\mathfrak{I}$. The restriction of p to $\mathfrak{I}$ is a linear isomorphism from $\mathfrak{I}$ onto $\frac{\mathcal{A}}{\mathfrak{J}}$, and so has an inverse q. We can consider q as a linear mapping from $\frac{\mathcal{A}}{\mathfrak{J}}$ into $\mathcal{A}$. Set $\xi = q\circ d_0\circ p$. Then, the mapping ξ is a linear from $\mathcal{A}$ into $\mathcal{A}$ so that lifts d_0. Putting $\rho = \delta^1\xi$ in $B^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}) \subseteq Z^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})$, we have

$$\begin{aligned} p(\rho(a,b)) &= p(\tau(a)\xi(b) - \xi(ab) + \xi(a)\sigma(b)) \\ &= p\tau(a)p\xi(b) - p\xi(ab) + p\xi(a)p\sigma(b) \\ &= \tau^* p(a)d_0p(b) - d_0p(ab) + d_0p(a)\sigma^* p(b) \\ &= 0, \end{aligned}$$

for all $a,b\in\mathcal{A}$. So, ρ takes all its values in the kernel $\mathfrak{J}$ of p and hence $\rho\in Z^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})$. Now, let $\eta\in C^1(\mathcal{A},\mathcal{A})$. We claim that η lifts d_0 if and only if $\eta-\xi\in C^1(\mathcal{A},\mathfrak{J})$. To prove this, note that η lifts d_0 if and only if $p\circ\eta = d_0\circ p = p\circ\xi$ which is equivalent to $p(\eta-\xi)=0$ and it means that $\eta-\xi\in C^1(\mathcal{A},\mathfrak{J})$. Thus, we can conclude that d_0 lifts to a (σ,τ)-derivation of $\mathcal{A}$ if and only if $\delta^1\xi\in B^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})$. Summing up we have the following theorem.

Theorem 2.4. *Let $\mathfrak{J}$ be a complemented closed ideal in a Banach algebra $\mathcal{A}$ and $\sigma,\tau\in \mathrm{Hom}(\mathcal{A})$ which leaves $\mathfrak{J}$ invariant. If $H^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})=0$, then every (σ^*,τ^*)-derivation of the quotient Banach algebra $\frac{\mathcal{A}}{\mathfrak{J}}$ lifts to a (σ,τ)-derivation of $\mathcal{A}$.*

We also have the partial converse of Theorem 2.4 as follows.

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra and $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$ such that $H^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})=0$. If $\mathfrak{J}$ is a closed ideal which is invariant under σ and τ, $\mathfrak{J}^2=\mathfrak{J}$, and every (σ^*,τ^*)-derivation of $\frac{\mathcal{A}}{\mathfrak{J}}$ lifts to a (σ,τ)-derivation of $\mathcal{A}$, then $H^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})=0$.*

Proof. Let $\rho\in Z^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})$ $(\subseteq Z^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}))$. Then, from $H^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})=0$, it deduces that $\rho=\delta^1\xi$ for some ξ in $C^1(\mathcal{A},\mathcal{A})$. Since $\delta^1\xi$ takes all its values in $\mathfrak{J}$, we have

$$\xi(j_1j_2) = \tau(j_1)\xi(j_2) + \xi(j_1)\sigma(j_2) - (\delta^1\xi)(j_1,j_2)\in\mathfrak{J},$$

for all $j_1,j_2\in\mathfrak{J}$. It follows that $\xi(\mathfrak{J})\subseteq\mathfrak{J}$. So, ξ induces a linear mapping

$$\begin{aligned} d_0 : \frac{\mathcal{A}}{\mathfrak{J}} &\to \frac{\mathcal{A}}{\mathfrak{J}}, \\ a+\mathfrak{J} &\mapsto \xi(a)+\mathfrak{J}. \end{aligned}$$

Thus, for all $a,b\in\mathcal{A}$, we get

$$\tau(a)\xi(b) - \xi(ab) + \xi(a)\sigma(b) = (\delta^1\xi)(a,b)\in\mathfrak{J}$$

and

$$p(\tau(a))p(\xi(b)) - p(\xi(ab)) + p(\xi(a))p(\sigma(b)) = 0,$$

$$\tau^*(p(a))d_0(p(b)) - d_0(p(ab)) + d_0(p(a))\sigma^*(p(b)) = 0.$$

This shows that d_0 is a (σ^*, τ^*)-derivation of $\frac{\mathcal{A}}{\mathcal{J}}$. By hypothesis, d_0 lifts to a derivation η on $\mathcal{A}$. So, $p \circ \eta = d_0 \circ p = p \circ \xi$ and hence $\eta - \xi \in C^1(\mathcal{A}, \mathcal{J})$. Therefore, $\rho = \delta^1\xi - \delta^1\eta \in B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{J})$ and $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{J}) = 0$. □

An *extension* of a Banach algebra $\mathcal{B}$ is a short-exact sequence of the form

$$\{0\} \to \ker\psi \to \mathcal{A} \xrightarrow{\psi} \mathcal{B} \to \{0\},$$

where $\mathcal{A}$ is a Banach algebra and $\psi : \mathcal{A} \to \mathcal{B}$ is a continuous, surjective algebra homomorphism. The extension is called *singular* if $\ker\psi$ has the trivial product, that is, $ab = 0$ for each $a, b \in \ker\psi$. We say that the extension *splits strongly* (resp. *admissible*) if ψ has a right inverse which is a continuous algebra homomorphism (resp. is bounded and linear).

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and $T \in Z^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$. Put $\mathcal{U}_T = \mathcal{A} \oplus_T \mathcal{X} = \{(a, x) : a \in \mathcal{A}, x \in \mathcal{X}\}$. Then, $\mathcal{U}_T$ equipped with the following product and norm is a Banach algebra:

$$\|(a, x)\| = \|a\| + \|x\|,$$

$$(a, x)(b, y) = (ab, \tau(a) \cdot y + x \cdot \sigma(b) + T(a, b)).$$

Further, $\sum(\mathcal{U}_T : \mathcal{X})$ is a singular, admissible Banach extension of $\mathcal{A}$.

The method of proof for the next consequence is similar the way for $H^2(\mathcal{A}, \mathcal{X})$ which was proved in [6, Theorem 2.8.12], so is omitted.

Theorem 2.6. *The map $T \mapsto \sum(\mathcal{U}_T, \mathcal{X})$ from $Z^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ induces a map from $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ to the family of equivalence classes of singular, admissible Banach extension of $\mathcal{A}$ by $\mathcal{X}$ with respect to strong equivalence.*

Theorem 2.7. $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = \{0\}$, *when* $H^2(\mathcal{A}, \mathcal{X}) = \{0\}$.

Proof. If $H^2(\mathcal{A}, \mathcal{X}) = \{0\}$, then each singular, admissible Banach extension of $\mathcal{A}$ by $\mathcal{X}$ splits strongly. Take $T \in Z^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$. Then, $\sum(\mathcal{U}_T, \mathcal{X})$ splits. So, there is a homomorphism $\theta : \mathcal{A} \to \mathcal{U}_T$ such that $\theta(a) = (a, -Sa)$ $(a \in \mathcal{A})$ for some $S \in C(\mathcal{A}, \mathcal{X})$ and hence $T = \delta^1 S$. Therefore, $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = \{0\}$. □

3. (σ, τ)-Super Weak Amenability of Banach Algebras

In this section, we introduce a concept of amenability which is a generalization of cyclic amenability on Banach algebras that help us to investigate the (σ, τ)-weak amenability of Banach algebras in more details.

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra, and $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. Then, $\mathcal{A}$ is called (σ, τ)-supper weakly amenable if for every Banach algebra $\mathcal{B}$ and every $\varphi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$, whenever $D : \mathcal{A} \to \mathcal{B}^*_\varphi$ is a (σ, τ)-derivation, then the equality $\langle D(a), \varphi(a)\rangle = 0$ holds for all $a \in \mathcal{A}$.

It is easily verified that $\mathcal{A}$ is (σ,τ)-super weakly amenable if and only if for every Banach algebra $\mathcal{B}$ and every $\varphi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ and each (σ,τ)-derivation $D : \mathcal{A} \to \mathcal{B}^*_\varphi$ with the following property is (σ,τ)-inner

$$\langle D(a), \varphi(b)\rangle + \langle D(b), \varphi(a)\rangle = 0 \quad (a, b \in \mathcal{A}).$$

It is obvious that:

- every (σ,τ)-supper weakly amenable Banach algebra is cyclically amenable;
- every (σ,τ)-weakly amenable Banach algebra $\mathcal{A}$ is (σ,τ)-supper weakly amenable when $\mathcal{B} = \mathcal{A}$ and φ is the identity map on $\mathcal{A}$. The converse is not true in general even for the special cases weak amenabiliy and cyclic amenability. In fact, any singly generated Banach algebra is cyclically amenable, as can be seen by looking at the values a continuous cyclic derivation must take on powers of the generator, while there are many examples of singly generated Banach algebras (even finite dimensional ones) that support bounded, non-zero point derivations, and hence are not weakly amenable [5].

However, we shall to show that two notions (σ,τ)-weak amenability and (σ,τ)-supper weak amenability coincide on commutative Banach algebras (Corollary 3.1). Before proceeding to the main results in this section, we bring the following lemma which is useful to achieve our purpose.

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\sigma,\tau \in \mathrm{Hom}(\mathcal{A})$ and $\varphi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$. If $d : \mathcal{A} \to \mathcal{B}^*_\varphi$ is a (σ,τ)-derivation, then $D : \mathcal{A} \to \mathcal{A}^*$ is a bounded (σ,τ)-derivation which is defined through*

$$\langle D(a), b\rangle := \langle d(a), \varphi(b)\rangle \quad (a, b \in \mathcal{A}).$$

Proof. Obviously, $D : \mathcal{A} \to \mathcal{A}^*$ is a bounded linear map. Also,

$$\begin{aligned}
\langle D(ab), c\rangle &= \langle d(ab), \varphi(c)\rangle \\
&= \langle d(a)\cdot\sigma(b) + \tau(a)\cdot d(b), \varphi(c)\rangle \\
&= \langle d(a)\cdot\varphi(\sigma(b)), \varphi(c)\rangle + \langle \varphi(\tau(a))\cdot d(b), \varphi(c)\rangle \\
&= \langle d(a), \varphi(\sigma(b)c)\rangle + \langle d(b), \varphi(c\tau(a))\rangle \\
&= \langle D(a), \sigma(b)c\rangle + \langle D(b), c\tau(a)\rangle \\
&= \langle D(a)\cdot\sigma(b), c\rangle + \langle \tau(a)\cdot D(b), c\rangle \\
&= \langle D(a)\cdot\sigma(b) + \tau(a)\cdot D(b), c\rangle \quad (a, b, c \in \mathcal{A}).
\end{aligned}$$

Therefore, D is a (σ,τ)-derivation. □

In [12], the authors have used from the Banach algebra introduced by Yong Zhang [13] to introduce a Banach algebra which is (σ,τ)-weak amenable for all homomorphisms σ,τ but not (σ,τ)-amenable for some homomorphisms σ and τ. In the oncoming example, we show that the menioned Banach algebra is a (σ,τ)-supper weakly amenable Banach algebra in which τ is the identity map.

Example 3.1. Firstly, we consider a product on Banach algebra $\ell^1 = l^1(\mathbb{N})$ as follows:

$$a \cdot b = a(1)b \qquad (a, b \in \ell^1).$$

Note that ℓ^1 has a left identity e_1 defined by

$$e_1(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

The dual space $(\ell^1)^* = \ell^\infty$ is a ℓ^1-bimodule via the ordinary actions as follows:

$$a \cdot f = f(a)e_1, \quad f \cdot a = a(1)f \qquad (a \in \ell(S), f \in \ell^\infty),$$

where e_1 is regarded as an element of ℓ^∞. Next let $\sigma : \ell^1 \to \ell^1$ be a bounded homomorphism. We have $a(1)\sigma(b) = \sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) = \sigma(a)(1)\sigma(b)$ and so $\sigma(b)(a(1) - \sigma(a)(1)) = 0$ for all $a, b \in \mathbb{N}$. Since $\sigma \neq 0$, we get

$$\sigma(a)(1) = a(1) \quad (a \in \ell^1).$$

Let $\mathcal{B}$ be an arbitrary Banach algebra and $\varphi \in \mathrm{Hom}(\ell^1, \mathcal{B})$. If $d : \ell^1 \to \mathcal{B}^*_\varphi$ is a bounded (σ, τ)-derivation, then by Lemma 3.1 the linear map $D : \ell^1 \to (\ell^1)^*$ defined through $\langle D(a), b\rangle = \langle d(a), \varphi(b)\rangle$, $a, b \in \ell^1$ is a (σ, τ)-derivation. Due to the (σ, τ)-weak amenability of ℓ^1 [12], there exists $f \in (\ell^1)^*$ such that $D(a) = f \cdot \sigma(a) - \tau(a) \cdot f$, $a \in \ell^1$. Hence,

$$\begin{aligned}
\langle d(a), \varphi(a)\rangle &= \langle D(a), a\rangle \\
&= \langle f \cdot \sigma(a) - \tau(a) \cdot f, a\rangle \\
&= \langle f, \sigma(a) \cdot a - a \cdot \tau(a)\rangle \\
&= \langle f(\sigma(a))(1)a - a(1)\tau(a)\rangle \\
&= \langle f, a(1)a - a(1)\tau(a)\rangle \\
&= a(1)\langle f, a - \tau(a)\rangle \\
&= 0 \quad (a \in \ell^1).
\end{aligned}$$

Here, we state the relationship between (σ, σ)-weak amenability and (σ, σ)-supper weak amenability on Banach algebras.

Proposition 3.1. *Let $\mathcal{A}$ be a Banach algebra and $\sigma \in \mathrm{Hom}(\mathcal{A})$ such that the range of σ commute with $\mathcal{A}$. Then, $\mathcal{A}$ is (σ, σ)-weakly amenable if and only if $\mathcal{A}$ is (σ, σ)-supper weakly amenable.*

Proof. We firstly assume that $\mathcal{A}$ is (σ, σ)-supper weakly amenable. Set $\mathcal{B} = \mathcal{A}$ and $\varphi = id$ (the identity map). Let $D : \mathcal{A} \to (\mathcal{A}_{id})^*$ be a bounded derivation. It follows from the (σ, σ)-super weak amenability of $\mathcal{A}$ that $\langle D(a), a\rangle = 0\, (a \in \mathcal{A})$ and hence D is (σ, σ)-inner.

Conversely, suppose that $\mathcal{A}$ is (σ, σ)-weakly amenable. Consider an arbitrary Banach algebra $\mathcal{B}$ and a $\varphi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Let $d : \mathcal{A} \to \mathcal{B}^*_\varphi$ be a (σ, σ)-derivation. By Lemma 3.1, the linear map $D : \mathcal{A} \to \mathcal{A}^*$ defined via $\langle D(a), b\rangle := \langle d(a), \varphi(b)\rangle$ is a (σ,σ)-derivation and so it is (σ,σ)-inner. Thus, there exists $f \in \mathcal{A}^*$ such that $D(a) = f \cdot \sigma(a) - \tau(a) \cdot f$. Hence, we have

$$\begin{aligned}\langle d(a), \varphi(a)\rangle &= \langle D(a), a\rangle \\ &= \langle f \cdot \sigma(a) - \sigma(a) \cdot f, a\rangle \\ &= \langle f, \sigma(a)a - a\sigma(a)\rangle = 0,\end{aligned}$$

for all $a, b \in \mathcal{A}$. This finishes the proof. □

One should remember that a commutative Banach algebra is weakly amenable if and only if it is cyclically amenable. We generalize this result as follows.

Corollary 3.1. *Let $\mathcal{A}$ be a commutative Banach algebra and $\sigma \in \mathrm{Hom}(\mathcal{A})$. Then, $\mathcal{A}$ is (σ,σ)-weakly amenable if and only if $\mathcal{A}$ is (σ,σ)-supper weakly amenable.*

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