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FROM CLASSICAL TO HIGHER-ORDER PARTIAL ISOMETRIES IN HILBERT SPACES

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Abstract. This paper extends the classical notion of partial isometries by introducing a new class of operators termed *partial isometries of order* n, where n is a positive integer. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a left (respectively, right) partial isometry of order n if

$$T^{*n}T^n = T^{*(n-1)}T^{n-1}P_{\mathsf{N}(T^n)^\perp} \quad \text{(respectively, } T^nT^{*n} = P_{\overline{\mathsf{R}(T^n)}}T^{n-1}T^{*(n-1)}\text{)},$$

where $P_{\mathsf{N}(T^n)^\perp}$ and $P_{\overline{\mathsf{R}(T^n)}}$ denote the orthogonal projections onto $\mathsf{N}(T^n)^\perp$ and $\overline{\mathsf{R}(T^n)}$, respectively. This generalization, formulated in terms of orthogonal projections, appears to be new and, to the best of our knowledge, has not been previously addressed in the literature. We investigate several fundamental properties of this new class of operators, focusing on their structural features and spectral behavior. Our analysis recovers and generalizes various known results concerning classical partial isometries. The paper concludes with a discussion of prospective research directions stemming from the framework developed herein.

1. Introduction and Notations

Let $\mathcal{H}$ denote a complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Let $\mathcal{B}(\mathcal{H})$ represent the algebra of all bounded linear operators on $\mathcal{H}$. For any operator $T \in \mathcal{B}(\mathcal{H})$, we denote by T^* its adjoint, by $\mathsf{N}(T)$ its kernel, and by $\mathsf{R}(T)$ its range. We denote by $\mathbb{D}$ the open unit disc in the complex plane and by $\partial\mathbb{D}$ its boundary.

An operator T is said to be an *isometry* if $\|Tx\| = \|x\|$ for all $x \in \mathcal{H}$, which is equivalent to the relation $T^*T = I$, where I denotes the identity operator on $\mathcal{H}$. An

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operator is called *unitary* if it satisfies $T^*T = TT^* = I$, and a *co-isometry* if its adjoint T^* is an isometry.

The study of linear operators, particularly isometries, has long constituted a central and dynamic area within operator theory, attracting sustained attention from numerous researchers in the field. This line of inquiry is often intimately connected with significant problems in modern mathematics and physics. A classical and highly significant generalization of isometries is given by the class of *partial isometries*. Within the algebra $\mathcal{B}(\mathcal{H})$, a partial isometry is defined as an operator whose restriction to the orthogonal complement of its kernel acts as an isometry. More precisely, an operator $T \in \mathcal{B}(\mathcal{H})$ is called a *partial isometry* if $\|T(x)\| = \|x\|$, for all $x \in \mathsf{N}(T)^{\perp}$. The collection of all partial isometries on $\mathcal{H}$ will be denoted by $\mathcal{PI}(\mathcal{H})$.

The following conditions are well known to be equivalent for any operator $T \in \mathcal{B}(\mathcal{H})$.

(1) $T \in \mathcal{PI}(\mathcal{H})$;
(2) $T^* \in \mathcal{PI}(\mathcal{H})$;
(3) $TT^*T = T$;
(4) T^*T is an orthogonal projection, specifically the projection onto $\mathsf{N}(T)^{\perp}$;
(5) TT^* is an orthogonal projection, specifically the projection onto $\mathsf{R}(T)$;
(6) The matrix representation of T, with respect to the decomposition $\mathcal{H} = \mathsf{N}(T)^{\perp} \oplus \mathsf{N}(T)$, has the form

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}, \quad \text{where } A^*A + B^*B = I.$$

A great deal of substantial work has been devoted to the study of partial isometries. This class of operators has played a fundamental role in the structural theory of Hilbert space operators, particularly in the polar decomposition of arbitrary operators and the dimension theory of von Neumann algebras. For a comprehensive overview of the theory and applications of partial isometries, we refer the interested reader to [1, 3, 5, 11, 14, 20, 21, 27, 29].

The characterizations and properties outlined above merely mark the beginning of a long and fruitful line of research that explores equivalent formulations of partial isometries across various frameworks that extend the algebraic and geometric structure of $\mathcal{B}(\mathcal{H})$. Chronologically, the first major direction originated with the works of R. Harte [23], R. Harte and M. Mbekhta [24], M. Mbekhta [27], and M. Mbekhta and L. Suciu [28], where partial isometries were characterized through the notion of regularity in the context of C^*-algebras, in connection with generalized inverses. A second characterization was established via the conorm of an element in a C^*-algebra. The concept of conorm based on the reduced minimum modulus of left and right multiplication operators was introduced by R. Harte and M. Mbekhta [24]. A third direction, due to C. A. Akemann and N. Weaver [2], provided a geometric characterization of partial isometries within a C^*-algebra $\mathcal{A}$, in terms of the Banach space structure of $\mathcal{A}$.

In recent years, there has been growing interest in generalizing the class of partial isometries. Notable examples include semi-partial isometries in Banach spaces [34], partial isometries in semi-Hilbertian spaces [5,9,15], m-partial isometries [30], $(m, \mathcal{N}_A)$-isometries [5], (k, m, n)-partial isometries [3] and semi-generalized partial isometries [16] in Hilbert spaces, among others. Other notable generalizations include (A, m)-partial isometries [32] and q-partial-(m, A)-isometries [31].

These developments motivate us to pursue this line of inquiry further by introducing a new class of operators that generalizes the notion of partial isometries. Our approach is based on characterizations involving orthogonal projections, as described in points (4) and (5) above for classical partial isometries. This generalization, grounded in the use of orthogonal projections, is new and has not been previously addressed in the literature.

This paper is organized into four sections. Section 2 introduces illustrative examples and foundational concepts, and establishes a decomposition theorem, thereby generalizing several known results concerning classical partial isometries. In Section 3, considerable attention is devoted to the spectral picture of quasicommuting left and right partial isometries of order n, culminating in Theorem 3.1. This result extends earlier work by Erdélyi et al. on classical partial isometries [12, Theorems 3 and 4]. Finally, Section 4 presents several open problems and future research directions that naturally arise from the framework developed in this paper.

2. Some Properties

We will consider the following definition.

Definition 2.1. We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is:

(1) a *left partial isometry of order n* if

$$T^{*n}T^n = T^{*n-1}T^{n-1}P_{\mathsf{N}(T^n)^\perp};$$

(2) a *right partial isometry of order n* if

$$T^nT^{*n} = P_{\overline{\mathsf{R}(T^n)}}T^{n-1}T^{*n-1}.$$

We denote by $\mathcal{PI}_n^\ell(\mathcal{H})$ the set of all left partial isometries of order n, and by $\mathcal{PI}_n^r(\mathcal{H})$ the set of all right partial isometries of order n; that is,

$$\mathcal{PI}_n^\ell(\mathcal{H}) := \left\{T \in \mathcal{B}(\mathcal{H}) : T^{*n}T^n = T^{*n-1}T^{n-1}P_{\mathsf{N}(T^n)^\perp}\right\},$$

$$\mathcal{PI}_n^r(\mathcal{H}) := \left\{T \in \mathcal{B}(\mathcal{H}) : T^nT^{*n} = P_{\overline{\mathsf{R}(T^n)}}T^{n-1}T^{*n-1}\right\}.$$

It is evident that $\mathcal{PI}_n^\ell(\mathcal{H})$ (resp. $\mathcal{PI}_n^r(\mathcal{H})$) contains all isometries (resp. co-isometries). The concept of a *left* (resp. *right*) *partial isometry of order n* serves as a generalization of the well-known notion of a partial isometry, as the class $\mathcal{PI}_1^\ell(\mathcal{H}) = \mathcal{PI}_1^r(\mathcal{H})$ coincides with the set $\mathcal{PI}(\mathcal{H})$ of all partial isometries.

We start with this preliminary result, which will be used throughout the paper.

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then, the following statements hold.*

(1) *$T \in \mathcal{PI}_n^\ell(\mathcal{H})$ if and only if $T^* \in \mathcal{PI}_n^r(\mathcal{H})$.*

(2) *The following assertions are equivalent:*

  (a) $T \in \mathcal{PI}_n^\ell(\mathcal{H})$,

  (b) $T^{*n}T^nT^{*n} = T^{*(n-1)}T^{n-1}T^{*n}$,

  (c) $T^nT^{*n}T^n = T^nT^{*(n-1)}T^{n-1}$.

(3) *The following assertions are equivalent:*

  (a) $T \in \mathcal{PI}_n^r(\mathcal{H})$,

  (b) $T^nT^{*n}T^n = T^{n-1}T^{*(n-1)}T^n$,

  (c) $T^{*n}T^nT^{*n} = T^{*n}T^{n-1}T^{*(n-1)}$.

Proof. (1) Let $S = T^*$. Then, the equivalence follows from the definitions:

$$\begin{aligned} T \in \mathcal{PI}_n^\ell(\mathcal{H}) &\Leftrightarrow T^{*n}T^n = T^{*(n-1)}T^{n-1}P_{\mathsf{N}(T^n)^\perp} \\ &\Leftrightarrow T^{*n}T^n = P_{\mathsf{N}(T^n)^\perp}T^{*(n-1)}T^{n-1} \text{ (by taking the adjoint of both sides)} \\ &\Leftrightarrow S^nS^{*n} = P_{\overline{\mathsf{R}(S^n)}}S^{n-1}S^{*(n-1)} \text{ (using } \mathsf{N}(T^n)^\perp = \overline{\mathsf{R}(T^{*n})} \text{ and } S = T^*) \\ &\Leftrightarrow S \in \mathcal{PI}_n^r(\mathcal{H}) \Leftrightarrow T^* \in \mathcal{PI}_n^r(\mathcal{H}). \end{aligned}$$

(2) (a) $\Rightarrow$ (b) Suppose $T \in \mathcal{PI}_n^\ell(\mathcal{H})$. Then,

$$T^{*n}T^nT^{*n} = T^{*(n-1)}T^{n-1}P_{\mathsf{N}(T^n)^\perp}T^{*n} = T^{*(n-1)}T^{n-1}T^{*n}.$$

(b) $\Rightarrow$ (a) Assume $T^{*n}T^nT^{*n} = T^{*(n-1)}T^{n-1}T^{*n}$, and define

$$A_n := T^{*n}T^n - T^{*(n-1)}T^{n-1}.$$

Then, $A_nT^{*n} = 0$, so for all $y \in \overline{\mathsf{R}(T^{*n})} = \mathsf{N}(T^n)^\perp$, we have $A_n(y) = 0$. It follows that

$$A_nP_{\mathsf{N}(T^n)^\perp} = 0,$$

which implies $T \in \mathcal{PI}_n^\ell(\mathcal{H})$.

(b) $\Leftrightarrow$ (c) This equivalence follows immediately by taking adjoints.

(3) The equivalences in part (3) follow from parts (1) and (2) by duality, completing the proof. □

Here are some illustrative examples.

Example 2.1. (1) If $T = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & -1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}$, then $T \in \mathcal{PI}_1^\ell(\mathcal{H}) \cap \mathcal{PI}_2^\ell(\mathcal{H}) \cap \mathcal{PI}_2^r(\mathcal{H})$.

(2) If $T = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $T \in \mathcal{PI}_2^\ell(\mathcal{H}) \cap \mathcal{PI}_2^r(\mathcal{H})$, but $T \notin \mathcal{PI}_1^\ell(\mathcal{H}) = \mathcal{PI}_1^r(\mathcal{H})$.

(3) If T is an isometry, a co-isometry, or a unitary operator, then $T \in \mathcal{PI}_n^\ell(\mathcal{H}) \cap \mathcal{PI}_n^r(\mathcal{H})$ for all $n \geq 1$. In particular, both orthogonal symmetries and the unilateral shift are contained in this class.

(4) If T is nilpotent of order k, then $T \in \mathcal{PI}_n^\ell(\mathcal{H}) \cap \mathcal{PI}_n^r(\mathcal{H})$ for all $n \geq k$.

(5) If T_1 and T_2 are left (respectively, right) partial isometries of order n, then $T = T_1 \oplus T_2$ is also a left (respectively, right) partial isometry of order n.

For $T \in \mathcal{B}(\mathcal{H})$, recall that the *reduced minimum modulus* of T is defined by

$$\gamma(T) := \inf \Big\{ \|Tx\| : \operatorname{dist}(x, \mathsf{N}(T)) = 1 \Big\},$$

with the convention that $\gamma(T) = +\infty$ if $T = 0$.

It is a classical result that $\gamma(T) > 0$ if and only if $\mathsf{R}(T)$ is closed [8, 19]. Moreover, if $T \in \mathcal{B}(\mathcal{H})$ is a nonzero operator, then by [8] we have

$$\gamma(T) = \inf \Big(\sigma(|T|) \setminus \{0\} \Big),$$

where $|T| = (T^*T)^{1/2}$ denotes the modulus of T. Consequently, $\gamma(T) = \gamma(T^*)$ (see [19, 25]).

From [27, Corollary 3.2], every non-zero partial isometry T satisfies $\gamma(T) = 1$, and thus has closed range. This classical property was originally noted by P. Halmos [22, Chapter 15, p. 69]. In the next result, we extend this fact to a special class of higher-order partial isometries by proving that any normal left (or right) partial isometry of order n must be of the form $T = U \oplus 0$, where U is unitary. In particular, such operators also have closed range.

Theorem 2.1. *Let T be a non-zero normal left (or right) partial isometry of order n. Then, with respect to the orthogonal decomposition $\mathcal{H} = \mathsf{N}(T)^\perp \oplus \mathsf{N}(T)$, the operator T has the form $T = U \oplus 0$, where U is a unitary operator acting on $\mathsf{N}(T)^\perp$.*

Proof. Assume without loss of generality that T is a normal left partial isometry of order n. By Proposition 2.1, we have $T^{*n}T^nT^{*n} = T^{*(n-1)}T^{n-1}T^{*n}$. Multiplying on the right by T^n yields $(|T^n|^2 - |T^{n-1}|^2)\,|T^n|^2 = 0$. Since T is normal, this relation reduces to

$$\Big(|T|^{2n} - |T|^{2(n-1)} \Big)\, |T|^{2n} = 0.$$

Hence, the polynomial $p(X) = X^{4n-2}(X^2 - 1)$ annihilates $|T|$. As $|T|$ is a positive operator, the spectral mapping theorem (see [18, Lemma 8.4.1]) implies

$$\sigma(|T|) \subset \{0, 1\}.$$

Therefore, $|T|$ has the form $|T| = I \oplus 0$ with respect to the orthogonal decomposition $\mathcal{H} = \mathsf{N}(T)^\perp \oplus \mathsf{N}(T)$. Since T is normal, we have $\mathsf{N}(T) = \mathsf{N}(T^*)$, and hence the partial isometry V in the polar decomposition $T = V|T|$ satisfies $V = U \oplus 0$, where U is unitary on $\mathsf{N}(T)^\perp$. It follows that

$$T = V|T| = (U \oplus 0)(I \oplus 0) = U \oplus 0,$$

where U is unitary on $\mathsf{N}(T)^\perp$. $\square$

Remark 2.1. As shown in the preceding proof, any non-zero normal left (or right) partial isometry T of order n satisfies

$$\gamma(T) := \inf\Big\{\sigma(|T|) \setminus \{0\}\Big\} = 1.$$

Example 2.2. Let $T \in \mathcal{B}(\mathcal{H})$ be injective (or surjective) and assume that $T \in \mathcal{PI}_1^{\ell}(\mathcal{H}) = \mathcal{PI}_1^{r}(\mathcal{H}) = \mathcal{PI}(\mathcal{H})$. Then, it is straightforward to verify that

$$T \in \bigcap_{n\geq 2} \mathcal{PI}_n^{\ell}(\mathcal{H}) \cap \mathcal{PI}_n^{r}(\mathcal{H}).$$

However, the converse does not hold in general. It is possible for an operator T to belong to this intersection

$$\bigcap_{n\geq 2} \mathcal{PI}_n^{\ell}(\mathcal{H}) \cap \mathcal{PI}_n^{r}(\mathcal{H}),$$

while failing to lie in $\mathcal{PI}(\mathcal{H})$ and being neither injective nor surjective.

To illustrate this, let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$, and define the operator $T \in \mathcal{B}(\mathcal{H})$ by

$$T(e_{2n}) = 0, \quad T(e_{2n+1}) = \frac{1}{n+1} e_{2n}, \quad \text{for all } n \in \mathbb{N}.$$

Clearly, $T^2 = 0$, and hence $T \in \bigcap_{n\geq 2} \mathcal{PI}_n^{\ell}(\mathcal{H}) \cap \mathcal{PI}_n^{r}(\mathcal{H})$. However, T is not injective. Moreover, the range of T is not closed, so T is not surjective, and by the classical property for partial isometries, we conclude that $T \notin \mathcal{PI}(\mathcal{H})$. We also note that T is not normal.

Here, we present an example of an injective operator that belongs to $\bigcap_{n\geq 2} \mathcal{PI}_n^{\ell}(\mathcal{H})$ but does not belong to $\mathcal{PI}_1^{\ell}(\mathcal{H})$.

Example 2.3. Consider the operators $A, B \in \mathcal{B}(\ell^2)$ defined by

$$\begin{aligned} A(x_1, x_2, x_3, \ldots) =& (0, x_1, 0, x_2, 0, x_3, \ldots), \\ B(x_1, x_2, x_3, \ldots) =& (x_1, 0, x_2/2, 0, x_3/3, 0, \ldots). \end{aligned}$$

Define the operator T on $\mathcal{H} = \ell^2 \oplus \ell^2$ by

$$T = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$

A straightforward computation shows that $B^*A = 0$, from which it follows that

$$T \in \bigcap_{n\geq 2} \mathcal{PI}_n^{\ell}(\mathcal{H}).$$

Moreover, since A is an isometry and B is injective with $B^*A = 0$, it follows that T is injective.

However, observe that

$$T^*T = \begin{bmatrix} I & 0 \\ 0 & B^*B \end{bmatrix},$$

and since B is not an isometry, $B^*B \neq I$, so $T^*T \neq P_{\mathsf{N}(T)^{\perp}}$. Therefore, $T \notin \mathcal{PI}_1^{\ell}(\mathcal{H})$.

We now provide a sufficient condition ensuring that a left (resp. right) partial isometry of order n is also a left (resp. right) partial isometry of order $n+1$.

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then, the following hold.*

1. *If $T \in \mathcal{PI}_n^\ell(\mathcal{H})$ and $T\Big(\mathsf{N}(T^{n+1})^\perp\Big) \subseteq \mathsf{N}(T^n)^\perp$, then $T \in \mathcal{PI}_{n+1}^\ell(\mathcal{H})$.*
2. *If $T \in \mathcal{PI}_n^r(\mathcal{H})$ and $T^*\Big(\mathsf{R}(T^{n+1})\Big) \subseteq \mathsf{R}(T^n)$, then $T \in \mathcal{PI}_{n+1}^r(\mathcal{H})$.*

Proof. (1) Since $T \in \mathcal{PI}_n^\ell(\mathcal{H})$, we have

$$T^{*n}T^n = T^{*n-1}T^{n-1}P_{\mathsf{N}(T^n)^\perp}.$$

Multiplying this identity on the right by T and on the left by T^*, we obtain

$$T^{*n+1}T^{n+1} = T^{*n}T^{n-1}P_{\mathsf{N}(T^n)^\perp}T. \tag{2.1}$$

We now claim that

$$P_{\mathsf{N}(T^n)^\perp}T = TP_{\mathsf{N}(T^{n+1})^\perp}. \tag{2.2}$$

Indeed, for any $x \in \mathsf{N}(T^{n+1})^\perp$, the assumption implies

$$T(x) \in \mathsf{N}(T^n)^\perp,$$

and therefore

$$TP_{\mathsf{N}(T^{n+1})^\perp}(x) = T(x) = P_{\mathsf{N}(T^n)^\perp}T(x), \quad \text{for all } x \in \mathsf{N}(T^{n+1})^\perp. \tag{2.3}$$

For $x \in \mathsf{N}(T^{n+1})$, we have $T(x) \in \mathsf{N}(T^n)$, so

$$TP_{\mathsf{N}(T^{n+1})^\perp}(x) = 0 = P_{\mathsf{N}(T^n)^\perp}T(x), \quad \text{for all } x \in \mathsf{N}(T^{n+1}). \tag{2.4}$$

Combining equations (2.3) and (2.4), we obtain (2.2). Next, using (2.2) together with (2.1), we deduce that

$$T^{*n+1}T^{n+1} = T^{*n}T^nP_{\mathsf{N}(T^{n+1})^\perp}.$$

Applying Proposition 2.1, we conclude that $T \in \mathcal{PI}_{n+1}^\ell(\mathcal{H})$.
(2) By continuity of T^*, we have

$$T^*\Big(\mathsf{N}(T^{*n+1})^\perp\Big) = T^*\Big(\overline{\mathsf{R}(T^{n+1})}\Big) = \overline{T^*(\mathsf{R}(T^{n+1}))} \subseteq \overline{\mathsf{R}(T^n)} = \mathsf{N}(T^{*n})^\perp.$$

Using (1) and Proposition 2.1 for T^*, it follows that

$$T \in \mathcal{PI}_{n+1}^r(\mathcal{H}).$$

This completes the proof. □

Corollary 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then, the following hold.*

1. *If $T \in \mathcal{PI}_n^\ell(\mathcal{H})$ and $\mathsf{N}(T^{n+1})$ is a reducing subspace for T, then $T \in \mathcal{PI}_{n+1}^\ell(\mathcal{H})$.*
2. *If $T \in \mathcal{PI}_n^r(\mathcal{H})$ and $\mathsf{R}(T^{n+1})$ is invariant by T^*, then $T \in \mathcal{PI}_{n+1}^r(\mathcal{H})$.*

It is well-known that powers of a partial isometry need not be partial isometries themselves. In the sense of Halmos, an operator T is called a *power partial isometry* if T^p is a partial isometry for all $p \in \mathbb{N}$. The class of power partial isometries has attracted considerable attention in the literature. For a comprehensive structural description, we refer to the classical work of P. R. Halmos and L. J. Wallen [21]. Further results can be found in [13] by H. Ezzahraoui et al., and in [5] by M. A. Aouichaoui.

The following proposition establishes a connection between left (resp. right) partial isometries of order n and power partial isometries.

Proposition 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then, the following hold.*

(1)

$$T \in \bigcap_{i=1}^{n} \mathcal{PI}_i^{\ell}(\mathcal{H}) \Leftrightarrow T^i \text{ is a partial isometry for all } 1 \leq i \leq n.$$

In particular,

$$T \in \bigcap_{i=1}^{+\infty} \mathcal{PI}_i^{\ell}(\mathcal{H}) \Leftrightarrow T \text{ is a power partial isometry.}$$

(2)

$$T \in \bigcap_{i=1}^{n} \mathcal{PI}_i^{r}(\mathcal{H}) \Leftrightarrow T^i \text{ is a partial isometry for all } 1 \leq i \leq n.$$

In particular,

$$T \in \bigcap_{i=1}^{+\infty} \mathcal{PI}_i^{r}(\mathcal{H}) \Leftrightarrow T \text{ is a power partial isometry.}$$

Proof. (1) We begin with the forward implication. The proof proceeds by induction on n. The base case $n = 1$ is immediate since $\mathcal{PI}_1^{\ell}(\mathcal{H}) = \mathcal{PI}(\mathcal{H})$ by definition, so T is a partial isometry.

Assume the statement holds for some $n \geq 1$. Suppose $T \in \mathcal{PI}_i^{\ell}(\mathcal{H})$ for all $1 \leq i \leq n+1$. Then, by Proposition 2.1, one has

$$T^{n+1}T^{*(n+1)}T^{n+1} = T^{n+1}T^{*n}T^n = TT^nT^{*n}T^n = TT^n = T^{n+1}.$$

This shows that T^{n+1} is a partial isometry, completing the induction.

The reverse implication follows by a similar argument.

(2) Since an operator T is a partial isometry if and only if its adjoint T^* is one, the assertion follows by applying the first part to T^* and using duality with Proposition 2.1. □

The reader may easily verify the following basic properties.

Proposition 2.4. *Let T, $S \in \mathcal{B}(\mathcal{H})$ and n, $m \in \mathbb{N}$. The following properties hold.*

1. *Assume that $T \in \mathcal{PI}_n^{\ell}(\mathcal{H})$ (resp. $T \in \mathcal{PI}_n^{r}(\mathcal{H})$) and $\lambda \in \mathbb{C} \setminus \{0\}$.*
   (a) *If $\lambda \in \partial\mathbb{D}$, then $\lambda T \in \mathcal{PI}_n^{\ell}(\mathcal{H})$ (resp. $\lambda T \in \mathcal{PI}_n^{r}(\mathcal{H})$).*

(b) *If* $T^n \neq 0$, *then*

$$\lambda T \in \mathcal{PI}_n^\ell(\mathcal{H}) \ \ (\textit{resp.}\ \lambda T \in \mathcal{PI}_n^r(\mathcal{H})) \Leftrightarrow \lambda \in \partial\mathbb{D}.$$

(2) *Assume that* $TS = ST$ *and* $TS^* = S^*T$.

(a) *If* T *and* S *are injective, then*

$$T \in \mathcal{PI}_n^\ell(\mathcal{H}),\ S \in \mathcal{PI}_m^\ell(\mathcal{H}) \Rightarrow TS \in \mathcal{PI}_k^\ell(\mathcal{H}), \quad \textit{with } k = \max\{n, m\}.$$

(b) *If*

$$T\Big(\mathsf{N}(T^{n+1})^\perp\Big) \subset \mathsf{N}(T^n)^\perp \quad \textit{and} \quad S\Big(\mathsf{N}(S^{n+1})^\perp\Big) \subset \mathsf{N}(S^n)^\perp,$$

then

$$T \in \mathcal{PI}_n^\ell(\mathcal{H}),\ S \in \mathcal{PI}_n^\ell(\mathcal{H}) \Rightarrow TS \in \mathcal{PI}_{n+1}^\ell(\mathcal{H}).$$

(c) *If* $\overline{\mathsf{R}(S)} = \overline{\mathsf{R}(T)} = \mathcal{H}$, *then*

$$T \in \mathcal{PI}_n^r(\mathcal{H}),\ S \in \mathcal{PI}_m^r(\mathcal{H}) \Rightarrow TS \in \mathcal{PI}_k^r(\mathcal{H}), \quad \textit{with } k = \max\{n, m\}.$$

(d) *If*

$$T^*\Big(\mathsf{R}(T^{n+1})\Big) \subset \mathsf{R}(T^n) \quad \textit{and} \quad S^*\Big(\mathsf{R}(S^{n+1})\Big) \subset \mathsf{R}(S^n),$$

then

$$T \in \mathcal{PI}_n^r(\mathcal{H}),\ S \in \mathcal{PI}_n^r(\mathcal{H}) \Rightarrow TS \in \mathcal{PI}_{n+1}^r(\mathcal{H}).$$

(3) (a) *If* S *is an isometry, then*

$$T \in \mathcal{PI}_n^\ell(\mathcal{H}) \Leftrightarrow STS^* \in \mathcal{PI}_n^\ell(\mathcal{H}), \quad T \in \mathcal{PI}_n^r(\mathcal{H}) \Leftrightarrow STS^* \in \mathcal{PI}_n^r(\mathcal{H}).$$

(b) *If* S *is a co-isometry,* $TS = ST$, *and* $TS^* = S^*T$, *then*

$$T \in \mathcal{PI}_n^\ell(\mathcal{H}) \Leftrightarrow ST \in \mathcal{PI}_n^\ell(\mathcal{H}).$$

(c) *If* S *is a co-isometry and* $TS = ST$, *then*

$$T \in \mathcal{PI}_n^r(\mathcal{H}) \Leftrightarrow ST \in \mathcal{PI}_n^r(\mathcal{H}).$$

(4) (a) *If* S *is a co-isometry, then*

$$T \in \mathcal{PI}_n^\ell(\mathcal{H}) \Leftrightarrow STS^* \in \mathcal{PI}_n^\ell(\mathcal{H}), \quad T \in \mathcal{PI}_n^r(\mathcal{H}) \Leftrightarrow STS^* \in \mathcal{PI}_n^r(\mathcal{H}).$$

(b) *If* S *is an isometry,* $TS = ST$, *and* $TS^* = S^*T$, *then*

$$T \in \mathcal{PI}_n^r(\mathcal{H}) \Leftrightarrow ST \in \mathcal{PI}_n^r(\mathcal{H}).$$

(c) *If* S *is an isometry and* $TS = ST$, *then*

$$T \in \mathcal{PI}_n^\ell(\mathcal{H}) \Leftrightarrow ST \in \mathcal{PI}_n^\ell(\mathcal{H}).$$

Recall that for $T \in \mathcal{B}(\mathcal{H})$, the *ascent* and *descent* of T, denoted respectively by $a(T)$ and $d(T)$, are defined as follows:

$$a(T) = \inf\{n \geq 0 : \mathsf{N}(T^n) = \mathsf{N}(T^{n+1})\} \quad \text{and} \quad d(T) = \inf\{n \geq 0 : \mathsf{R}(T^n) = \mathsf{R}(T^{n+1})\}.$$

If the sets in the infima are empty, the infimum is taken to be $+\infty$. Furthermore, if $a(T) < +\infty$ (resp. $d(T) < +\infty$), then for all $k \in \mathbb{N}$, one has:

$$\mathsf{N}(T^{a(T)+k}) = \mathsf{N}(T^{a(T)}) \quad \text{and} \quad \mathsf{R}(T^{d(T)+k}) = \mathsf{R}(T^{d(T)}).$$

Proposition 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$.*

(1) *Suppose $a(T) \leq n-1$ and $\mathsf{R}(T^{n-1})$ is invariant under T^*T. Then,*

$$T \in \mathcal{PI}_n^{\ell}(\mathcal{H}) \Leftrightarrow T^*T\Big|_{\overline{\mathsf{R}(T^{n-1})}} = I\Big|_{\overline{\mathsf{R}(T^{n-1})}}.$$

In this case, it follows that $T^*T^n = T^{n-1}$.

(2) *Suppose $d(T) \leq n-1$ and $\mathsf{N}(T^{n-1})$ is invariant under TT^*. Then,*

$$T \in \mathcal{PI}_n^{r}(\mathcal{H}) \Longleftrightarrow TT^*\Big|_{\mathsf{N}(T^{n-1})^{\perp}} = I\Big|_{\mathsf{N}(T^{n-1})^{\perp}}.$$

In this case, it follows that $T^nT^* = T^{n-1}$.

Proof. (1) Since $a(T) \leq n-1$, we have $\mathsf{N}(T^{n-1}) = \mathsf{N}(T^n)$. Put

$$A_n := T^{*n-1}(T^*T - I)T^{n-1}.$$

Then, we observe

$$\begin{aligned}
T \in \mathcal{PI}_n^{\ell}(\mathcal{H}) &\Longleftrightarrow T^nA_n = 0 \\
&\Longleftrightarrow \mathsf{R}(A_n) \subset \mathsf{N}(T^n) \cap \mathsf{R}(T^{*n-1}) \\
&\Longleftrightarrow A_n = 0 \\
&\Longleftrightarrow \text{for all } x \in \mathcal{H}, \quad \langle T^{*n-1}(T^*T - I)T^{n-1}x, x\rangle = 0 \\
&\Longleftrightarrow \text{for all } y \in \overline{\mathsf{R}(T^{n-1})}, \quad \langle (T^*T - I)y, y\rangle = 0 \\
&\Longleftrightarrow T^*T\Big|_{\overline{\mathsf{R}(T^{n-1})}} = I\Big|_{\overline{\mathsf{R}(T^{n-1})}}.
\end{aligned}$$

Once this condition holds, we directly obtain $T^*T^n = T^{n-1}$.

(2) Since $d(T) \leq n-1$, it follows that $a(T^*) \leq n-1$. Moreover, using the assumed invariance of $\mathsf{N}(T^{n-1})$ under TT^*, we have

$$TT^*\left(\mathsf{R}(T^{*n-1})\right) \subset TT^*\left(\mathsf{N}(T^{n-1})^{\perp}\right) \subset \mathsf{N}(T^{n-1})^{\perp}.$$

Hence, the result follows from part (1) by duality and Proposition 2.1. ☐

We conclude this section by presenting a decomposition result.

Theorem 2.2. (1) *Let $T \in \mathcal{B}(\mathcal{H})$ such that $a(T) \leq n-1$ and $\mathsf{R}(T^{n-1})$ is invariant by T^*. If T is a left partial isometry of order $n \geq 2$, then there exist a partial isometry $\mathcal{P} \in \mathcal{B}(\mathcal{H})$ and a nilpotent operator $\mathcal{N} \in \mathcal{B}(\mathcal{H})$ such that*

$$T = \mathcal{P} + \mathcal{N},$$

with

$$\mathcal{N}^{n-1} = 0 \quad \text{and} \quad \mathcal{N}\mathcal{P}^* = \mathcal{P}^*\mathcal{N} = \mathcal{N}\mathcal{P} = 0.$$

Furthermore, $T_{|\overline{\mathsf{R}(T^{n-1})}}$ *is unitary.*

(2) *Let* $T \in \mathcal{B}(\mathcal{H})$ *such that* $d(T) \leq n-1$ *and* $\mathsf{N}(T^{n-1})$ *is a reducing subspace of* T*. If* T *is a right partial isometry of order* $n \geq 2$*, then there exist a partial isometry* $\mathcal{P} \in \mathcal{B}(\mathcal{H})$ *and a nilpotent operator* $\mathcal{N} \in \mathcal{B}(\mathcal{H})$ *such that* $T = \mathcal{P} + \mathcal{N}$*, with* $\mathcal{N}^{n-1} = 0$ *and* $\mathcal{P}\mathcal{N} = \mathcal{P}\mathcal{N}^* = \mathcal{N}^*\mathcal{P} = 0$*. Furthermore,* $T_{|\mathsf{N}(T^{n-1})^\perp}$ *is unitary.*

Proof. We first prove the first assertion. Consider the orthogonal projection $Q \in \mathcal{B}(\mathcal{H})$ onto $\overline{\mathsf{R}(T^{n-1})}$, and define

$$\mathcal{P} := TQ, \quad \mathcal{N} := T(I-Q).$$

By Proposition 2.5, we have

$$T^*T_{|\overline{\mathsf{R}(T^{n-1})}} = I_{|\overline{\mathsf{R}(T^{n-1})}}, \tag{2.5}$$

which implies

$$\mathcal{P}^*\mathcal{N} = \mathcal{N}\mathcal{P} = \mathcal{N}\mathcal{P}^* = 0, \quad (I-Q)T(I-Q) = T(I-Q).$$

Consequently,

$$\mathcal{N}^{n-1} = T^{n-1}(I-Q) = 0.$$

Moreover, $\mathcal{P}\mathcal{P}^*\mathcal{P} = TQT^*TQ = TQ = \mathcal{P}$, so that $\mathcal{P}$ is a partial isometry. Since $\mathcal{P}^*\mathcal{N} = \mathcal{N}\mathcal{P} = \mathcal{N}^{n-1} = 0$, we obtain

$$T^{n-1} = \sum_{k=1}^{n-1} \mathcal{P}^k\mathcal{N}^{n-1-k}, \quad \mathcal{N}\mathcal{N}^*T^{n-1} = 0.$$

As $\mathcal{P}$ is a partial isometry, it follows that

$$\mathcal{P}\mathcal{P}^*T^{n-1} = \sum_{k=1}^{n-1} \mathcal{P}^k\mathcal{N}^{n-1-k} = T^{n-1}.$$

Hence,

$$TT^*T^{n-1} = \Big(\mathcal{P}\mathcal{P}^* + \mathcal{N}\mathcal{N}^*\Big)T^{n-1} = T^{n-1}.$$

Therefore,

$$TT^*_{|\overline{\mathsf{R}(T^{n-1})}} = I_{|\overline{\mathsf{R}(T^{n-1})}}. \tag{2.6}$$

Since $\overline{\mathsf{R}(T^{n-1})}$ is a reducing subspace of T, combining (2.5) and (2.6) we obtain that

$$T_{|\overline{\mathsf{R}(T^{n-1})}} \text{ is unitary.}$$

Finally, using that T is nilpotent (resp. a partial isometry, $T \in \mathcal{PI}^r_n(\mathcal{H})$) if and only if T^* is nilpotent (resp. a partial isometry, $T^* \in \mathcal{PI}^\ell_n(\mathcal{H})$), together with

$$d(T) \leq n-1 \Rightarrow a(T^*) \leq n-1$$

and $T^*_{|\overline{\mathsf{R}\left(T^{*(n-1)}\right)}} = T^*_{|\mathsf{N}(T^{n-1})^\perp}$ is unitary if and only if $T_{|\mathsf{N}(T^{n-1})^\perp}$ is unitary, all these properties imply that the second assertion follows by duality. This completes the proof. □

3. Spectral Picture

Recall that the *spectrum* $\sigma(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set of all complex scalars $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is not invertible in the algebra $\mathcal{B}(\mathcal{H})$. In other words,

$$\sigma(T) = \{\lambda \in \mathbb{C} : \mathsf{N}(\lambda I - T) \neq \{0\} \text{ or } \mathsf{R}(\lambda I - T) \neq \mathcal{H}\}\,.$$

The subset of the spectrum consisting of those λ for which $\lambda I - T$ is not injective is called the *point spectrum* of T, and is denoted by $\sigma_p(T)$:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \mathsf{N}(\lambda I - T) \neq \{0\}\}\,.$$

Thus, the point spectrum is precisely the set of eigenvalues of T.

Another important subset of the spectrum is the *approximate point spectrum*, denoted $\sigma_{ap}(T)$, which is defined as

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \text{ exists}\,(x_n) \subset \mathcal{H},\ \|x_n\| = 1,\ (\lambda I - T)x_n \to 0 \text{ as } n \to +\infty\}\,.$$

We now recall the notion of *quasicommuting operators*, which satisfy a certain asymptotic commutation condition.

Definition 3.1 ([7])**.** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be quasicommuting if

$$\lim_{n\to+\infty} \|T^*T^n - T^nT^*\|^{1/n} = 0.$$

Quasicommuting operators have been studied extensively in the literature, and they exhibit some interesting properties. One such property is stated in the following lemma.

Lemma 3.1 ([7, 12])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasicommuting operator, and define*

$$\mathcal{H}_0 = \left\{x \in \mathcal{H} : \lim_{n\to+\infty} \|T^n x\|^{1/n} = 0\right\}.$$

Then, $\mathcal{H}_0$ is a linear subspace of $\mathcal{H}$ that is invariant under both T and T^*, and the restriction $T_{|\mathcal{H}_0^\perp}$ is normal.

Recall that the point spectrum (as well as the approximate point spectrum) of every quasicommuting partial isometry is contained in the union of the unit circle and the singleton set $\{0\}$, as established in [12, Theorems 3 and 4]. The aim of the following main theorem is to establish that this result remains valid for quasicommuting partial isometries of order n.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasicommuting right partial isometry of order $n \geq 2$ such that $d(T) \leq n-1$ and $\mathsf{N}(T^{n-1})$ is invariant under TT^*. Then,*

$$\sigma(T) \subset \partial\mathbb{D} \cup \{0\}.$$

In particular, $\overline{\sigma_p(T)} \subset \sigma_{ap}(T) \subset \partial\mathbb{D} \cup \{0\}$.

Proof. **Case 1.** Suppose that $\|T\| < 1$. By Proposition 2.5, we have

$$\|T^{n-1}\| = \|T^nT^*\| \leq \|T^{n-1}\| \cdot \|T^*T\| \leq \|T^{n-1}\| \cdot \|T\|^2.$$

Since $\|T\| < 1$, it follows that $\|T^{n-1}\| < \|T^{n-1}\|$, unless $T^{n-1} = 0$. Hence, T must be nilpotent. Consequently, $\sigma(T) = \sigma_p(T) = \sigma_{ap}(T) = \{0\}$, and the conclusion follows.

Case 2. Suppose that $\|T\| \geq 1$. Set

$$\Gamma_T := \left\{\lambda \in \mathbb{C} : \frac{1}{\|T\|} \leq |\lambda| \leq \|T\|\right\}.$$

The proof proceeds in four steps.

Step 1. (Localization of the point spectrum) We claim that $\sigma_p(T) \subset \Gamma_T \cup \{0\}$.

Let $\lambda \in \sigma_p(T) \setminus \{0\}$, and let $x \in \mathcal{H} \setminus \{0\}$ be an eigenvector, i.e., $Tx = \lambda x$. Iterating, we get $T^kx = \lambda^kx$, and therefore $x = \lambda^{-k}T^kx$ for all $k \in \mathbb{N}$.

Using again Proposition 2.5, we have $T^kT^* = T^{k-1}$ for all $k \geq n$. Hence,

$$\begin{aligned}\lim_{k\to+\infty} \|T^*T^kx - T^kT^*x\|^{1/k} &= \lim_{k\to+\infty} \|\lambda^kT^*x - \lambda^{k-1}x\|^{1/k} \\ &= |\lambda| \cdot \lim_{k\to+\infty} \|T^*x - \lambda^{-1}x\|^{1/k} = 0.\end{aligned}$$

This implies $T^*x = \lambda^{-1}x$, so $\lambda^{-1} \in \sigma(T^*)$, and therefore

$$|\lambda| \leq \|T\|,\ |\lambda^{-1}| \leq \|T\| \quad \Rightarrow \quad \lambda \in \Gamma_T.$$

Step 2. (Localization of the approximate point spectrum) We now show that $\sigma_{ap}(T) \subset \Gamma_T \cup \{0\}$, using Berberian's method [10].

Let $T' \in \mathcal{B}(\mathsf{L})$ be the Berberian extension of T, acting on a larger Hilbert space $\mathsf{L} \supset \mathcal{H}$, as constructed in [10]. The operator T' retains the properties of being a quasicommuting right partial isometry of order n. By [10, Theorem 1], we have

$$\sigma_{ap}(T) = \sigma_p(T') \subset \Gamma_{T'} \cup \{0\} = \Gamma_T \cup \{0\}.$$

Step 3. (Decomposition into unitary and quasinilpotent parts) We claim that $T = \mathcal{U} \oplus \mathcal{Q}$, where $\mathcal{U}$ is unitary and $\mathcal{Q}$ is quasinilpotent.

Let $\mathcal{Q} := T|_{\overline{\mathcal{H}_0}}$, where

$$\mathcal{H}_0 := \left\{x \in \mathcal{H} : \lim_{k\to+\infty} \|T^kx\|^{1/k} = 0\right\}.$$

Then, $\mathcal{Q}^k\mathcal{Q}^* = \mathcal{Q}^{k-1}$ for all $k \geq n$.

Indeed, by Proposition 2.5, this property holds for T on the original space $\mathcal{H}$. Since, by Lemma 3.2, $\overline{\mathcal{H}_0}$ is a reducing subspace of T, it passes to the restriction $\mathcal{Q} = T|_{\overline{\mathcal{H}_0}}$. It follows that $\mathcal{Q}$ is quasicommuting. From [12, Theorem 6] and Steps 1 and 2, we deduce $\sigma(\mathcal{Q}) \subset \Gamma_{\mathcal{Q}} \cup \{0\}$, so 0 is an isolated point of the spectrum of $\mathcal{Q}$, and hence, by [12, Theorem 2], $\mathcal{Q}$ is quasinilpotent.

Let $\mathcal{U} := T|_{\mathcal{H}_0^\perp}$. By Lemma 3.1, $\mathcal{U}$ is normal. Furthermore, Proposition 2.5 implies that

$$\|Tw\|^2 = \langle T^*Tw, w\rangle = \|w\|^2, \quad \text{for all } w \in \mathsf{N}(T^n)^\perp.$$

Since $\mathcal{H}_0^\perp \subset \mathsf{N}(T^n)^\perp$, we conclude that $\mathcal{U}$ is unitary.

Step 4. (Conclusion) From the above decomposition, we have

$$\sigma(T) = \sigma(\mathcal{U}) \cup \sigma(\mathcal{Q}) \subset \partial\mathbb{D} \cup \{0\},$$

because $\mathcal{U}$ is unitary and $\mathcal{Q}$ is quasinilpotent. This concludes the proof. □

Remark 3.1. Suppose that $a(T) \leq n-1$ and $\mathsf{R}\left(T^{n-1}\right)$ is invariant under T^*T. If $T \in \mathcal{PI}_n^\ell(\mathcal{H})$ is quasicommuting, then one similarly obtains the following results:

$$\sigma(T) \subset \partial\mathbb{D} \cup \{0\}.$$

In particular, $\overline{\sigma_p(T)} \subset \sigma_{ap}(T) \subset \partial\mathbb{D} \cup \{0\}$.

4. Conclusion and Perspectives

In this paper, we introduced the notions of left and right partial isometries of order n, which extend the classical theory of partial isometric operators. We examined their structural properties, ranges and powers, as well as their spectral characteristics.

Despite these advances, several fundamental questions remain open and deserve further investigation. We list below some promising directions for future research.

(1) **Relaxation of assumptions.** Some results in this work rely on conditions such as $a(T) \leq n-1$ with $\mathsf{R}(T^{n-1})$ invariant under T^*T, or $d(T) \leq n-1$ with $\mathsf{N}(T^{n-1})$ invariant under TT^*. It remains an open question whether analogous conclusions hold without these restrictions.

(2) **Linear preservers for $n \geq 2$.** Motivated by the results in [6, 33], a natural problem is to determine whether one can characterize linear maps on $\mathcal{B}(\mathcal{H})$ that preserve left or right partial isometries of order $n \geq 2$ in both directions.

(3) **Finite-dimensional theory.** Developing the theory of left and right partial isometries of order n in finite-dimensional Hilbert spaces to complement results such as those by S. R. Garcia et al. [17].

(4) **Extension to C^*-algebras.** Investigating partial isometries of order n in the setting of C^*-algebras, aiming to generalize the results of F. J. Fernández-Polo and A. Peralta [14].

(5) **Semi-Hilbertian spaces.** Exploring the notion of left and right partial isometries of order n within semi-Hilbertian spaces, following the approaches developed for the case $n = 1$ in [9, 15].

(6) **Factorization and topology.** Studying factorization properties and the connected components of sets of left and right partial isometries of order n, generalizing results by Mbekhta and Skhiri [26] and improving upon earlier works by P. R. Halmos and J. E. McLaughlin.

We expect that the results presented here will provide a solid foundation for continued research on left and right partial isometries of order n and inspire further developments in the theory of partial isometries and related operator classes.

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