

INTRODUCTION TO CONTINUOUS BIFRAMES IN HILBERT SPACES AND THEIR TENSOR PRODUCTS

PRASENJIT GHOSH¹ AND TAPAS K. SAMANTA²

ABSTRACT. We introduce the notion of a continuous biframe in a Hilbert space which is a generalization of discrete biframe in Hilbert space. Representation theorem for this type of generalized frame is verified and some characterizations of this biframe with the help of a invertible operator is given. Here we also introduce the concept of continuous biframe for the tensor products of Hilbert spaces and give an example. Further, we study dual continuous biframe and continuous biframe Bessel multiplier in Hilbert spaces and their tensor products.

1. INTRODUCTION AND PRELIMINARIES

The notion of a frame in Hilbert space was first introduced by Duffin and Schaeffer [5] in connection with some fundamental problem in non-harmonic analysis. Thereafter, it was further developed and popularized by Daubechies et al. [4] in 1986. A discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame element. A sequence $\{f_i\}_{i=1}^{+\infty}$ in a separable Hilbert space H is called a frame for H , if there exist positive constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{+\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The constants A and B are called lower and upper bounds, respectively.

P. Balazs [1] developed weighted and controlled frame in Hilbert space. Biframe is also a generalization of controlled frame in Hilbert space which was studied by M. F.

Key words and phrases. Frame, Dual frame, Continuous frame, biframe, tensor product.
2020 *Mathematics Subject Classification.* Primary: 42C15. Secondary: 94A12, 46C07.
DOI

Received: January 29, 2026.

Accepted: July 23, 2026.

Parizi et al. [9]. To define frame in Hilbert space, only one sequence is needed, but for a biframe, two sequences are needed. A pair of sequences $(\{f_i\}_{i=1}^{+\infty}, \{g_i\}_{i=1}^{+\infty})$ in H is called a biframe for H if there exist positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i=1}^{+\infty} \langle f, f_i \rangle \langle g_i, f \rangle \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The constants A and B are called lower and upper biframe bounds, respectively.

Continuous frames extend the concept of discrete frames when the indices are related to some measurable space. Continuous frame in Hilbert space was studied by A. Rahimi et al. [11]. M. H. Faroughi and E. Osgooei [7] also studied continuous frame and continuous Bessel mapping. Continuous frame and discrete frame have been used in image processing, coding theory, wavelet analysis, signal denoising, feature extraction, robust signal processing etc.

In this paper, we give the notion of continuous biframes in Hilbert spaces and their tensor products and then discuss some examples of this type of frame. A characterization of continuous biframe using its frame operator is established. We will see that the image of a continuous biframe under a bounded invertible operator in Hilbert space is also a continuous biframe in Hilbert space. Continuous biframe Bessel multipliers in Hilbert spaces and their tensor product are presented.

Throughout this paper, H is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$. (Ω, μ) denotes abstract measure space with positive measure μ . I_H is the identity operator on H . $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2 . In particular, $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H .

Definition 1.1 ([11]). Let H be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ . A mapping $F : \Omega \rightarrow H$ is called a continuous frame with respect to (Ω, μ) if

- (i) F is weakly-measurable, i.e., for all $f \in H$, $w \rightarrow \langle f, F(w) \rangle$ is a measurable function on Ω ,
- (ii) there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu \leq B\|f\|^2,$$

for all $f \in H$. The constants A and B are called continuous frame bounds. If $A = B$, then it is called a tight continuous frame. If the mapping F satisfies only the right inequality, then it is called continuous Bessel mapping with Bessel bound B .

Let $L^2(\Omega, \mu)$ be the class of all measurable functions $f : \Omega \rightarrow H$ such that $\|f\|_2^2 = \int_{\Omega} \|f(w)\|^2 d\mu < +\infty$. It can be proved that $L^2(\Omega, \mu)$ is a Hilbert space with respect to the inner product defined by

$$\langle f, g \rangle_{L^2} = \int_{\Omega} \langle f(w), g(w) \rangle d\mu, \quad f, g \in L^2(\Omega, \mu).$$

Theorem 1.1 ([7]). Let $F : \Omega \rightarrow H$ be a Bessel mapping. Then, the operator $T_C : L^2(\Omega, \mu) \rightarrow H$ is defined by

$$\langle T_C(\varphi), h \rangle = \int_{\Omega} \varphi(w) \langle F(w), h \rangle d\mu,$$

where $\varphi \in L^2(\Omega, \mu)$ and $h \in H$ is well-defined, linear, bounded and its adjoint operator is given by

$$T_C^* : H \rightarrow L^2(\Omega, \mu), \quad T_C^* f(w) = \langle f, F(w) \rangle, \quad f \in H, w \in \Omega.$$

The operator T_C is called a pre-frame operator or synthesis operator and its adjoint operator is called analysis operator of F .

Definition 1.2 ([7]). Let $F : \Omega \rightarrow H$ be a continuous frame for H . Then, the operator $S_C : H \rightarrow H$ defined by

$$\langle S_C(f), h \rangle = \int_{\Omega} \langle f, F(w) \rangle \langle F(w), h \rangle d\mu, \quad \text{for all } f, h \in H,$$

is called the frame operator of F .

The tensor product of Hilbert spaces are introduced by several ways and it is a certain linear space of operators which was represented by Folland in [6].

Definition 1.3 ([10]). The tensor product of Hilbert spaces $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ is denoted by $H_1 \otimes H_2$ and it is defined to be an inner product space associated with the inner product

$$(1.1) \quad \langle f \otimes g, f' \otimes g' \rangle = \langle f, f' \rangle_1 \langle g, g' \rangle_2,$$

for all $f, f' \in H_1$ and $g, g' \in H_2$. The norm on $H_1 \otimes H_2$ is given by

$$(1.2) \quad \|f \otimes g\| = \|f\|_1 \|g\|_2, \quad \text{for all } f \in H_1 \text{ and } g \in H_2.$$

The space $H_1 \otimes H_2$ is complete with respect to the above inner product. Therefore, the space $H_1 \otimes H_2$ is a Hilbert space.

For $Q \in \mathcal{B}(H_1)$ and $T \in \mathcal{B}(H_2)$, the tensor product of operators Q and T is denoted by $Q \otimes T$ and defined as

$$(Q \otimes T) A = Q A T^*, \quad \text{for all } A \in H_1 \otimes H_2.$$

It can be easily verified that $Q \otimes T \in \mathcal{B}(H_1 \otimes H_2)$ (see [6]).

Theorem 1.2 ([6]). Suppose $Q, Q' \in \mathcal{B}(H_1)$ and $T, T' \in \mathcal{B}(H_2)$. Then,

- (i) $Q \otimes T \in \mathcal{B}(H_1 \otimes H_2)$ and $\|Q \otimes T\| = \|Q\| \cdot \|T\|$,
- (ii) $(Q \otimes T)(f \otimes g) = Q(f) \otimes T(g)$ for all $f \in H_1, g \in H_2$,
- (iii) $(Q \otimes T)(Q' \otimes T') = (QQ') \otimes (TT')$,
- (iv) $Q \otimes T$ is invertible if and only if Q and T are invertible, in which case $(Q \otimes T)^{-1} = (Q^{-1} \otimes T^{-1})$,
- (v) $(Q \otimes T)^* = (Q^* \otimes T^*)$.

2. CONTINUOUS BIFRAME IN HILBERT SPACE

In this section, first we give the definition of a continuous biframe in Hilbert space and then discuss some of its properties.

Definition 2.1. Let H be a Hilbert space and (Ω, μ) be a measure space with positive measure μ . A pair $(F, G) = (F : \Omega \rightarrow H, G : \Omega \rightarrow H)$ of mappings is called a continuous biframe for H with respect to (Ω, μ) if

- (i) F, G are weakly-measurable, i. e., for all $f \in H$, $w \mapsto \langle f, F(w) \rangle$ and $w \mapsto \langle f, G(w) \rangle$ are measurable functions on Ω ,
- (ii) there exist constants $0 < A \leq B < +\infty$ such that

$$(2.1) \quad A \|f\|^2 \leq \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu \leq B \|f\|^2,$$

for all $f \in H$. The constants A and B are called continuous biframe bounds. If $A = B$, then it is called a tight continuous biframe and it is called Parseval continuous biframe if $A = B = 1$. If the pair (F, G) satisfies only the right inequality, then it is called continuous biframe Bessel mapping with Bessel bound B .

In particular, if μ is the counting measure and $\Omega = \mathbb{N}$, then (F, G) is a discrete biframe for H .

Now, we validate the above definition by some examples.

Example 2.1. Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for H . Consider

$$\Omega = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}.$$

Then, it is a measure space equipped with the Lebesgue measure μ . Suppose $\{B_1, B_2, B_3\}$ is a partition of Ω where $\mu(B_1) \geq \mu(B_2) \geq \mu(B_3) > 1$. Define $F : \Omega \rightarrow H$ by

$$F(w) = \begin{cases} \frac{e_1}{\sqrt{\mu(B_1)}}, & \text{if } w \in B_1, \\ \frac{e_2}{\sqrt{\mu(B_2)}}, & \text{if } w \in B_2, \\ \frac{2e_3}{\sqrt{\mu(B_3)}}, & \text{if } w \in B_3, \end{cases}$$

and $G : \Omega \rightarrow H$ by

$$G(w) = \begin{cases} \frac{2e_1}{\sqrt{\mu(B_1)}}, & \text{if } w \in B_1, \\ \frac{e_2}{\sqrt{\mu(B_2)}}, & \text{if } w \in B_2, \\ \frac{e_3}{2\sqrt{\mu(B_3)}}, & \text{if } w \in B_3. \end{cases}$$

It is easy to verify that for all $f \in H$, $w \mapsto \langle f, F(w) \rangle$ and $w \mapsto \langle f, G(w) \rangle$ are measurable functions on Ω . Now, for $f \in H$, we have

$$\begin{aligned} \int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu &= \int_{B_1} \left| \left\langle f, \frac{1}{\sqrt{\mu(B_1)}} e_1 \right\rangle \right|^2 d\mu + \int_{B_2} \left| \left\langle f, \frac{1}{\sqrt{\mu(B_2)}} e_2 \right\rangle \right|^2 d\mu \\ &\quad + \int_{B_3} \left| \left\langle f, \frac{1}{\sqrt{\mu(B_3)}} 2e_1 \right\rangle \right|^2 d\mu \\ &= |\langle f, e_1 \rangle|^2 + |\langle f, e_2 \rangle|^2 + 4 |\langle f, e_3 \rangle|^2 \\ &= \|f\|^2 + 3 |\langle f, e_3 \rangle|^2. \end{aligned}$$

Therefore, $F : \Omega \rightarrow H$ is a continuous frame for H with bounds 1 and 4. Similarly, it can be shown that $G : \Omega \rightarrow H$ is a continuous frame for H with bounds 1/4 and 4. On the other hand, for $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu &= \int_{B_1} \left\langle f, \frac{e_1}{\sqrt{\mu(B_1)}} \right\rangle \left\langle \frac{2e_1}{\sqrt{\mu(B_1)}}, f \right\rangle d\mu \\ &\quad + \int_{B_2} \left\langle f, \frac{e_2}{\sqrt{\mu(B_2)}} \right\rangle \left\langle \frac{e_2}{\sqrt{\mu(B_2)}}, f \right\rangle d\mu \\ &\quad + \int_{B_3} \left\langle f, \frac{2e_3}{\sqrt{\mu(B_3)}} \right\rangle \left\langle \frac{e_3}{2\sqrt{\mu(B_3)}}, f \right\rangle d\mu \\ &= 2 |\langle f, e_1 \rangle|^2 + |\langle f, e_2 \rangle|^2 + |\langle f, e_3 \rangle|^2 \\ &= \|f\|^2 + |\langle f, e_1 \rangle|^2. \end{aligned}$$

Therefore, (F, G) is a continuous biframe for H with bounds 1 and 2.

Example 2.2. Let H be an infinite dimensional separable Hilbert space and $\{e_i\}_{i=1}^{+\infty}$ be an orthonormal basis for H . Suppose

$$\begin{aligned} \{f_i\}_{i=1}^{+\infty} &= \{e_1, e_1, e_1, e_2, e_3, \dots\}, \\ \{g_i\}_{i=1}^{+\infty} &= \{0, e_1, e_1, e_2, e_3, \dots\}. \end{aligned}$$

Now, let (Ω, μ) be a measure space with μ is σ -finite. Then, we can write $\Omega = \bigcup_{i=1}^{+\infty} \Omega_i$, where $\{\Omega_i\}_{i=1}^{+\infty}$ is a sequence of disjoint measurable subsets of Ω with $\mu(\Omega_i) < +\infty$. We define the mappings $F : \Omega \rightarrow H$ by $F(w) = \frac{1}{\sqrt{\mu(\Omega_i)}} f_i$ and $G : \Omega \rightarrow H$ by $G(w) = \frac{1}{\sqrt{\mu(\Omega_i)}} g_i$, for $w \in \Omega_i$. Then, for $f \in H$, we have

$$\begin{aligned} \int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu &= \sum_{i=1}^{+\infty} \int_{\Omega_i} |\langle f, f_i \rangle|^2 d\mu = 2 |\langle f, e_1 \rangle|^2 + \sum_{i=1}^{+\infty} |\langle f, e_i \rangle|^2 \\ &= 2 |\langle f, e_1 \rangle|^2 + \|f\|^2. \end{aligned}$$

Therefore, F is a continuous frame for H with bounds 1 and 3. Similarly, it can be shown that G is a continuous frame for H with bounds 1 and 2. Now, for $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu &= \sum_{i=1}^{+\infty} \int_{\Omega_i} \langle f, f_i \rangle \langle g_i, f \rangle d\mu \\ &= \langle f, e_1 \rangle \langle e_1, f \rangle + \langle f, e_1 \rangle \langle e_1, f \rangle + \langle f, e_2 \rangle \langle e_2, f \rangle + \cdots \\ &= \langle f, e_1 \rangle \langle e_1, f \rangle + \langle f, \langle f, e_1 \rangle e_1 \rangle + \langle f, \langle f, e_2 \rangle e_2 \rangle + \cdots \\ &= |\langle f, e_1 \rangle|^2 + \left\langle f, \sum_{i=1}^{+\infty} \langle f, e_i \rangle e_i \right\rangle \\ &= |\langle f, e_1 \rangle|^2 + \langle f, f \rangle = |\langle f, e_1 \rangle|^2 + \|f\|^2. \end{aligned}$$

Thus, (F, G) is a continuous biframe for H with bounds 1 and 2.

Now, in this example, we take

$$\begin{aligned} \{f_i\}_{i=1}^{+\infty} &= \left\{ \frac{1}{3}e_1, \frac{1}{3}e_1, e_2, e_2, e_3, e_4, \dots \right\}, \\ \{g_i\}_{i=1}^{+\infty} &= \left\{ -\frac{1}{2}e_1, \frac{1}{2}e_1, -\frac{1}{5}e_2, \frac{1}{5}e_2, e_3, e_4, \dots \right\}, \end{aligned}$$

and define $F, G : \Omega \rightarrow H$ as above. Then, it is easy to verify that F is a continuous frame for H with bounds $2/9$, 2 and G is a continuous frame for H with bounds $2/25$, 1 . Now, we see that (F, G) is not a continuous biframe for H . For $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu &= \sum_{i=1}^{+\infty} \int_{\Omega_i} \langle f, f_i \rangle \langle g_i, f \rangle d\mu \\ &= \left\langle f, \frac{1}{3}e_1 \right\rangle \left\langle -\frac{1}{2}e_1, f \right\rangle + \left\langle f, \frac{1}{3}e_1 \right\rangle \left\langle \frac{1}{2}e_1, f \right\rangle \\ &\quad + \langle f, e_2 \rangle \left\langle -\frac{1}{5}e_2, f \right\rangle + \langle f, e_2 \rangle \left\langle \frac{1}{5}e_2, f \right\rangle \\ &\quad + \langle f, e_3 \rangle \langle e_3, f \rangle + \langle f, e_4 \rangle \langle e_4, f \rangle + \cdots \\ &= \sum_{i=3}^{+\infty} |\langle f, e_i \rangle|^2. \end{aligned}$$

Now, for $f = e_1, e_2$, we have $\int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu = 0$. This shows that (F, G) does not have a non-zero lower bound. Thus, (F, G) is not a continuous biframe for H .

Next, we give an example of a continuous biframe for a real inner product space.

Example 2.3. Let

$$V = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}^+ \right\}.$$

Define $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}^+$ by $\langle M, N \rangle = \text{trace}(MN^t)$, for all $M, N \in V$. Then, it is easy to verify that $\langle \cdot, \cdot \rangle$ is an real inner product on V . Now, we consider the measure space $(\Omega = [0, 1], \mu)$ where μ is the Lebesgue measure. Define $F : \Omega \rightarrow V$ by

$$F(w) = \begin{pmatrix} 2(1-w) & 0 \\ 0 & (1+w) \end{pmatrix}, \quad w \in \Omega,$$

and $G : \Omega \rightarrow V$ by

$$G(w) = \begin{pmatrix} 5(1-w) & 0 \\ 0 & 5(1-w) \end{pmatrix}, \quad w \in \Omega.$$

It is easy to verify that for all $M \in V$, $w \mapsto \langle M, F(w) \rangle$ and $w \mapsto \langle M, G(w) \rangle$ are measurable functions on Ω . Now, for each $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in V$, we get

$$\begin{aligned} \langle M, F(w) \rangle &= \text{trace} \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 2(1-w) & 0 \\ 0 & (1+w) \end{pmatrix}^t \right\} \\ &= \text{trace} \left\{ \begin{pmatrix} 2a(1-w) & 0 \\ 0 & b(1+w) \end{pmatrix} \right\} \\ &= 2a(1-w) + b(1+w), \\ \langle G(w), M \rangle &= \text{trace} \left\{ \begin{pmatrix} 5(1-w) & 0 \\ 0 & 5(1-w) \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^t \right\} \\ &= 5(1-w)(a+b). \end{aligned}$$

Thus, for each $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in V$, we have

$$\begin{aligned} \int_{[0,1]} \langle M, F(w) \rangle \langle G(w), M \rangle d\mu &= \int_{[0,1]} 5(a+b)(1-w) \{2a(1-w) + b(1+w)\} d\mu \\ &= \frac{10}{3}(a+b)^2. \end{aligned}$$

Also, it is easy to verify that $\|M\|^2 = a^2 + b^2$. Thus, for each $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in V$, we have

$$(a^2 + b^2) \leq \int_{[0,1]} \langle M, F(w) \rangle \langle G(w), M \rangle d\mu \leq \frac{20}{3}(a^2 + b^2).$$

Therefore, (F, G) is a continuous biframe for H with bounds 1 and $20/3$.

Let $(F, G) = (F : \Omega \rightarrow H, G : \Omega \rightarrow H)$ be a continuous biframe for H with respect to (Ω, μ) . Then, the mapping $\Psi : H \times H \rightarrow \mathbb{C}$ defined by

$$\Psi(f, g) = \int_{\Omega} \langle f, F(w) \rangle \langle G(w), g \rangle d\mu$$

is well-defined, a sesquilinear form (i.e., linear in the first and conjugate-linear in the second variable) and is bounded. By Cauchy-Schwartz inequality, we get

$$\begin{aligned} |\Psi(f, g)| &\leq \int_{\Omega} |\langle f, F(w) \rangle \langle G(w), g \rangle| d\mu \\ &\leq \left(\int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu \right)^{1/2} \left(\int_{\Omega} |\langle G(w), g \rangle|^2 d\mu \right)^{1/2} \\ &\leq B \|f\| \cdot \|g\|. \end{aligned}$$

By Theorem 2.3.6 in [8], there exists a unique operator $S_{F,G} : H \rightarrow H$ such that

$$\Psi(f, g) = \langle S_{F,G} f, g \rangle, \quad \text{for all } f, g \in H,$$

and moreover $\|\Psi\| = \|S_{F,G}\|$.

Now, we introduce the continuous biframe operator and give some properties.

Definition 2.2. Let $(F, G) = (F : \Omega \rightarrow H, G : \Omega \rightarrow H)$ be a continuous biframe for H with respect to (Ω, μ) . Then, the operator $S_{F,G} : H \rightarrow H$ defined by

$$S_{F,G} f = \int_{\Omega} \langle f, F(w) \rangle G(w) d\mu,$$

for all $f \in H$ is called the biframe operator.

Now, for each $f \in H$, we have

$$(2.2) \quad \langle S_{F,G} f, f \rangle = \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu.$$

Thus, for each $f \in H$, we get

$$A \|f\|^2 \leq \langle S_{F,G} f, f \rangle \leq B \|f\|^2.$$

This implies that $AI \leq S_{F,G} \leq BI$, where I is the identity operator on H . Hence, $S_{F,G}$ is positive and invertible. In frame theory, reconstruction of the elements of a Hilbert space is a most important fact. Now, using continuous biframe operator $S_{F,G}$, we give the reconstruction formula, i.e., every $f \in H$ has the representations

$$\begin{aligned} f &= S_{F,G} S_{F,G}^{-1} f = \int_{\Omega} \langle f, S_{F,G}^{-1} F(w) \rangle G(w) d\mu, \\ f &= S_{F,G}^{-1} S_{F,G} f = \int_{\Omega} \langle f, F(w) \rangle S_{F,G}^{-1} G(w) d\mu. \end{aligned}$$

Example 2.4. Let $H = \mathbb{R}^3$. Suppose

$$\begin{aligned} \{f_i\}_{i=1}^3 &= \{(2, 1, 1), (-1, 3, -1), (-1, 1, 4)\}, \\ \{g_i\}_{i=1}^3 &= \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}. \end{aligned}$$

Consider

$$\Omega = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}.$$

Then, (Ω, μ) is a measure space, where μ is the Lebesgue measure. Suppose $\{B_1, B_2, B_3\}$ is a partition of Ω where $\mu(B_1) \geq \mu(B_2) \geq \mu(B_3) > 1$. Now, we define $F : \Omega \rightarrow H$ by

$$F(w) = \begin{cases} \frac{f_1}{\sqrt{\mu(B_1)}}, & \text{if } w \in B_1, \\ \frac{f_2}{\sqrt{\mu(B_2)}}, & \text{if } w \in B_2, \\ \frac{f_3}{\sqrt{\mu(B_3)}}, & \text{if } w \in B_3, \end{cases}$$

and $G : \Omega \rightarrow H$ by

$$G(w) = \begin{cases} \frac{g_1}{\sqrt{\mu(B_1)}}, & \text{if } w \in B_1, \\ \frac{g_2}{\sqrt{\mu(B_2)}}, & \text{if } w \in B_2, \\ \frac{g_3}{\sqrt{\mu(B_3)}}, & \text{if } w \in B_3. \end{cases}$$

It is easy to verify that for all $f \in H$, $w \mapsto \langle f, F(w) \rangle$ and $w \mapsto \langle f, G(w) \rangle$ are measurable functions on Ω . Now, for $f = (x, y, z) \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu &= \int_{B_1} \left\langle f, \frac{f_1}{\sqrt{\mu(B_1)}} \right\rangle \left\langle \frac{g_1}{\sqrt{\mu(B_1)}}, f \right\rangle d\mu \\ &\quad + \int_{B_2} \left\langle f, \frac{f_2}{\sqrt{\mu(B_2)}} \right\rangle \left\langle \frac{g_2}{\sqrt{\mu(B_2)}}, f \right\rangle d\mu \\ &\quad + \int_{B_3} \left\langle f, \frac{f_3}{\sqrt{\mu(B_3)}} \right\rangle \left\langle \frac{g_3}{\sqrt{\mu(B_3)}}, f \right\rangle d\mu \\ &= \langle f, f_1 \rangle \langle g_1, f \rangle + \langle f, f_2 \rangle \langle g_2, f \rangle + \langle f, f_3 \rangle \langle g_3, f \rangle \\ &= \langle (x, y, z), (2, 1, 1) \rangle \langle (1, 0, 0), (x, y, z) \rangle + \\ &\quad + \langle (x, y, z), (-1, 3, -1) \rangle \langle (0, 1, 0), (x, y, z) \rangle \\ &\quad + \langle (x, y, z), (-1, 1, 4) \rangle \langle (0, 0, 1), (x, y, z) \rangle \\ &= (2x + y + z)x + (-x + 3y - z)y + (-x + y + 4z)z \\ &= 2x^2 + 3y^2 + 4z^2 \\ &\leq 4(x^2 + y^2 + z^2) = 4\|(x, y, z)\|^2 = 4\|f\|^2. \end{aligned}$$

Thus, for each $f \in H$, we get

$$2\|f\|^2 \leq \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu \leq 4\|f\|^2.$$

Therefore, (F, G) is a continuous biframe for H with bounds 2 and 4.

For $(x, y, z) \in \mathbb{R}^3$, the continuous biframe operator $S_{F,G}$ is given by

$$\begin{aligned} S_{F,G}(x, y, z) &= \langle (x, y, z), (2, 1, 1) \rangle (1, 0, 0) + \\ &\quad + \langle (x, y, z), (-1, 3, -1) \rangle (0, 1, 0) \\ &\quad + \langle (x, y, z), (-1, 1, 4) \rangle (0, 0, 1) \\ &= (2x + y + z, -x + 3y - z, -x + y + 4z). \end{aligned}$$

The matrix associated with the operator $S_{F,G}$ is given by

$$[S_{F,G}] = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 3 & -1 \\ -1 & 1 & 4 \end{pmatrix}.$$

Since $\det([S_{F,G}]) = 33 \neq 0$, the matrix $[S_{F,G}]$ is invertible. Thus, the operator $S_{F,G}$ is well defined and invertible bounded linear operator on $\mathbb{R}^3$. Here, $S_{F,G}S_{F,G}^* \neq S_{F,G}^*S_{F,G}$ and hence the operator $S_{F,G}$ is not self-adjoint.

The inverse of the matrix $[S_{F,G}]$ is given by

$$[S_{F,G}]^{-1} = \frac{1}{33} \begin{pmatrix} 13 & -3 & -4 \\ 5 & 9 & 1 \\ 2 & -3 & 7 \end{pmatrix}.$$

Therefore, for $(x, y, z) \in \mathbb{R}^3$, $S_{F,G}^{-1}$ is given by

$$S_{F,G}^{-1}(x, y, z) = \frac{1}{33} (13x - 3y - 4z, 5x + 9y + z, 2x - 3y + 7z).$$

Now, for $f = (x, y, z) \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, F(w) \rangle S_{F,G}^{-1} G(w) d\mu &= \langle f, f_1 \rangle S_{F,G}^{-1} g_1 + \langle f, f_2 \rangle S_{F,G}^{-1} g_2 + \langle f, f_3 \rangle S_{F,G}^{-1} g_3 \\ &= (2x + y + z) \frac{1}{33} (13, 5, 2) + (-x + 3y - z) \frac{1}{33} (-3, 9, -3) \\ &\quad + (-x + y + 4z) \frac{1}{33} (-4, 1, 7) \\ &= \frac{1}{33} (33x, 33y, 33z) = (x, y, z) = f \end{aligned}$$

Similarly, it can be verified that

$$\int_{\Omega} \langle f, S_{F,G}^{-1} F(w) \rangle G(w) d\mu = f, \quad f \in H.$$

Thus, the representation theorem is verified in this example.

We know that every positive operator in a complex Hilbert space is self-adjoint. Thus, we have the self-adjointness property for the positive operator $S_{F,G}$ in complex Hilbert space.

Theorem 2.1. *The pair (F, G) is a continuous biframe for a complex Hilbert space H with respect to (Ω, μ) if and only if (G, F) is a continuous biframe for H with respect to (Ω, μ) .*

Proof. Let (F, G) is a continuous biframe for H with bounds A and B . Then for each $f \in H$, we have

$$A \|f\|^2 \leq \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu \leq B \|f\|^2.$$

Since $S_{F,G}$ is self adjoint, using (2.2), we can write

$$\begin{aligned} \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu &= \langle S_{F,G} f, f \rangle = \overline{\langle S_{F,G} f, f \rangle} \\ &= \overline{\int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu} \\ &= \int_{\Omega} \langle f, G(w) \rangle \langle F(w), f \rangle d\mu. \end{aligned}$$

Thus, for each $f \in H$, we have

$$A \|f\|^2 \leq \int_{\Omega} \langle f, G(w) \rangle \langle F(w), f \rangle d\mu \leq B \|f\|^2.$$

Therefore, (G, F) is a continuous biframe for H .

Similarly, we can prove the converse part of this theorem. $\square$

In the next theorem, we establish a characterization of a continuous biframe using its biframe operator.

Theorem 2.2. *Let (F, G) be a continuous biframe Bessel mapping for H with respect to (Ω, μ) . Then, (F, G) is a continuous biframe for H if and only if there exists $\alpha > 0$ such that $S_{F,G} \geq \alpha I$, where $S_{F,G}$ is the continuous biframe operator for (F, G) .*

Proof. Let (F, G) be a continuous biframe for H with bounds A and B . Then, using (2.1) and (2.2), for each $f \in H$, we get

$$A \|f\|^2 \leq \langle S_{F,G} f, f \rangle \leq B \|f\|^2.$$

Thus, $A \langle f, f \rangle \leq \langle S_{F,G} f, f \rangle$ implies $S_{F,G} \geq \alpha I$, where $\alpha = A$.

Conversely, suppose that $S_{F,G} \geq \alpha I$. Thus, for each $f \in H$, we have

$$\alpha \|f\|^2 \leq \langle S_{F,G} f, f \rangle = \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu.$$

Hence, (F, G) is a continuous biframe for H with lower biframe bound α^2 . This completes the proof. $\square$

Next, also we give a characterization of a continuous biframe with the help of a invertible operator on H .

Theorem 2.3. *Let T be an invertible bounded linear operator on H . Then, (F, G) is a continuous biframe for H with respect to (Ω, μ) if and only if (TF, TG) is a continuous biframe for H with respect to (Ω, μ) .*

Proof. For each $f \in H$, $w \mapsto \langle f, TF(w) \rangle = \langle T^*f, F(w) \rangle$ and $w \mapsto \langle f, TG(w) \rangle = \langle T^*f, G(w) \rangle$ are measurable functions on Ω . Let (F, G) is a continuous biframe for H with bounds A and B . Since T is invertible, for $f \in H$, we have

$$\|f\|^2 = \|(T^{-1})^* T^* f\|^2 \leq \|T^{-1}\|^2 \|T^* f\|^2.$$

Now, for each $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, TF(w) \rangle \langle TG(w), f \rangle d\mu &= \int_{\Omega} \langle T^* f, F(w) \rangle \langle G(w), T^* f \rangle d\mu \\ &\leq B \|T^* f\|^2 \leq B \|T\|^2 \|f\|^2. \end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, TF(w) \rangle \langle TG(w), f \rangle d\mu &= \int_{\Omega} \langle T^* f, F(w) \rangle \langle G(w), T^* f \rangle d\mu \\ &\geq A \|T^* f\|^2 \geq A \|T^{-1}\|^{-2} \|f\|^2. \end{aligned}$$

Hence, (TF, TG) is a continuous biframe for H with bounds $A \|T^{-1}\|^{-2}$ and $B \|T\|^2$.

Conversely, suppose that (TF, TG) is a continuous biframe for H with bounds A and B . Now, for each $f \in H$, we have

$$\begin{aligned} \frac{A}{\|T\|^2} \|f\|^2 &= \frac{A}{\|T\|^2} \|(T^{-1}T)^* f\|^2 \leq A \|(T^{-1})^* f\|^2 \\ &\leq \int_{\Omega} \langle (T^{-1})^* f, TF(w) \rangle \langle TG(w), (T^{-1})^* f \rangle d\mu \\ &= \int_{\Omega} \langle T^* (T^{-1})^* f, F(w) \rangle \langle G(w), T^* (T^{-1})^* f \rangle d\mu \\ &= \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu. \end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned} \int_{\Omega} \langle f, F(w) \rangle \langle G(w), f \rangle d\mu &= \int_{\Omega} \langle T^* (T^{-1})^* f, F(w) \rangle \langle G(w), T^* (T^{-1})^* f \rangle d\mu \\ &= \int_{\Omega} \langle (T^{-1})^* f, TF(w) \rangle \langle TG(w), (T^{-1})^* f \rangle d\mu \\ &\leq B \|(T^{-1})^* f\|^2 \leq B \|T^{-1}\|^2 \|f\|^2. \end{aligned}$$

Thus, (F, G) is a continuous biframe for H with bounds $\frac{A}{\|T\|^2}$ and $B \|T^{-1}\|^2$. This completes the proof. $\square$

The notion of multipliers for continuous frame for the first time introduced in [2]. In this section, we improve and extend it the framework to biframes.

Definition 2.3. Let (F, F) and (G, G) be continuous biframe Bessel mappings for H with respect to (Ω, μ) and let $m : \Omega \rightarrow \mathbb{C}$ be a measurable function. Then, the operator $M_{m,F,G} : H \rightarrow H$ defined by

$$\langle M_{m,F,G}f, g \rangle = \int_{\Omega} m(w) \langle f, F(w) \rangle \langle G(w), g \rangle d\mu,$$

for all $f, g \in H$, is called continuous biframe Bessel multiplier of F and G with respect to m .

Theorem 2.4. *The continuous biframe Bessel multiplier of F and G with respect to m is well defined and bounded.*

Proof. Let (F, F) and (G, G) be continuous biframe Bessel mappings for H with bounds B_1 and B_2 . Then for any $f, g \in H$, we have

$$\begin{aligned} |\langle M_{m,F,G}f, g \rangle| &= \left| \int_{\Omega} m(w) \langle f, F(w) \rangle \langle G(w), g \rangle d\mu \right| \\ &\leq \|m\|_{\infty} \left(\int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu \right)^{1/2} \left(\int_{\Omega} |\langle g, G(w) \rangle|^2 d\mu \right)^{1/2} \\ &\leq \|m\|_{\infty} \sqrt{B_1 B_2} \|f\| \cdot \|g\|. \end{aligned}$$

This shows that $\|M_{m,F,G}\| \leq \|m\|_{\infty} \sqrt{B_1 B_2}$ and so $M_{m,F,G}$ is well-defined and bounded. This completes the proof. $\square$

Following the proof of the Theorem 2.4, for each $f \in H$, we have

$$(2.3) \quad \|M_{m,F,G}f\| = \sup_{\|g\|=1} |\langle M_{m,F,G}f, g \rangle| \leq \|m\|_{\infty} \sqrt{B_2} \left(\int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu \right)^{1/2}$$

and similarly it can be shown that

$$(2.4) \quad \|M_{m,F,G}^*g\| \leq \|m\|_{\infty} \sqrt{B_1} \left(\int_{\Omega} |\langle G(w), g \rangle|^2 d\mu \right)^{1/2}.$$

Theorem 2.5. *Let $M_{m,F,G}$ be the continuous biframe Bessel multiplier of F and G with respect to m . Then, $M_{m,F,G}^* = M_{\bar{m},F,G}$.*

Proof. For $f, g \in H$, we have

$$\begin{aligned}\langle f, M_{m,F,G}^* g \rangle &= \langle M_{m,F,G} f, g \rangle = \int_{\Omega} m(w) \langle f, F(w) \rangle \langle G(w), g \rangle d\mu \\ &= \int_{\Omega} \langle f, \overline{m}(w) \langle g, G(w) \rangle F(w) \rangle d\mu \\ &= \langle f, M_{\overline{m},F,G} g \rangle.\end{aligned}$$

This completes the proof. $\square$

Theorem 2.6. *Let $M_{m,F,G}$ be the continuous biframe Bessel multiplier of F and G with respect to m . Then, (F, F) is a continuous biframe for H provided for each $f \in H$, there exists $D > 0$ such that $\|M_{m,F,G} f\| \geq D \|f\|$.*

Proof. For each $f \in H$, using (2.3), we get

$$\begin{aligned}D^2 \|f\|^2 &\leq \|M_{m,F,G} f\|^2, \\ D^2 \|f\|^2 &\leq \|m\|_{\infty}^2 B_2 \int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu, \\ \frac{D^2}{\|m\|_{\infty}^2 B_2} \|f\|^2 &\leq \int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu, \\ \frac{D^2}{\|m\|_{\infty}^2 B_2} \|f\|^2 &\leq \int_{\Omega} \langle f, F(w) \rangle \langle F(w), f \rangle d\mu.\end{aligned}$$

Thus, (F, F) is a continuous biframe for H with bounds $\frac{D^2}{\|m\|_{\infty}^2 B_2}$ and B_1 . This completes the proof. $\square$

Theorem 2.7. *Let $M_{m,F,G}$ be the continuous biframe Bessel multiplier of F and G with respect to m . Suppose $\lambda_1 < 1$, $\lambda_2 > -1$ such that for each $f \in H$, we have*

$$\|f - M_{m,F,G} f\| \leq \lambda_1 \|f\| + \lambda_2 \|M_{m,F,G} f\|.$$

Then, (F, F) is a continuous biframe for H .

Proof. For each $f \in H$, we have

$$\|f\| - \|M_{m,F,G} f\| \leq \|f - M_{m,F,G} f\| \leq \lambda_1 \|f\| + \lambda_2 \|M_{m,F,G} f\|$$

implies $(1 - \lambda_1) \|f\| \leq (1 + \lambda_2) \|M_{m,F,G} f\|$. Now, using (2.3), we get

$$\left(\frac{1 - \lambda_1}{1 + \lambda_2} \right) \|f\| \leq \|m\|_{\infty} \sqrt{B_2} \left(\int_{\Omega} |\langle f, F(w) \rangle|^2 d\mu \right)^{1/2}$$

implies

$$(2.5) \quad \frac{(1 - \lambda_1)^2}{\|m\|_{\infty}^2 B_2 (1 + \lambda_2)^2} \|f\|^2 \leq \int_{\Omega} \langle f, F(w) \rangle \langle F(w), f \rangle d\mu.$$

Thus, (F, F) is a continuous biframe for H with bounds $\frac{(1-\lambda_1)^2}{\|m\|_\infty^2 B_2 (1+\lambda_2)^2}$ and B_1 . This completes the proof. $\square$

Theorem 2.8. *Let $M_{m,F,G}$ be the continuous biframe Bessel multiplier of F and G with respect to m . Suppose $\lambda \in [0, 1)$ such that for each $f \in H$, we have*

$$\|f - M_{m,F,G}f\| \leq \lambda \|f\|.$$

Then, (F, G) and (G, G) are continuous biframes for H .

Proof. Putting $\lambda_1 = \lambda$ and $\lambda_2 = 0$ in (2.5), we get

$$\frac{(1-\lambda)^2}{\|m\|_\infty^2 B_2} \|f\|^2 \leq \int_{\Omega} \langle f, F(w) \rangle \langle F(w), f \rangle d\mu.$$

Thus, (F, F) is a continuous biframe for H . On the other hand, for each $f \in H$, we have

$$\|f - M_{m,F,G}^*f\| = \|(I - M_{m,F,G})^*f\| \leq \|I - M_{m,F,G}\| \cdot \|f\| \leq \lambda \|f\|$$

implies $(1-\lambda)\|f\| \leq \|M_{m,F,G}^*f\|$. Now, using (2.4), we get

$$\frac{(1-\lambda)^2}{\|m\|_\infty^2 B_1} \|f\|^2 \leq \int_{\Omega} |\langle G(w), f \rangle|^2 d\mu = \int_{\Omega} \langle f, G(w) \rangle \langle G(w), f \rangle d\mu.$$

Thus, (G, G) is a continuous biframe for H with bounds $\frac{(1-\lambda)^2}{\|m\|_\infty^2 B_1}$ and B_2 . This completes the proof. $\square$

Definition 2.4. Let (F, G) be a continuous biframe for H . If

$$\langle f, g \rangle = \int_{\Omega} \langle f, F(w) \rangle \langle G(w), g \rangle d\mu,$$

holds for all $f, g \in H$, then (F, G) is called dual continuous biframe for H .

Theorem 2.9. *Let (F, G) be a continuous biframe for H with continuous biframe operator $S_{F,G}$. Then, $(S_{F,G}^{-1}F, G)$ and $(F, S_{F,G}^{-1}G)$ are dual continuous biframes for H .*

Proof. For each $f, g \in H$, we have

$$\begin{aligned} \langle f, g \rangle &= \int_{\Omega} \langle f, S_{F,G}^{-1}F(w) \rangle \langle G(w), g \rangle d\mu, \\ \langle f, g \rangle &= \int_{\Omega} \langle f, F(w) \rangle \langle S_{F,G}^{-1}G(w), g \rangle d\mu. \end{aligned}$$

This verifies that $(S_{F,G}^{-1}F, G)$ and $(F, S_{F,G}^{-1}G)$ are dual continuous biframes for H . $\square$

In the following theorem, we will find a dual continuous biframe for H with respect to the multiplier operator.

Theorem 2.10. *Let $M_{m,F,G}$ be invertible and (F, G) be a continuous biframe for H . Then, $\left((M_{m,F,G}^{-1}\overline{m}F)^*, G\right)$ is a dual continuous biframe for H .*

Proof. From the definition of $M_{m,F,G}$, we can write

$$\langle M_{m,F,G}f, g \rangle = \int_{\Omega} m(w) \langle f, F(w) \rangle \langle G(w), g \rangle d\mu.$$

Now, by replacing f with $M_{m,F,G}^{-1}f$, we get

$$\begin{aligned} \langle f, g \rangle &= \int_{\Omega} m(w) \langle M_{m,F,G}^{-1}f, F(w) \rangle \langle G(w), g \rangle d\mu \\ &= \int_{\Omega} \langle f, (M_{m,F,G}^{-1})^* \overline{m}(w)F(w) \rangle \langle G(w), g \rangle d\mu. \end{aligned}$$

Thus, $\left((M_{m,F,G}^{-1}\overline{m}F)^*, G\right)$ is a dual continuous biframe for H . This completes the proof. $\square$

3. CONTINUOUS BIFRAME IN $H_1 \otimes H_2$

In this section, we introduce the concept of continuous biframe in tensor product of Hilbert spaces $H_1 \otimes H_2$ and give a characterization.

Definition 3.1. Let $(X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)$ be the product of measure spaces with σ -finite positive measures μ_1, μ_2 on X_1, X_2 , respectively.

A pair $(\mathbf{F}, \mathbf{G}) = (\mathbb{F} : X \rightarrow H_1 \otimes H_2, \mathbb{G} : X \rightarrow H_1 \otimes H_2)$ is called a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) if the following hold.

- (i) $\mathbb{F}, \mathbb{G}$ is weakly-measurable, i.e., for all $f \otimes g \in H_1 \otimes H_2$, $x = (x_1, x_2) \mapsto \langle f \otimes g, \mathbb{F}(x) \rangle$ and $(x_1, x_2) \mapsto \langle f \otimes g, \mathbb{G}(x) \rangle$ are measurable functions on X .
- (ii) There exist constants $A, B > 0$ such that

$$(3.1) \quad A \|f \otimes g\|^2 \leq \int_X \langle f \otimes g, \mathbb{F}(x) \rangle \langle \mathbb{G}(x), f \otimes g \rangle d\mu \leq B \|f \otimes g\|^2,$$

for all $f \otimes g \in H_1 \otimes H_2$. The constants A and B are called continuous biframe bounds. If $A = B$, then the pair $(\mathbf{F}, \mathbf{G})$ is called a tight continuous biframe for $H_1 \otimes H_2$. If $(\mathbf{F}, \mathbf{G})$ satisfies only the right inequality of (1.3), then it is called continuous biframe Bessel mapping in $H_1 \otimes H_2$ with Bessel bound B .

Theorem 3.1. *The pair of mappings $(\mathbf{F}, \mathbf{G}) = (F_1 \otimes F_2, G_1 \otimes G_2) = \mathbb{F}, \mathbb{G} : X \rightarrow H_1 \otimes H_2$ is a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) if and only if $F_1, G_1 : X_1 \rightarrow H_1$ is a continuous biframe for H_1 with respect to (X_1, μ_1) and $F_2, G_2 : X_2 \rightarrow H_2$ is a continuous biframe for H_2 with respect to (X_2, μ_2) .*

Proof. Suppose that $(\mathbf{F}, \mathbf{G}) = (F_1 \otimes F_2, G_1 \otimes G_2)$ is a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) having bounds A and B . Let $f \in H_1 - \{\theta\}$ and fix $g \in H_2 - \{\theta\}$.

Then, $f \otimes g \in H_1 \otimes H_2 - \{\theta \otimes \theta\}$ and by Fubini's theorem we have

$$\begin{aligned} & \int_X \langle f \otimes g, F_1(x_1) \otimes F_2(x_2) \rangle \langle G_1(x_1) \otimes G_2(x_2), f \otimes g \rangle d\mu \\ &= \int_{X_1} \langle f, F_1(x_1) \rangle_1 \langle G_1(x_1), f \rangle_1 d\mu_1 \int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2. \end{aligned}$$

Therefore, for each $f \otimes g \in H_1 \otimes H_2$, (3.1) can be written as

$$\begin{aligned} A \|f\|_1^2 \|g\|_2^2 &\leq \int_{X_1} \langle f, F_1(x_1) \rangle_1 \langle G_1(x_1), f \rangle_1 d\mu_1 \int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2 \\ &\leq B \|f\|_1^2 \|g\|_2^2. \end{aligned}$$

Since f and g are non-zero and therefore

$$\int_{X_1} \langle f, F_1(x_1) \rangle_1 \langle G_1(x_1), f \rangle_1 d\mu_1, \int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2$$

are non-zero. Thus, from the above inequality, we can write

$$\begin{aligned} \frac{A \|g\|_2^2}{\int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2} \|f\|_1^2 &\leq \int_{X_1} \langle f, F_1(x_1) \rangle_1 \langle G_1(x_1), f \rangle_1 d\mu_1 \\ &\leq \frac{B \|g\|_2^2}{\int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2} \|f\|_1^2. \end{aligned}$$

Thus, for each $f \in H_1 - \{\theta\}$, we have

$$A_1 \|f\|_1^2 \leq \int_{X_1} \langle f, F_1(x_1) \rangle_1 \langle G_1(x_1), f \rangle_1 d\mu_1 \leq B_1 \|f\|_1^2,$$

where

$$A_1 = \inf_{g \in H_2, \|g\|_2=1} \left\{ \frac{A \|g\|_2^2}{\int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2} \right\}$$

and

$$B_1 = \sup_{g \in H_2, \|g\|_2=1} \left\{ \frac{A \|g\|_2^2}{\int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2} \right\}.$$

This shows that (F_1, G_1) is a continuous biframe for H_1 with respect to (X_1, μ_1) . Similarly, it can be shown that (F_2, G_2) is a continuous biframe for H_2 with respect to (X_2, μ_2) .

Conversely, suppose that (F_1, G_1) is a continuous biframe for H_1 with respect to (X_1, μ_1) having bounds A, B and (F_2, G_2) is a continuous biframe for H_2 with respect to (X_2, μ_2) having bounds C, D . By the assumption it is easy to verify that

$(\mathbf{F}, \mathbf{G}) = (F_1 \otimes F_2, G_1 \otimes G_2)$ is weakly measurable on $H_1 \otimes H_2$ with respect to (X, μ) . Now, for each $f \in H_1 - \{\theta\}$ and $g \in H_2 - \{\theta\}$, we have

$$\begin{aligned} A \|f\|_1^2 &\leq \int_{X_1} \langle f, F_1(x_1) \rangle_1 \langle G_1(x_1), f \rangle_1 d\mu_1 \leq B \|f\|_1^2, \\ C \|g\|_2^2 &\leq \int_{X_2} \langle g, F_2(x_2) \rangle_2 \langle G_2(x_2), g \rangle_2 d\mu_2 \leq B \|g\|_2^2. \end{aligned}$$

Multiplying the above two inequalities and using Fubini's theorem we get

$$\begin{aligned} AC \|f \otimes g\|^2 &\leq \int_X \langle f \otimes g, F_1(x_1) \otimes F_2(x_2) \rangle \langle G_1(x_1) \otimes G_2(x_2), f \otimes g \rangle d\mu \\ &\leq BD \|f \otimes g\|^2, \end{aligned}$$

for all $f \otimes g \in H_1 \otimes H_2$. Thus, for each $f \otimes g \in H_1 \otimes H_2$, we have

$$AC \|f \otimes g\|^2 \leq \int_X \langle f \otimes g, \mathbb{F}(x) \rangle \langle \mathbb{G}(x), f \otimes g \rangle d\mu \leq BD \|f \otimes g\|^2.$$

Hence, $(\mathbf{F}, \mathbf{G}) = (F_1 \otimes F_2, G_1 \otimes G_2)$ is a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) having bounds AC and BD . This completes the proof. $\square$

Example 3.1. Let $\{e_i\}_{i=1}^{+\infty}$ be an orthonormal basis for H_1 and (X_1, μ_1) be a measure space where μ_1 is σ -finite. Then we can write $X_1 = \bigcup_{i=1}^{+\infty} \Omega_i$, where $\{\Omega_i\}_{i=1}^{+\infty}$ is a sequence of disjoint measurable subsets of X_1 with $\mu(\Omega_i) < +\infty$. Suppose

$$\begin{aligned} \{f_i\}_{i=1}^{+\infty} &= \{e_1, e_1, e_1, e_2, e_3, \dots\}, \\ \{g_i\}_{i=1}^{+\infty} &= \{0, e_1, e_1, e_2, e_3, \dots\}. \end{aligned}$$

For each $x_1 \in X_1$, we define the mappings $F_1 : X_1 \rightarrow H_1$ by $F_1(x_1) = \frac{1}{\sqrt{\mu(\Omega_i)}} f_i$ and $G_1 : X_1 \rightarrow H_1$ by $G_1(x_1) = \frac{1}{\sqrt{\mu(\Omega_i)}} g_i$, for $x_1 \in \Omega_i$. Then, by Example 2.2, (F_1, G_1) is a continuous biframe for H with bounds 1 and 2.

On the other hand, let $\{e'_j\}_{j=1}^{+\infty}$ be an orthonormal basis for H_2 and (X_2, μ_2) be a measure space with μ_2 is σ -finite. Then, $X_2 = \bigcup_{j=1}^{+\infty} \Omega'_j$, and $\{\Omega'_j\}_{j=1}^{+\infty}$ is a sequence of disjoint measurable subsets of X_2 with $\mu(\Omega'_j) < +\infty$. Suppose

$$\begin{aligned} \{f'_j\}_{j=1}^{+\infty} &= \{5e_1, 3e_2, 2e_3, 2e_4, \dots\}, \\ \{g'_j\}_{j=1}^{+\infty} &= \{0, 0, 3e_1, 0, 2e_2, 2e_3, \dots\}. \end{aligned}$$

Now, we define $F_2 : X_2 \rightarrow H_2$ by $F_2(x_2) = \frac{1}{\sqrt{\mu(\Omega'_j)}} f'_j$ and $G_2 : X_2 \rightarrow H_2$ by $G_2(x_2) = \frac{1}{\sqrt{\mu(\Omega'_j)}} g'_j$, for $x_2 \in \Omega'_j$. Now, for $f, g \in H_2$, we have

$$\begin{aligned} \int_{X_2} \langle f, F_2(x_2) \rangle \langle G_2(x_2), f \rangle d\mu_2 &= \sum_{j=1}^{+\infty} \int_{\Omega'_j} \langle f, f'_j \rangle \langle g'_j, f \rangle d\mu_2 \\ &= \langle f, e_1 \rangle \langle e_1, f \rangle + 2 \langle f, e_1 \rangle \langle e_1, f \rangle + 2 \langle f, e_2 \rangle \langle e_2, f \rangle + \cdots \\ &= |\langle f, e_1 \rangle|^2 + 2 |\langle f, e_1 \rangle|^2 + 2 |\langle f, e_2 \rangle|^2 + \cdots \\ &= |\langle f, e_1 \rangle|^2 + 2 \|f\|^2. \end{aligned}$$

Thus, (F_2, G_2) is a continuous biframe for H with bounds 2 and 3. Thus, by the Theorem 3.1, $(\mathbf{F}, \mathbf{G}) = (F_1 \otimes F_2, G_1 \otimes G_2)$ is a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) having bounds 2 and 6.

Let $(X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)$ be the product of measure spaces with σ -finite positive measures μ_1, μ_2 . Let $L^2(X, \mu)$ be the class of all measurable functions $\Psi : X \rightarrow H_1 \otimes H_2$ such that

$$\int_X \|\Psi(x)\|^2 d\mu = \int_{X_1} \|\varphi_1(x_1)\|_1 d\mu_1 \int_{X_2} \|\varphi_2(x_2)\|_2 d\mu_2 < +\infty,$$

for $\varphi_1 \in L^2_1(X_1, \mu_1)$, $\varphi_2 \in L^2_2(X_2, \mu_2)$, with the inner product

$$\begin{aligned} \langle \Psi, \Phi \rangle_{L^2} &= \int_X \langle \Psi(x), \Phi(x) \rangle d\mu \\ &= \int_{X_1} \langle \varphi_1(x_1), \psi_1(x_1) \rangle_1 d\mu_1 \int_{X_2} \langle \varphi_2(x_2), \psi_2(x_2) \rangle_2 d\mu_2 \\ &= \langle \varphi_1, \psi_1 \rangle_{L^2_1} \langle \varphi_2, \psi_2 \rangle_{L^2_2}, \end{aligned}$$

where $\Psi = \varphi_1 \otimes \varphi_2$, $\Phi = \psi_1 \otimes \psi_2 \in L^2(X, \mu)$, for $\varphi_1, \psi_1 \in L^2_1(X_1, \mu_1)$ and $\varphi_2, \psi_2 \in L^2_2(X_2, \mu_2)$. The space $L^2(X, \mu)$ is complete with respect to the above inner product. Therefore, it is an Hilbert space.

Definition 3.2. Let $(\mathbf{F}, \mathbf{G}) = (\mathbb{F} : X \rightarrow H_1 \otimes H_2, \mathbb{G} : X \rightarrow H_1 \otimes H_2)$ be a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) . Then, the operator $S_{\mathbb{F} \otimes \mathbb{G}} : H_1 \otimes H_2 \rightarrow H_1 \otimes H_2$ given by

$$S_{\mathbb{F} \otimes \mathbb{G}}(f \otimes g) = \int_X \langle f \otimes g, \mathbb{F}(x) \rangle \mathbb{G}(x) d\mu$$

is called continuous biframe operator.

Theorem 3.2. Let $(\mathbf{F}, \mathbf{G}) = (F_1 \otimes F_2, G_1 \otimes G_2) = \mathbb{F}, \mathbb{G} : X \rightarrow H_1 \otimes H_2$ is a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) having continuous biframe operator $S_{\mathbb{F} \otimes \mathbb{G}}$. Then, $S_{\mathbb{F} \otimes \mathbb{G}} = S_{F_1, G_1} \otimes S_{F_2, G_2}$, where S_{F_1, G_1} and S_{F_2, G_2} are continuous biframe operators of (F_1, G_1) and (F_2, G_2) , respectively.

Proof. For each $f \otimes g \in H_1 \otimes H_2$, we have

$$\begin{aligned} S_{\mathbb{F} \otimes \mathbb{G}}(f \otimes g) &= \int_X \langle f \otimes g, F_1(x_1) \otimes F_2(x_2) \rangle (G_1(x_1) \otimes G_2(x_2)) d\mu \\ &= \left(\int_{X_1} \langle f, F_1(x_1) \rangle_1 G_1(x_1) d\mu_1 \right) \otimes \left(\int_{X_2} \langle g, F_2(x_2) \rangle_2 G_2(x_2) d\mu_2 \right) \\ &= S_{F_1, G_1} f \otimes S_{F_2, G_2} g = (S_{F_1, G_1} \otimes S_{F_2, G_2})(f \otimes g). \end{aligned}$$

Thus, $S_{\mathbb{F} \otimes \mathbb{G}} = S_{F_1, G_1} \otimes S_{F_2, G_2}$. $\square$

Lemma 3.1. *Let (F_1, G_1) be a continuous biframe for H_1 with respect to (X_1, μ_1) having bounds A, B and (F_2, G_2) be a continuous biframe for H_2 with respect to (X_2, μ_2) having bounds C, D . Then, $ACI_{H_1 \otimes H_2} \leq S_{\mathbb{F} \otimes \mathbb{G}} \leq BDI_{H_1 \otimes H_2}$, where $I_{H_1 \otimes H_2}$ is the identity operator on $H_1 \otimes H_2$ and $S_{F_1, G_1}, S_{F_2, G_2}$ are continuous biframe operators of $(F_1, G_1), (F_2, G_2)$, respectively.*

Proof. Since S_{F_1, G_1} and S_{F_2, G_2} are continuous biframe operators, we have

$$AI_{H_1} \leq S_{F_1, G_1} \leq BI_{H_1}, \quad CI_{H_2} \leq S_{F_2, G_2} \leq DI_{H_2},$$

where I_{H_1} and I_{H_2} are the identity operators on H_1 and H_2 , respectively. Taking tensor product on the above two inequalities, we get

$$AC(I_{H_1} \otimes I_{H_2}) \leq (S_{F_1, G_1} \otimes S_{F_2, G_1}) \leq BD(I_{H_1} \otimes I_{H_2})$$

implies $ACI_{H_1 \otimes H_2} \leq S_{\mathbb{F} \otimes \mathbb{G}} \leq BDI_{H_1 \otimes H_2}$.

This completes the proof. $\square$

Theorem 3.3. *Let (F_1, G_1) be a continuous biframe for H_1 with respect to (X_1, μ_1) and (F_2, G_2) be a continuous biframe for H_2 with respect to (X_2, μ_2) . Then, $\Delta = ((T_1 \otimes T_2)(F_1 \otimes F_2), (T_1 \otimes T_2)(G_1 \otimes G_2))$ is a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) if and only if $T_1 \otimes T_2$ is an invertible bounded linear operator on $H_1 \otimes H_2$.*

Proof. Let (F_1, G_1) be a continuous biframe for H_1 with bounds A, B and (F_2, G_2) be a continuous biframe for H_2 with bounds C, D . First we suppose that $T_1 \otimes T_2$ is an invertible bounded linear operator on $H_1 \otimes H_2$. Then, by Theorem 1.2, T_1 and T_2 are invertible bounded linear operators on H_1 and H_2 , respectively. Now, by Theorem 2.3, $(T_1 F_1, T_1 G_1)$ is a continuous biframe for H_1 with bounds $A \|T_1^{-1}\|^{-2}$, $B \|T_1\|^2$ and $(T_2 F_2, T_2 G_2)$ is a continuous biframe for H_2 with bounds $C \|T_2^{-1}\|^{-2}$, $D \|T_2\|^2$. Hence, by Theorem 3.1, the pair

$$(T_1 F_1 \otimes T_2 F_2, T_1 G_1 \otimes T_2 G_2) = ((T_1 \otimes T_2)(F_1 \otimes F_2), (T_1 \otimes T_2)(G_1 \otimes G_2))$$

is a continuous biframe for $H_1 \otimes H_2$ with bounds

$$\frac{AC}{\|T_1^{-1}\|^2 \|T_2^{-1}\|^2} = \frac{AC}{\|T_1^{-1} \otimes T_2^{-1}\|^2} = \frac{AC}{\|(T_1 \otimes T_2)^{-1}\|^2} = AC \|(T_1 \otimes T_2)^{-1}\|^{-2}$$

and $BD \|T_1\|^2 \|T_2\|^2 = BD \|T_1 \otimes T_2\|^2$.

Conversely, suppose that Δ is a continuous biframe for $H_1 \otimes H_2$ with respect to (X, μ) . Let S_{F_1, G_1} and S_{F_2, G_2} be the continuous biframe operators of (F_1, G_1) and (F_2, G_2) , respectively. By Theorem 3.1, $(T_1 F_1, T_1 G_1)$ and $(T_2 F_2, T_2 G_2)$ are continuous biframes for H_1 and H_2 , respectively. Now, for $f \in H_1$, we have

$$\int_{X_1} \langle f, T_1 F_1(x_1) \rangle_1 T_1 G_1(x_1) d\mu_1 = T_1 \left(\int_{X_1} \langle T_1^* f, F_1(x_1) \rangle_1 G_1(x_1) d\mu_1 \right) = T_1 S_{F_1, G_1} T_1^* f.$$

This shows that $T_1 S_{F_1, G_1} T_1^*$ is the corresponding continuous biframe operator of $(T_1 F_1, T_1 G_1)$. Thus, $T_1 S_{F_1, G_1} T_1^*$ is invertible on H_1 and hence T_1 is invertible on H_1 . Similarly, it can be shown that $T_2 S_{F_2, G_2} T_2^*$ is the corresponding continuous biframe operator of $(T_2 F_2, T_2 G_2)$ and hence T_2 is invertible on H_2 . Therefore, $T_1 \otimes T_2$ is an invertible bounded linear operator on $H_1 \otimes H_2$. This completes the proof. $\square$

Now, we would complete this section by introducing the idea of a continuous biframe Bessel multiplier in $H_1 \otimes H_2$.

Definition 3.3. Let $(\mathbb{F}, \mathbb{F})$ and $(\mathbb{G}, \mathbb{G})$ be continuous biframe Bessel mappings in $H_1 \otimes H_2$ with respect to (X, μ) having bounds B_1 and B_2 , respectively and $m : X \rightarrow \mathbb{C}$ be a measurable function. The operator $\mathcal{M}_{m, \mathbb{F}, \mathbb{G}} : H_1 \otimes H_2 \rightarrow H_1 \otimes H_2$ defined by

$$(3.2) \quad \mathcal{M}_{m, \mathbb{F}, \mathbb{G}}(f \otimes g) = \int_X m(x) \langle f \otimes g, \mathbb{F}(x) \rangle \mathbb{G}(x) d\mu,$$

for all $f \otimes g \in H_1 \otimes H_2$, is called continuous biframe Bessel multiplier of $\mathbb{F}$ and $\mathbb{G}$ with respect to m .

Remark 3.1. Let $F_1, G_1 : X_1 \rightarrow H_1$ be continuous biframe Bessel mappings in H_1 with respect to (X_1, μ_1) and $F_2, G_2 : X_2 \rightarrow H_2$ be continuous biframe Bessel mappings in H_2 with respect to (X_2, μ_2) and $m_1 : X_1 \rightarrow \mathbb{C}$, $m_2 : X_2 \rightarrow \mathbb{C}$ be two measurable functions. Suppose $M_{m_1, F_1, G_1} : H_1 \rightarrow H_1$ be the continuous biframe Bessel multiplier of F_1 and G_1 with respect to m_1 and $M_{m_2, F_2, G_2} : H_2 \rightarrow H_2$ be the continuous biframe Bessel multiplier of F_2 and G_2 with respect to m_2 . Now, by Theorem 3.1, $\mathcal{F} = F_1 \otimes F_2$, $\mathcal{G} = G_1 \otimes G_2 : X \rightarrow H_1 \otimes H_2$ are continuous biframe Bessel mappings in $H_1 \otimes H_2$ with respect to (X, μ) . Also, $m(x) = m_1(x_1)m_2(x_2)$ is a measurable function. From (3.2), for each $f \otimes g \in H_1 \otimes H_2$, we can write

$$\begin{aligned} \mathcal{M}_{m, \mathbb{F}, \mathbb{G}}(f \otimes g) &= \int_X m(x) \langle f \otimes g, \mathbb{F}(x) \rangle \mathbb{G}(x) d\mu \\ &= \int_{X_1} m_1(x_1) \langle f, F_1(x_1) \rangle_1 G_1(x_1) d\mu_1 \otimes \int_{X_2} m_2(x_2) \langle g, F_2(x_2) \rangle_2 G_2(x_2) d\mu_2 \\ &= M_{m_1, F_1, G_1} f \otimes M_{m_2, F_2, G_2} g = (M_{m_1, F_1, G_1} \otimes M_{m_2, F_2, G_2})(f \otimes g). \end{aligned}$$

Thus, $\mathcal{M}_{m, \mathbb{F}, \mathbb{G}} = M_{m_1, F_1, G_1} \otimes M_{m_2, F_2, G_2}$.

Remark 3.2. Following the Theorem 2.4, the continuous biframe Bessel multiplier of $\mathbb{F}$ and $\mathbb{G}$ with respect to m is well defined and bounded.

Acknowledgement. The authors would like to thank the editor and the referees for their helpful suggestions and comments to improve this paper.

REFERENCES

- [1] P. Balazs, J. P. Antonie and A. Grybos, *Weighted and controlled frames: mutual relationship and first numerical properties*, Int. J. Wavelets Multiresolut. Inf. Process. **14**(1) (2010), 109–132.
- [2] P. Balazs, D. Bayer and A. Rahimi, *Multipliers for continuous frames in Hilbert spaces*, J. Phys. A **45**(24) (2012), Article ID 244023.
- [3] I. Bogdanova, P. Vandergheynst, J. P. Antoine, L. Jacques and M. Morrividone, *Stereographic wavelet frames on the sphere*, Appl. Comput. Harmon. Anal. **16** (2005), 223–252.
- [4] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27**(5) (1986), 1271–1283.
- [5] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.
- [6] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press BOCA Raton, Florida, 1995.
- [7] M. H. Faroughi and E. Osgoei, *c-frames and c-Bessel mappings*, Bull. Iranian Math. Soc. **38**(1) (2012), 203–222.
- [8] G. J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, San Diego, 1990.
- [9] M. F. Parizi, A. Alijani and M. A. Dehghan, *Biframes and some their properties*, J. Inequal. Appl. **2022** (2022), Article ID 104. <https://doi.org/10.1186/s13660-022-02844-7>
- [10] G. U. Reddy, N. G. Reddy and B. K. Reddy, *Frame operator and Hilbert-Schmidt operator in tensor product of Hilbert spaces*, J. Dyn. Syst. Geom. Theor. **7**(1) (2009), 61–70.
- [11] A. Rahimi, A. Najati and Y. N. Dehghan, *Continuous frames in Hilbert spaces*, Methods Funct. Anal. Topology **12**(2) (2006), 170–182.

¹DEPARTMENT OF MATHEMATICS,
BARWAN N. S. HIGH SCHOOL (HS),
BARWAN, MURSHIDABAD, 742132, WEST BENGAL, INDIA
Email address: prasenjitpuremath@gmail.com

²DEPARTMENT OF MATHEMATICS,
ULUBERIA COLLEGE,
ULUBERIA, HOWRAH, 711315, WEST BENGAL, INDIA
Email address: mumpu_tapas@yahoo.co.in