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DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR A NEW DIFFERENTIAL OPERATOR CONTAINING MITTAG-LEFFLER FUNCTION

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ABSTRACT. Owning to the importance and great interest of linear operators, a generalisation of linear derivative operator $\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)$ is newly introduced in this study. The main objective of this paper is to investigate various subordination and superordination related to the aforementioned generalised linear derivative operator. Additionally, the resultant sandwich-type of this operator is also considered.

1. DEFINITION AND PRELIMINARIES

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{H} = \mathcal{H}(\Delta)$ indicate the family of analytic functions within Δ. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ containing the functions of the form

$$\mathcal{H}[a,n] = \left\{ f \in \mathcal{H}(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}, \quad z \in \Delta.$$

Furthermore, let $\mathcal{A}(p)$ indicate the subclass of $\mathcal{H}$ containing the functions having the following form

$$f(z) = z^p + \sum_{i=p+1}^{\infty} a_i z^i, \quad p \in \mathbb{N}, \tag{1.1}$$

which are analytic and p-valent in Δ. For clarity, we write $\mathcal{A}(1) = \mathcal{A}$.

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The convolution (or Hadamard product) $f * g$ for two analytic functions f defined by (1.1) and

$$g(z) = z^p + \sum_{i=p+1}^{\infty} b_i z^i$$

is given by

$$f(z) * g(z) = z^p + \sum_{i=p+1}^{\infty} a_i b_i z^i.$$

For the two analytic functions f and g in $\mathcal{H}(\Delta)$, we are saying that $f(z)$ is subordinate to $g(z)$ usually denoted by $f(z) \prec g(z)$ in case if there is a Schwarz function ω with $\omega(z) = 0$, $|\omega(z)| < 1$, $z \in \Delta$, such that $f(z) = g(\omega(z))$ for all $z \in \Delta$.

Especially, if $g(z)$ is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

Let $\mathcal{S}^*_\alpha(p)$ and $\mathcal{K}_\alpha(p)$ denote the familiar subclasses of the class $\mathcal{A}(p)$ consisting of the functions which are p-valently starlike and p-valently convex of order α in Δ, respectively,

$$\mathcal{S}^*_\alpha(p) = \left\{ f \in \mathcal{A}(p) : \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha,\, z \in \Delta \right\},$$
$$\mathcal{K}_\alpha(p) = \left\{ f \in \mathcal{A}(p) : \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha,\, z \in \Delta \right\}.$$

The method of differential subordinations ,which is additionally called the admissible functions method, was maybe the first one presented by Miller and Mocanu in 1978 [13]. From that point onward and roughly in 1981 [14] the theory started to proliferate and progressively develop. Relevant details are epitomized in a book written by Miller and Mocanu [15].

Definition 1.1 (see [15]). Let $\varphi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ and $h(z)$ be univalent in Δ. If $\zeta(z)$ is analytic function in Δ and also satisfies the second-order differential subordination

$$\varphi(\zeta(z), z\zeta'(z), z^2\zeta''(z); z \in \Delta) \prec h(z), \quad z \in \Delta, \tag{1.2}$$

then $\zeta(z)$ is defined as a solution of the differential subordination (1.2). A univalent function $q(z)$ is called a dominant if $\zeta(z) \prec q(z)$ for all $\zeta(z)$ satisfying (1.2). A dominant $\tilde{q}$ is called the best dominant when $\tilde{q} \prec q$ for all dominants q of (1.2).

Definition 1.2 (see [16]). Let $\phi : \mathbb{C}^3 \times \Delta \to \mathbb{U}$ let $h(z)$ be analytic function in Δ. If $\zeta(z)$ and $\phi(\zeta(z), z\zeta'(z), z^2\zeta''(z); z)$ are univalent in Δ and $\zeta(z)$ satisfies the (second-order) differential subordination

$$h(z) \prec \phi(\zeta(z), z\zeta'(z), z^2\zeta''(z)), \quad z \in \Delta, \tag{1.3}$$

then $\zeta(z)$ is defined as a solution of the differential subordination (1.3). An analytic function $q(z)$ is called a a subordinates, if $q(z) \prec \zeta(z)$ for all $\zeta(z)$ satisfying (1.3). A univalent subordinate $\tilde{q}$ is called the best subordinate when $q \prec \tilde{q}$ for all subordinates q of (1.3).

Definition 1.3 (see [16]). Let G denote the set of functions f which are analytic and injective on $\overline{\Delta}\backslash B(f)$, where

$$B(f) = \left\{\xi \in \partial\Delta : \lim_{z\to\xi} f(z) = \infty\right\},$$

and $f'(\xi) \neq 0$, $\xi \in \partial\Delta\backslash B(f)$.

In 1999, Dziok and Srivastava [6] introduced the function $g_p(a_1, \ldots, a_r, b_1, \ldots, b_s; z)$, which defined by generalized hypergeometric function as following

$$g_p(a_1, \ldots, a_r, b_1, \ldots, b_s; z) = z^p + \sum_{i=p+1}^{\infty} \frac{(a_1)_{i-p} \cdots (a_r)_{i-p}}{(b_1)_{i-p} \cdots (b_s)_{i-p}} \frac{z^i}{(i-p)!}, \quad p \in \mathbb{N}, \tag{1.4}$$

where $a_k \in \mathbb{C}$, $b_n \in \mathbb{C} \setminus \{0, -1, \ldots\}$, $k = 1, \ldots, r$, $n = 1, \ldots, s$ and $r \leq 1 + s$, $r, s \in \mathbb{N}_0$ and $(v)_i$ is the Pochhammer symbol defined by

$$(v)_i = \frac{\Gamma(v+i)}{\Gamma(v)} = \begin{cases} v(v+1)\cdots(v+i-1), & i = 1, 2, \ldots, \\ 1, & i = 0. \end{cases}$$

For convenience, we write $g_p(a_1, \ldots, a_r, b_1, \ldots, b_s; z) = \mathcal{G}_p(a_1, b_1; z)$.

The well known Mittag-Leffler function $E_\alpha(z)$ which is introduced by Mittag- Leffler [17] and [18] is defined hereunder. Similarly, the first two parametric generalization $E_{\alpha,\beta}(z)$ of the same function by Wiman [27] is defined as well

$$E_\alpha(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + 1)}$$

and

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)},$$

where $\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

The above mentioned resulted in plenty of valuable work has been made by numerous authors in an endeavor to clarify Mittag-Leffler function and its first two parametric generalization, see for instance [4, 8–10, 20, 23, 25] and [26].

Now, we define the function $\mathcal{F}_{\alpha,\beta}(z)$ by

$$\mathcal{F}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{i=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(i-1)+\beta)} z^i.$$

Having use of the function $\mathcal{F}_{\alpha,\beta}(z)$, Elhaddad et al. [7] defined the differential operator $\mathcal{D}_\delta^m(\alpha, \beta)f : \mathcal{A} \longrightarrow \mathcal{A}$ as illustrated below:

$$\mathcal{D}_\delta^m(\alpha, \beta)f(z) = z + \sum_{i=2}^{\infty} [1 + (i-1)\delta]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(i-1)+\beta)} a_i z^i, \tag{1.5}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\delta > 0$.

Now, we define the operator $\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)$ in (1.5) of a function $f \in \mathcal{A}(p)$ given by (1.1) as below:

$$\mathcal{D}_{\delta,p}^{m}(\alpha,\beta)f(z) = z^p + \sum_{i=p+1}^{\infty}\left[\frac{p+(i-p)\delta}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(i-p)+\beta)}a_i z^i, \quad p \in \mathbb{N}, \tag{1.6}$$

where $m \in \mathbb{N}_0$, $\delta > 0$.

Corresponding to $\mathcal{G}_p(a_1, b_1; z)$ which defined in (1.4), $\mathcal{D}_{\delta,p}^{m}(\alpha,\beta)f(z)$ defined in (1.6) and utilizing Hadamard product, we define a new generalized derivative operator $\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)$ as follows.

Definition 1.4. Let $f \in \mathcal{A}(p)$, then the generalized derivative operator $\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z) : \mathcal{A}(p) \to \mathcal{A}(p)$ is given by

$$\begin{aligned} &\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z) \\ =&\mathcal{G}_p(a_1,b_1;z) * \mathcal{D}_{\delta,p}^{m}(\alpha,\beta)f(z) \\ =&z^p + \sum_{i=p+1}^{\infty}\left[\frac{p+(i-p)\delta}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(i-p)+\beta)} \frac{(a_1)_{i-p}\cdots(a_r)_{i-p}}{(b_1)_{i-p}\cdots(b_s)_{i-p}} \frac{a_i z^i}{(i-p)!}. \end{aligned} \tag{1.7}$$

We can easily verify from (1.7) that

$$\begin{aligned} p\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z) =&(p-p\delta)\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z) \\ &+ \delta z(\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z))'. \end{aligned} \tag{1.8}$$

Remark 1.1.
- For $s=0,\ r=1,\ a_1=1,\ \alpha=0,\ \beta=1$ and $p=1$, we get Al-Oboudi operator [1].
- For $s=0,\ r=1,\ a_1=1,\ \beta=1,\ \alpha=0,\ \delta=1$ and $p=1$, we get Sălăgean operator [22].
- For $s=0,\ r=1,\ a_1=1,\ m=0$ and $p=1$, we get $\mathbb{E}_{\alpha,\beta}(z)$ [25].
- For $m=0,\ \alpha=0$ and $\beta=1$, we get the operator studied by Dziok and Srivastava [6].
- For $m=0,\ \alpha=0,\ p=1,\ r=1,\ s=0,\ a_1=\lambda+1$ and $\beta=1$, we get the operator examined byby Ruscheweyh [21].
- For $m=0,\ \alpha=0,\ p=1,\ r=2,\ s=1$ and $\beta=1$, we get the operator which was introduced by Hohlov [11].
- For $m=0,\ \alpha=0,\ p=1,\ r=2,\ s=1,\ a_2=1$ and $\beta=1$, we get the operator investigated by Carlson and Shaffer [5].

So as to demonstrate and approve above results, following primer results are required.

Lemma 1.1 (see [24])**.** *Let $g(z)$ be convex function within the open unit disk Δ and let ν and μ be complex numbers, $\nu \in \mathbb{C}$ and $\mu \in \mathbb{C}/\{0\}$, with*

$$\operatorname{Re}\left\{\frac{zg''(z)}{g'(z)}+1\right\} > \max\left\{-\operatorname{Re}\left(\frac{\nu}{\mu}\right), 0\right\}.$$

If $h(z)$ is analytic within Δ and

$$\nu h(z) + \mu z h'(z) \prec \nu g(z) + \mu z g'(z). \tag{1.9}$$

Thus, $h(z) \prec g(z)$, $z \in \Delta$, and $g(z)$ is the best dominant of (1.9).

Lemma 1.2 (see [16])**.** *Let μ be a complex number with* $\mathrm{Re}(\mu) > 0$ *and g be a convex function within Δ. If $h(z) \in \mathcal{H}[g(0), 1] \cap G$ and $h(z) + \mu z h'(z)$ is univalent in Δ, thus*

$$g(z) + \mu z g'(z) \prec h(z) + \mu z h'(z), \tag{1.10}$$

consequently, $g(z) \prec h(z)$ and $g(z)$ is the best subordinant of (1.10).

2. Main Results

Theorem 2.1. *Let $m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\delta > 0$, $\sigma \in \mathbb{C}/\{0\}$ and $\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)$ defined by* (1.7)*. Let $g(z)$ be univalent in Δ, with $g(0) = 1$, and assume that*

$$\mathrm{Re}\left\{\frac{zg''(z)}{g'(z)} + 1\right\} > \max\left\{-\frac{p}{\delta}\mathrm{Re}\left(\frac{1}{\sigma}\right), 0\right\}. \tag{2.1}$$

If f in the class $\mathcal{A}(p)$ satisfies the subordination condition

$$\sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p}\right) \prec g(z) + \frac{\sigma\delta}{p} z g'(z), \tag{2.2}$$

then

$$\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \prec g(z)$$

and $g(z)$ is the best dominant of (2.2).

Proof. Define the function $\zeta(z)$ by

$$\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} = \zeta(z). \tag{2.3}$$

Differentiating (2.3) logarithmically with respect to z, we have

$$\frac{z\zeta'(z)}{\zeta(z)} = \frac{z(\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z))'}{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)} - p. \tag{2.4}$$

Using (1.8) in the resulting equation (2.4), we get

$$\begin{aligned}
\frac{z\zeta'(z)}{\zeta(z)} &= \left(\frac{p}{\delta}\right)\left\{\frac{z(\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z))'}{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)} - 1\right\}\\
&=\sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p}\right)\\
&=\zeta(z) + \frac{\sigma\delta}{p} z\zeta'(z),
\end{aligned}$$

then the differential subordination from hypothesis (2.2) is equivalent to

$$\zeta(z) + \frac{\sigma\delta}{p} z\zeta'(z) \prec g(z) + \frac{\sigma\delta}{p} z g'(z).$$

To prove our result, we need to use Lemma 1.1. For that purpose, let $\nu = 1$, $\mu = \frac{\sigma\delta}{p}$. We get

$$\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p} \prec g(z),$$

which is the required result. $\square$

Setting $g(z) = \frac{1+Cz}{1+Dz}$ in Theorem 2.1, where $-1 \leq D < C \leq 1$. Then, the condition (2.1) turn into

$$\text{Re}\left\{\frac{1-Dz}{1+Dz}\right\} > \max\left\{0, -\frac{p}{\delta}\text{Re}\left(\frac{1}{\sigma}\right)\right\}, \quad z \in \Delta. \tag{2.5}$$

The function

$$\Psi(\gamma) = \frac{1-\gamma}{1+\gamma}, \quad |\gamma| < |D|,$$

is convex in Δ and since $\Psi(\overline{\gamma}) = \overline{\Psi(\gamma)}$ for all $|\gamma| < |D|$, then the image $\Psi(\Delta)$ is a convex domain symmetric with respect to the real axis. Thus,

$$\inf\left\{\text{Re}\left(\frac{1-Dz}{1+Dz}\right), z \in \Delta\right\} = \frac{1-|D|}{1+|D|} > 0.$$

Then, the relation (2.5) is identical to

$$\frac{p}{\delta}\text{Re}\left(\frac{1}{\sigma}\right) \geq \frac{|D|-1}{|D|+1},$$

as a result, we get the following corollary.

Corollary 2.1. *Let $m \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $\delta > 0$, $-1 \leq D < C \leq 1$ and $\sigma \in \mathbb{C}/\{0\}$ with*

$$\max\left\{0, -\frac{p}{\delta}\text{Re}\left(\frac{1}{\sigma}\right)\right\} \leq \frac{1-|D|}{1+|D|}.$$

If f in the class $\mathcal{A}(p)$ and

$$\begin{aligned}\sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right) \\ \prec \frac{1+Cz}{1+Dz} + \frac{\sigma\delta}{p}\frac{(C-D)z}{(1+D)^2}z,\end{aligned} \tag{2.6}$$

then

$$\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p} \prec \frac{1+Cz}{1+Dz}$$

and $\dfrac{1+Cz}{1+Dz}$ is the best dominant of (2.6).

Theorem 2.2. *Let* $m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\delta > 0$, $\sigma \in \mathbb{C}/\{0\}$ *and* $\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)$ *defined by* (1.7). *Let* $h(z)$ *be a convex function in* Δ, *with* $h(0) = 1$. *Let* f *in the class* $\mathcal{A}(p)$ *and*

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \in \mathcal{H}[1, 1] \cap G.$$

If

$$\sigma \left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \right) + (1 - \sigma) \left(\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \right)$$

in univalent in Δ, *and*

$$h(z) + \frac{\sigma\delta}{p} z h'(z) \prec \sigma \left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \right) + (1 - \sigma) \left(\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \right), \tag{2.7}$$

then

$$h(z) \prec \frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p}$$

and $h(z)$ *is the best subordinant of* (2.7).

Proof. Define the function $\chi(z)$ by

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} = \chi(z). \tag{2.8}$$

From the presumption of Theorem 2.2, we note that the function χ is analytic in the open unit disk Δ. Differentiating (2.8) logarithmically with respect to z, we get

$$\frac{z\chi'(z)}{\chi(z)} = \frac{z(\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z))'}{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)} - p. \tag{2.9}$$

Using (1.8) in (2.9) and after some calculations, we get

$$\chi(z) + \frac{\sigma\delta}{p} z\chi'(z) = \sigma \left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \right) + (1 - \sigma) \left(\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \right)$$

and presently, by utilizing Lemma 1.2, we have the specified result. □

Setting $h(z) = \frac{1+Cz}{1+Dz}$ in Theorem 2.2, where $-1 \leq D < C \leq 1$, we get the following result.

Corollary 2.2. *Let* $m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\delta > 0$, $\sigma \in \mathbb{C}/\{0\}$, $-1 \leq D < C \leq 1$ *and* $\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)$ *defined by* (1.7). *Let* f *in the class* $\mathcal{A}(p)$ *and*

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1)f(z)}{z^p} \in \mathcal{H}[1, 1] \cap G.$$

If

$$\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)$$

is univalent in Δ*, and*

$$\frac{1+Cz}{1+Dz}+\frac{\sigma\delta}{p}\frac{(C-D)z}{(1+Dz)^2}\prec\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right), \tag{2.10}$$

then

$$\frac{1+Cz}{1+Dz}\prec\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}$$

and $\frac{1+Cz}{1+Dz}$ *is the best subordinant of* (2.10)*.*

Combining Theorem 2.1 and Theorem 2.2, we get the following sandwich result.

Theorem 2.3. *Let* $m\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\},\ \delta>0,\ \sigma\in\mathbb{C}/\{0\}$ *and* $\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)$ *defined by* (1.7)*. Let* $h(z)$ *and* $g(z)$ *be a convex function in* Δ*, with* $h(0)=g(z)=1$*. Let* f *in the class* $\mathcal{A}(p)$ *and*

$$\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\in\mathcal{H}[1,1]\cap G.$$

If

$$\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)$$

is univalent in Δ *and*

$$h(z)+\frac{\sigma\delta}{p}zh'(z)\prec\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)\prec g(z)+\frac{\sigma\delta}{p}zg'(z), \tag{2.11}$$

then

$$h(z)\prec\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\prec g(z),$$

and $h(z)$ *and* $g(z)$ *is the best subordinant and the best dominant respectively of* (2.11)*.*

We skip the proofing because it is the same as in the proof of the last theorem.

Remark 2.1. Other work associated with the derivative and integral operator for different issues can be determined in [2, 3, 12] and [19].

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References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. **27** (2004), 1429–1436.

[2] O. Al-Refai and M. Darus, *Main differential sandwich theorem with some applications*, Lobachevskii J. Math. **30** (2009), 1–11.

[3] M. K. Aouf, A. O. Mostafa and R. El-Ashwah, *Sandwich theorems for p-valent functions defined by a certain integral operator*, Math. Comput. Modelling **53** (2011), 1647–1653.

[4] A. A. Attiya, *Some applications of Mittag-Leffler function in the unit disk,* Filomat **30** (2016), 2075–2081.

[5] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), 737–745.

[6] J. Dziok and H. S. Srivastava, *Classes of analytic functions associated with the generalised hypergeometric function*, Appl. Math. Comput. **103** (1999), 1–13.

[7] S. Elhaddad, H. Aldweby and M. Darus, *On certain subclasses of analytic functions involving differential operator*, Jnanabha **48**(I) (2018), 55–64.

[8] S. Elhaddad, H. Aldweby and M. Darus, *Majorization properties for subclass of analytic p-valent functions associated with generalized differential operator involving Mittag-Leffler function*, Nonlinear Functional Analysis and Applications **23**(4) (2018), 743–753.

[9] S. Elhaddad and M. Darus, *On meromorphic functions defined by a new operator containing the Mittag-Leffler function*, Symmetry **11**(2) (2019), Article ID 210.

[10] S. Elhaddad and M. Darus, *Some properties of certain subclasses of analytic function associated with generalized differential operator involving Mittag-Leffler function*, Transylvanian Journal of Mathematics and Mechanics **10**(1) (2018), 1–7.

[11] J. E. Hohlov, *Operators and operations on the class of univalent functions*, Izvestiya Vysshikh Uchebnykh Zavedenii Matematika **10** (1978), 83–89.

[12] R.W. Ibrahim and M. Darus, *Subordination and superordination for functions based on Dziok-Srivastava linear operator*, Bull. Math. Anal. Appl. **2** (2010), 15–26.

[13] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** (1978), 298–305.

[14] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions,* Michigan Math. J. **28** (1981), 157–171.

[15] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker, New York, 2000.

[16] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, Theory and Application **84** (2003), 815–826.

[17] G. M. Mittag-Leffler, *Sur la nouvelle fonction $E_\alpha(x)$*, C. R. Math. Acad. Sci. Paris **137**(2) (1903), 554–558.

[18] G. M. Mittag-Leffler, *Sur la representation analytique d'une branche uniforme d'une fonction monogene*, Acta Math. **29**(1) (1905), 101–181.

[19] N. M. Mustafa and M. Darus, *Differential subordination and superordination for a new linear derivative operator*, International Journal of Pure and Applied Mathematics **70** (2011), 825–835.

[20] H. Rehman, M. Darus, and J. Salah, *Coefficient properties involving the generalized K-Mittag-Leffler functions*, Transylvanian Journal of Mathematics and Mechanics **9**(2) (2017), 155–164.

[21] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[22] G. S. Sălăgean, *Subclasses of Univalent Functions*, Lecture Notes in Mathematics **1013**, Springer-Verlag, Heidelberg, 1983, 362–372.
[23] J. Salah and M. Darus, *A note on generalized Mittag-Leffler function and application*, Far East Journal of Mathematical Sciences **48**(1) (2011), 33–46.
[24] T. N. Shanmugam, S. Sivasubramanian and H. M. Srivastava, *Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations*, Integral Transforms Spec. Funct. **17**(12) (2006), 889–899.
[25] H. M. Srivastava, B. A. Frasin and V. Pescar, *Univalence of integral operators involving Mittag-Leffler functions,* Appl. Math. Inf. Sci. **11**(3) (2017), 635–641.
[26] H. M. Srivastava and Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel,* Appl. Math. Comput. **211** (2009), 198–210.
[27] A. Wiman, *Über den fundamentalsatz in der teorie der funktionen* $E_\alpha(x)$, Acta Math. **29**(1) (1905), 191–201.

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