Kragujevac Journal of Mathematics
Volume 46(4) (2022), Pages 595–603.

# BOUNDEDNESS OF L-INDEX IN JOINT VARIABLES FOR SUM OF ENTIRE FUNCTIONS

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Abstract. In the paper, we present sufficient conditions of boundedness of **L**-index in joint variables for a sum of entire functions, where $\mathbf{L}:\mathbb{C}^n\to\mathbb{R}^n_+$ is a continuous function, $\mathbb{R}_+=(0,+\infty)$. They are applicable to a very wide class of entire functions because for every entire function $F$ in $\mathbb{C}^n$ with bounded multiplicities of zero points there exists a positive continuous function **L** such that $F$ has bounded **L**-index in joint variables. Our propositions are generalizations of Pugh's result obtained for entire functions of one variable of bounded index.

## 1. Introduction

Let us introduce a main definition. Let $l:\mathbb{C}\to\mathbb{R}_+$ be a fixed positive continuous function, where $\mathbb{R}_+=(0,+\infty)$. An entire function $f$ is said to be of bounded $l$-index [15,25] if there exists an integer $m$, independent of $z$, such that for all $p$ and all $z\in\mathbb{C}$, $\frac{|f^{(p)}(z)|}{l^p(z)p!}\leq\max\left\{\frac{|f^{(s)}(z)|}{l^s(z)s!}:0\leq s\leq m\right\}$. The least such integer $m$ is called the $l$-index of $f(z)$ and is denoted by $N(f,l)$. If $l(z)\equiv 1$, then we obtain a definition of function of *bounded index* [16] and in this case we denote $N(f):=N(f,1)$.

In 1970, W. J. Pugh and S. M. Shah [22] posed some questions on properties of entire functions of bounded index. One of these questions is following. II. Classes of functions of bounded index: is the sum (or product) of two functions of bounded index also of bounded index?

Later W. J. Pugh [21] proved that class of entire functions of bounded index is not closed under the operation of addition of the functions. He presented an example of two functions, for which its sum is a function of unbounded index. Also, there were

*Key words and phrases.* Entire function of several variables, bounded **L**-index in joint variables, sum of entire functions.

2010 *Mathematics Subject Classification.* Primary: 32A15. Secondary: 32A17, 30D20.

*Received*: July 19, 2019.

*Accepted*: February 27, 2020.

deduced conditions providing index boundedness for sum of entire functions, when one addend is a function of bounded index. His example was based on the fact that every entire function with bounded multiplicities of zeros has unbounded index. Moreover, bounded multiplicities of zeros of the entire function $f : \mathbb{C} \to \mathbb{C}$ is necessary and sufficient condition for existence of some positive continuous function $l : \mathbb{C} \to \mathbb{R}_+$ such that $f$ has bounded $l$-index [13].

There are two approaches to introduce concept of index boundedness in multidimensional complex space. The first approach uses directional derivatives in the definition. It generates a concept of entire function of bounded $L$-index in direction [4, 7], where $L : \mathbb{C}^n \to \mathbb{R}^n_+$ is a positive continuous function. And the second approach uses all possible partial derivatives in the definition. It leads to a concept of entire function of bounded $\mathbf{L}$-index in joint variables [3, 9], where $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ is a positive continuous vector-valued function. Pugh's example and his theorem was generalized for entire functions of bounded $L$-index in direction (see [8, 11]).

Of course, the similar question can be posed for entire functions of bounded $\mathbf{L}$-index in joint variables: What are sufficient conditions that sum of entire functions of bounded $\mathbf{L}$-index in joint variables is also a function of bounded $\mathbf{L}$-index in joint variables?

In [10], there were generalized Pugh's example and the sufficient conditions for this class of functions, if $\mathbf{L} \equiv \mathbf{1}$, i.e., for entire functions of bounded index in joint variables. Here we will formulate and prove theorems which contain sufficient conditions for arbitrary positive continuous vector-function $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}_+$.

Note that for every entire function $F$ with bounded multiplicities of zero points [12, 13] there exists a positive continuous function $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ such that $F$ is of bounded $\mathbf{L}$-index in joint variables. Thus, the concept of bounded $\mathbf{L}$-index in joint variables allows studying properties of very wide class of entire functions.

The concepts of bounded $L$-index in a direction and bounded $\mathbf{L}$-index in joint variables have applications in analytic theory of partial differential equations. A connection between these classses of entire functions is partially established in [6, 9]. They allow investigating properties of entire solutions of partial differential equations [4, 7] and their system [19]. Index boundedness of entire solution yields some sharp growth estimates, uniform distribution of zeros, regular behavior of its derivatives, etc. There is also known such a result [23] that if entire functions $f$ and $g$ satisfy differential equations with some additional conditions, then $f + g$ will be of bounded index. Besides, another objects of investigations in theory of bounded index are functions analytic in a polydisc [2], in a ball [5] or in Cartesian product of a disc and a complex plane [1].

## 2. Notations, Definitions and Auxiliary Results

Let us introduce some standard notations in theory of entire functions of several variables. Let $\mathbb{R}^n$ and $\mathbb{C}^n$ be $n$-dimensional real and complex vector spaces, respectively, $n \in \mathbb{N}$. Denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$. For $K =$

$(k_1,\dots,k_n)\in\mathbb{Z}^n_+$ let us write $\|K\|=k_1+\dots+k_n$, $K!=k_1!\cdots k_n!$. For $A=(a_1,\dots,a_n)\in\mathbb{C}^n$, $B=(b_1,\dots,b_n)\in\mathbb{C}^n$, we will use formal notations without violation of the existence of these expressions $A\pm B=(a_1\pm b_1,\dots,a_n\pm b_n)$, $AB=(a_1b_1,\dots,a_nb_n)$, $A/B=(a_1/b_1,\dots,a_n/b_n)$, $A^B=a_1^{b_1}a_2^{b_2}\cdots a_n^{b_n}$. For $A$, $B\in\mathbb{R}^n$ $\max\{A,B\}=(\max\{a_1,b_1\},\dots,\max\{a_n,b_n\})$, a notation $A<B$ means that $a_j<b_j$ for all $j\in\{1,\dots,n\}$. Similarly, the relation $A\le B$ is defined.

For $R=(r_1,\dots,r_n)$ we denote by $\mathbb{D}^n(z^0,R):=\{z\in\mathbb{C}^n: |z_j-z_j^0|<r_j, j\in\{1,\dots,n\}\}$ the polydisc, by $\mathbb{T}^n(z^0,R):=\{z\in\mathbb{C}^n: |z_j-z_j^0|=r_j, j\in\{1,\dots,n\}\}$ its skeleton and by $\mathbb{D}^n[z^0,R]:=\{z\in\mathbb{C}^n: |z_j-z_j^0|\le r_j, j\in\{1,\dots,n\}\}$ the closed polydisc.

For a partial derivative of entire function $F(z)=F(z_1,\dots,z_n)$ we will use the notation

$$F^{(K)}(z)=\frac{\partial^{\|K\|}F}{\partial z^K}=\frac{\partial^{k_1+\dots+k_n}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}},\quad \text{where } K=(k_1,\dots,k_n)\in\mathbb{Z}^n_+.$$

Let $\mathbf{L}(z)=(l_1(z),\dots,l_n(z))$, where $l_j(z)$ are positive continuous functions of variable $z\in\mathbb{C}^n$, $j\in\{1,2,\dots,n\}$.

An entire function $F(z)$ is called a function of bounded $\mathbf{L}$-index in joint variables [3,9], if there exists a number $m\in\mathbb{Z}_+$ such that for all $z\in\mathbb{C}^n$ and $J=(j_1,j_2,\dots,j_n)\in\mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}\le\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}: K\in\mathbb{Z}^n_+, \|K\|\le m\right\}. \tag{2.1}$$

The least integer $m$ for which inequality (2.1) holds is called $\mathbf{L}$-index in joint variables of the function $F$ and is denoted by $N(F,\mathbf{L})$. If $l_j(z_j)\equiv 1$, $j\in\{1,2,\dots,n\}$, then the entire function is called a function of bounded index (in joint variables) [14,17,18,20,24].

For $R\in\mathbb{R}^n_+$, $j\in\{1,\dots,n\}$ and $\mathbf{L}(z)=(l_1(z),\dots,l_n(z))$ we define

$$\lambda_{1,j}(z_0,R)=\inf\left\{l_j(z)/l_j(z^0)\colon z\in D^n\left[z^0,R/\mathbf{L}(z^0)\right]\right\},\quad \lambda_{1,j}(R)=\inf_{z^0\in\mathbb{C}^n}\lambda_{1,j}(z_0,R),$$

$$\lambda_{2,j}(z_0,R)=\sup\left\{l_j(z)/l_j(z^0)\colon z\in D^n\left[z^0,R/\mathbf{L}(z^0)\right]\right\},\quad \lambda_{2,j}(R)=\sup_{z^0\in\mathbb{C}^n}\lambda_{2,j}(z_0,R),$$

$$\Lambda_k(R)=(\lambda_{k,j}(R),\dots,\lambda_{k,n}(R)),\quad k\in\{1,2\}.$$

By $Q^n$ we denote a class of functions $\mathbf{L}(z)$ which for some $R^0\in\mathbb{R}^n_+$ satisfy the condition

$$0<\Lambda_1(R^0)\le\Lambda_2(R^0)<+\infty. \tag{2.2}$$

Note that if (2.2) holds for some $R_0$ then it is valid for all $R\in\mathbb{R}^n_+$.

We need the following proposition.

**Theorem 2.1** ([9])**.** *Let $\mathbf{L}\in Q^n$. An entire function $F$ has bounded $\mathbf{L}$-index in joint variables if and only if for any $R'$, $R''\in\mathbb{R}^n_+$, $\mathbf{0}<R'<R''$, there exists a number*

$p_1 = p_1(R', R'') \geq 1$ *such that for every* $z^0 \in \mathbb{C}^n$

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R''}{\mathbf{L}(z^0)}\right)\right\} \leq p_1 \max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R'}{\mathbf{L}(z^0)}\right)\right\}. \tag{2.3}$$

**Lemma 2.1** ([3])**.** *If* $\mathbf{L} \in Q^n$, *then for every* $j \in \{1, \ldots, n\}$ *and for every fixed* $z^* \in \mathbb{C}^n$ $|z_j| l_j(z^* + z_j \mathbf{1}_j) \to \infty$ *as* $|z_j| \to \infty$.

## 3. Main Result

**Theorem 3.1.** *Let* $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ *be a continuous function,* $F$, $G$ *be entire functions in* $\mathbb{C}^n$, *which obey the following conditions:*

a) $G(z)$ *has bounded* $\mathbf{L}$*-index in joint variables with* $N(G, \mathbf{L}) = N < +\infty$;

b) *there exists* $\alpha \in (0, 1)$ *such that for all* $z \in \mathbb{C}^n$ *and for every* $\|P\| \geq N + 1$, $P \in \mathbb{N}^n$,

$$\frac{|G^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq \alpha \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N\right\}; \tag{3.1}$$

c) *for some* $z^0 \in \mathbb{C}^n$, $F(z^0) \neq 0$, *and every* $z \in \mathbb{C}^n$ *one has*

$$\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, \frac{2R}{\mathbf{L}(z)}\right)\right\} \leq \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N\right\}, \tag{3.2}$$

*where* $r_j = |z_j - z_j^0| l_j(z)$, $R = (r_1, \ldots, r_n)$;

d) *one of the following conditions is valid: either exists* $c \geq 1$ *for all* $z \in \mathbb{C}^n$ *such that* $|z_j - z_j^0| l_j(z) \leq 1$ *for some* $j \in \{1, \ldots, n\}$ *one has*

$$\frac{\max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}}{|F(z^0)|} \leq c < +\infty,$$

*or* $\mathbf{L} \in Q^n$.

*Then for every* $\varepsilon \in \mathbb{C}$, $|\varepsilon| \leq \frac{1-\alpha}{2c}$ *the function*

$$H(z) = G(z) + \varepsilon F(z) \tag{3.3}$$

*has bounded* $\mathbf{L}$*-index in joint variables and* $N(H, \mathbf{L}) \leq N$.

*Proof.* The proof uses methods and ideas from [8, 10, 21]. One should observe that for $\mathbf{L} \in Q^n$ by Lemma 2.1 the set $A := \{z \in \mathbb{C}^n\colon |z_j - z_j^0| l_j(z) \leq 1$ for some $j \in \{1, \ldots, n\}\}$ is bounded. Then there exits $c \geq 1$ such that for every $z \in A$ the inequality

$$\frac{\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, 2\max\left\{\frac{\Lambda_2(\mathbf{1})}{\mathbf{L}(z^0)}, \frac{R}{\mathbf{L}(z)}\right\}\right)\right\}}{|F(z^0)|} \leq c < +\infty \tag{3.4}$$

holds.

We write Cauchy's formula for the entire function $F(z)$

$$\frac{F^{(P)}(z)}{P!} = \frac{1}{(2\pi i)^n} \int\limits_{z'\in\mathbb{T}^n(z,R/\mathbf{L}(z))} \frac{F(z')}{(z'-z)^{P+\mathbf{1}}} dz'. \tag{3.5}$$

For the chosen $r_j = |z_j - z_j^0| l_j(z)$ we have

$$\frac{r_j}{l_j(z)} = |z_j' - z_j| \geq |z_j' - z_j^0| - |z_j - z_j^0| = |z_j' - z_j^0| - \frac{r_j}{l_j(z)}.$$

Hence,

$$|z_j' - z_j^0| \leq \frac{2r_j}{l_j(z)}. \tag{3.6}$$

From (3.5) it follows that

$$\begin{aligned}\frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq & \frac{1}{(2\pi)^n \mathbf{L}^P(z)} \cdot \frac{\mathbf{L}^{P+\mathbf{1}}(z)}{R^{P+\mathbf{1}}} \prod_{j=1}^n \frac{2\pi r_j}{l_j(z)} \max\left\{|F(z')| : z' \in \mathbb{T}^n(z, R/\mathbf{L}(z))\right\} \\ \leq & \frac{1}{R^P} \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}. \end{aligned} \tag{3.7}$$

If $R^P > 1$, then (3.7) means that

$$\frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}. \tag{3.8}$$

Let $R^P \in (0,1]$. Then for some $j \in \{1,\dots,n\}$, $r_j = |z_j - z_j^0| l_j(z) \in (0,1]$. Putting $r_j = 1$ for those $j$ in (3.5) and (3.6) and $R' = \max\{\mathbf{1}, R\}$, we similarly deduce

$$\begin{aligned}\frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq & \frac{1}{R'^P} \max\left\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R'/\mathbf{L}(z))\right\} \\ = & \frac{1}{R'^P} \frac{\max\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R'/\mathbf{L}(z))\}}{\max\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\}} \\ & \times \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\} \\ \leq & \frac{\max\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\}}{R'^P |F(z^0)|} \\ & \times \max\left\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\} \\ \leq & c \max\left\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}, \end{aligned} \tag{3.9}$$

where

$$c = \sup_{\substack{z\in\mathbb{C}^n, \exists j \in\{1,\dots,n\} \\ |(z_j - z_j^0) l_j(z)| \leq 1}} \frac{\max\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\}}{|F(z^0)|}.$$

If $\mathbf{L} \in Q^n$, then $\sup\left\{\frac{l_j(z^0)}{l_j(z)} : z^0 \in \mathbb{D}^n(z, \frac{\mathbf{1}}{\mathbf{L}(z)})\right\} \le \lambda_{2,j}(\mathbf{1})$. This means that $l_j(z) \ge \frac{l_j(z^0)}{\lambda_{2,j}(\mathbf{1})}$. Using the inequality, we choose

$$c := \sup_{\substack{z\in\mathbb{C}^n, \exists j\in\{1,\ldots,n\} \\ |(z_j - z_j^0)l_j(z)|\le 1}} \frac{\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, 2\max\left\{\frac{\Lambda_2(\mathbf{1})}{\mathbf{L}(z^0)}, \frac{R}{\mathbf{L}(z)}\right\}\right)\right\}}{|F(z^0)|} \ge 1$$

in (3.9). In view of (3.8) and (3.9), one has

$$\frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le c \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}, \tag{3.10}$$

for all $P \in \mathbb{Z}_+^n$.

We differentiate equality (3.3) $p$ times, $\|P\| = p \ge N+1$, and apply consequently (3.1), (3.10) and (3.2) to the obtained equality

$$\begin{aligned} \frac{|H^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le& \frac{|G^{(P)}(z)|}{P!\mathbf{L}^P(z)} + \frac{|\varepsilon||F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le \alpha \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le N\right\} \\ &+ c|\varepsilon| \max\left\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\} \\ \le& (\alpha + c|\varepsilon|) \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le N\right\}. \end{aligned} \tag{3.11}$$

If $\|S\| \le N$ then (3.10) is true with $P = S$, but (3.1) does not hold. Therefore, the differentiation of (3.3) give us the lower estimate

$$\begin{aligned} \frac{|H^{(S)}(z)|}{S!\mathbf{L}^S(z)} \ge& \frac{|G^{(S)}(z)|}{S!\mathbf{L}^S(z)} - \frac{|\varepsilon||F^{(S)}(z)|}{S!\mathbf{L}^S(z)} \\ \ge& \frac{|G^{(S)}(z)|}{S!\mathbf{L}^S(z)} - c|\varepsilon| \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}, \end{aligned} \tag{3.12}$$

where $\|S\| \le N$. From (3.2) and (3.12) we conclude

$$\max\left\{\frac{|H^{(S)}(z)|}{S!\mathbf{L}^S(z)} : \|S\| \le N\right\} \ge (1 - c|\varepsilon|) \max\left\{\frac{|G^{(S)}(z)|}{S!\mathbf{L}^S(z)} : \|S\| \le N\right\}. \tag{3.13}$$

If $c|\varepsilon| < 1$, then (3.11) and (3.13) yield

$$\frac{|H^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le \frac{\alpha + c|\varepsilon|}{1 - c|\varepsilon|} \max_{0\le s\le N} \left\{\frac{|H^{(S)}(z)|}{S!\mathbf{L}^S(z)} : \|S\| \le N\right\},$$

for $\|P\| \ge N+1$. Assume that $\frac{\alpha + c|\varepsilon|}{1 - c|\varepsilon|} \le 1$. Then $|\varepsilon| \le \frac{1-\alpha}{2c}$. For this $\varepsilon$ the function $H$ has bounded $\mathbf{L}$-index in joint variables with $N(H, \mathbf{L}) \le N$. Proof of Theorem 3.1 is complete. □

*Remark* 3.1. Every entire function $F$ with $N(F, \mathbf{L}) = 0$ obeys inequality (3.4) (see proof of necessity of Theorem 3 in [9]).

If $\mathbf{L} \in Q^n$, then condition b) in Theorem 3.1 is always satisfied. The next theorem is valid.

**Theorem 3.2.** *Let $\mathbf{L} \in Q^n$, $\alpha \in (0, 1)$ and $F$, $G$ be entire functions in $\mathbb{C}^n$, which satisfy the following conditions:*

*a) $G(z)$ has bounded $\mathbf{L}$-index in joint variables;*

*b) for some $z^0 \in \mathbb{C}^n$, $F(z^0) \neq 0$, and every $z \in \mathbb{C}^n$ the inequality holds*

$$\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, \frac{2R}{\mathbf{L}(z)}\right)\right\} \leq \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\},$$

*where $r_j = |z_j - z_j^0| l_j(z)$, $R = (r_1, \dots, r_n)$.*

*If $|\varepsilon| \leq \frac{1-\alpha}{2c}$, then the function*

$$H(z) = G(z) + \varepsilon F(z)$$

*has bounded $\mathbf{L}$-index in joint variables, with $N(H, L) \leq N(G_\alpha, \mathbf{L}_\alpha)$, where $G_\alpha(z) = G(z/\alpha)$, $\mathbf{L}_\alpha(z) = \mathbf{L}(z/\alpha)$.*

*Proof.* Condition b) in Theorem 3.1 always is obeyed with $N(G_\alpha, \mathbf{L}_\alpha)$ instead $N = N(G, \mathbf{L})$, where $G_\alpha(z) = G(z/\alpha)$, $\mathbf{L}_\alpha(z) = \mathbf{L}(z/\alpha)$, $\alpha \in (0, 1)$. Indeed, by Theorem 2.1, inequality (2.3) holds for the function $G$. Substituting $\frac{z^0}{\alpha}$, $\frac{z}{\alpha}$ instead $z^0$, $z$ respectively in (2.3), we obtain

$$\begin{aligned} &\max\left\{|G(z/\alpha)| \colon z \in \mathbb{T}^n\left(z^0, \frac{R''\alpha}{\mathbf{L}(z^0/\alpha)}\right)\right\} \\ \leq &p_1 \max\left\{|G(z/\alpha)| \colon z \in \mathbb{T}^n\left(z^0, \frac{R'\alpha}{\mathbf{L}(z^0/\alpha)}\right)\right\}. \end{aligned} \tag{3.14}$$

By Theorem 2.1 inequality (3.14) yields that $G_\alpha(z) = G(z/\alpha)$ is of bounded $\mathbf{L}_\alpha$-index in joint variables and vice versa. Therefore, for each $\|P\| \geq N(G_\alpha, \mathbf{L}_\alpha) + 1$ and $\alpha \in (0, 1)$ we have

$$\begin{aligned} \frac{|G_\alpha^{(P)}(z)|}{P!\mathbf{L}_\alpha^P(z)} =&\frac{|G^{(P)}(z/\alpha)|}{\alpha^{\|P\|}P!\mathbf{L}^P(z/\alpha)} \leq \max\left\{\frac{|G_\alpha^{(S)}(z)|}{S!\mathbf{L}_\alpha^S(z)} \colon \ \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\} \\ =&\max\left\{\frac{|G^{(S)}(z/\alpha)|}{\alpha^{\|S\|}S!\mathbf{L}^S(z/\alpha)} : \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\}. \end{aligned}$$

Hence, for all $z \in \mathbb{C}^n$

$$\begin{aligned} \frac{|G^{(P)}(z/\alpha)|}{P!\mathbf{L}^p(z/\alpha)} \leq &\max\left\{\frac{\alpha^{\|P\|-\|S\|}|G(z/\alpha)|}{S!\mathbf{L}^S(z/\alpha)} : \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\} \\ \leq &\alpha \max\left\{\frac{|G^{(S)}(z/\alpha)|}{S!\mathbf{L}^S(z/\alpha)} : \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\}. \end{aligned} \tag{3.15}$$

Thus, from inequality (3.15) it follows (3.1). □

It is easy to verify that $N(G_\alpha, \mathbf{L}_\alpha) \leq N(G, \mathbf{L})$ for $\alpha \in (0, 1)$. Thus, $N(G_\alpha, \mathbf{L}_\alpha)$ in Theorem 3.2 can be replaced by $N(G, \mathbf{L})$.

**Acknowledgements.** Author cordially thanks Prof. O. B. Skaskiv (Lviv, Ukraine) for his questions and interesting ideas which help my studies.

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