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A NOTE ON THE BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF FRACTIONAL INTEGRALS IN GRAND VARIABLE HERZ-MORREY SPACES

MEHVISH SULTAN[1] AND BABAR SULTAN[2]

Abstract. In this paper we obtain the boundedness of the higher order commutators of the fractional integral operator of variable order on the grand variable Herz-Morrey spaces.

1. Introduction

We consider the Riesz potential operator

$$I^{\lambda(x)}f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\lambda(x)}}dy,\quad 0<\lambda(x)<n.$$

Note that $\lambda(x)$ is the order of the Riesz potential operator which is variable. Researchers in harmonic analysis, functional analysis, and related areas often use these variable exponent function spaces to address challenging problems that cannot be adequately handled by classical function spaces. They offer a powerful framework for studying functions with diverse characteristics, making them a valuable tool in modern mathematical research. This field has seen substantial advancements in recent times. This progress involves the development of new techniques, deeper understanding of the structure and properties of these function spaces, some instances of this work are in [6, 7, 14, 15].

In [16], authors describe a Sobolev-type theorem that concerns the potential operator $I^{\lambda(x)}$ mapping from a Lebesgue space $L^{p(\cdot)}$ into a weighted Lebesgue space $L^{p(\cdot)}_w$

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in $\mathbb{R}^n$. The theorem is established under specific conditions on the exponent function $p(x)$, which is allowed to vary with the point x.

Grand variable Herz spaces is the generalization of Herz spaces, the boundedeness of some operators can be checked in [19, 25–27]. Moreover the grand variable Herz-Morrey spaces are the generalization of grand variable Herz spaces and they were introduced in [20, 21]. Grand weighted Herz spaces and grand weighted Herz-Morrey spaces are the generalizations of weighted Herz spaces and weighted Herz-Morrey spaces respectively, see for instance [22, 23]. For more results in these spaces see [1–4, 9, 10, 18, 24].

Inspired by the concept, in this article we will obtain the boundedness of higher order commutators of fractional integrals of variable order in grand variable Herz-Morrey spaces. We divided this article into different sections. Apart from introduction, a section is dedicated for providing necessary definitions, lemmas, and preliminary results. Then in the last section we will prove our main results.

2. Preliminaries

This section is dedicated to some necessary definitions and important lemmas to prove our main results.

2.1. **Function spaces with variable exponent.** For this section we refer to [7, 14, 15].

Definition 2.1. Let Q be a measurable set in $\mathbb{R}^n$ and $r(\cdot)\colon\ Q \to [1, +\infty)$ be a measurable function. We suppose that

$$1 \leq r_-(Q) \leq r(q) \leq r_+(Q) < +\infty, \tag{2.1}$$

where $r_- := \operatorname{ess}\inf_{q\in Q} r(q)$, $r_+ := \operatorname{ess}\sup_{q\in Q} r(q)$.

(a) Variable Lebesgue spaces $L^{r(\cdot)}(Q)$ are defined as

$$L^{r(\cdot)}(Q) = \left\{ h \text{ measurable} : \int_Q \left(\frac{|h(x)|}{\gamma}\right)^{r(x)} dx < +\infty, \text{ where } \gamma \text{ is a constant} \right\}.$$

Norm in $L^{r(\cdot)}(Q)$ can be defined as

$$\|h\|_{L^{r(\cdot)}(Q)} = \inf\left\{ \gamma > 0 : \int_Q \left(\frac{|h(x)|}{\gamma}\right)^{r(x)} dx \leq 1 \right\}.$$

(b) The space $L^{r(\cdot)}_{\text{loc}}(Q)$ is defined as

$$L^{r(\cdot)}_{\text{loc}}(Q) := \left\{ h : h \in L^{r(\cdot)}(K) \text{ for all compact subsets } K \subset Q \right\}.$$

The log-condition is stated as follows:

$$|r(x) - r(y)| \leq \frac{C(r)}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2},\ x, y \in Q, \tag{2.2}$$

where $C(r) > 0$.

And the decay condition: there exists a number $r_\infty \in (1, +\infty)$, such that

$$|r(x) - r_\infty| \le \frac{C}{\ln(e + |x|)}, \tag{2.3}$$

and also decay condition

$$|r(x) - r_0| \le \frac{C}{\ln|x|}, \quad |x| \le \frac{1}{2}, \tag{2.4}$$

holds for some $r_0 \in (1, +\infty)$.

We use these notations in this article.

(i) $B(x, r)$ is the ball of radius r and center at the point x.
(ii) $B_k := B(0, 2^k) = \{x \in \mathbb{R}^n : |x| < 2^k\}$ for all $k \in \mathbb{Z}$.
(iii) $R_{t,\tau} := B_\tau \setminus B_t = \{x : t < |x| < \tau\}$ is a spherical layer.
(iv) $R_k := B_k \setminus B_{k-1}$.
(v) $\chi_k := \chi_{R_k}$, $\chi_{t,\tau} := \chi_{R_{t,\tau}}$.
(vi) Let $f \in L^1_{\mathrm{loc}}(H)$ be a locally integrable function, then the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined as

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n} \int\limits_{B(x,r)} |f(x)| dx, \quad x \in H,$$

where

$$B(x, r) := \{y \in H : |x - y| < r\}.$$

(vii) The set $\mathcal{P}(H)$ consists of all measurable functions $q(\cdot)$ satisfying $q_- > 1$ and $q_+ < +\infty$.
(viii) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(H)$ consists of all functions $q \in \mathcal{P}(H)$ satisfying (2.1) and (2.2).
(ix) $\mathcal{P}_\infty(H)$ and $\mathcal{P}_{0,+\infty}(H)$ are subsets of $\mathcal{P}(H)$ and the values of these subsets lie in $[1, +\infty)$ which satisfy the condition (2.3) and both conditions (2.3) and (2.4), respectively.
(x) In what follows, we denote $\chi_l = \chi_{R_l}$, $R_l = B_l \setminus B_{l-1}$, $B_l = B(0, 2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}$ for all $l \in \mathbb{Z}$.
(xi) By $p'(x) = p(x)/(p(x) - 1)$, we denote the conjugate exponent of $p(\cdot)$.
(xii) C is a constant, its value varies from line to line and is independent of the main parameters involved.

2.2. **Variable exponent Herz spaces.** In this section we will define variable exponent Herz spaces.

Definition 2.2. Let $p, q \in [1, +\infty)$ and $\alpha \in \mathbb{R}$, the norms of classical versions of non-homgeneous and homgeneous Herz spaces are given below,

$$\|f\|_{K^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(B(0,1))} + \left\{ \sum_{i\in\mathbb{N}} 2^{i\alpha q} \left(\int_{F_{2^{i-1},2^i}} |f(y)|^p dy \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

where

$$\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{i\in\mathbb{Z}} 2^{i\alpha q} \left(\int_{F_{2^{i-1},2^i}} |f(y)|^p dy \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

respectively, where $F_{2^{i-1},2^i} := B(0, 2^i) \backslash B(0, 2^{i-1})$.

Definition 2.3. Let $q \in [1, +\infty)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$. The homgeneous version of variable exponent Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined as

$$\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} = \left(\sum_{i=-\infty}^{+\infty} \|2^{\alpha i} f \chi_i\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}.$$

Definition 2.4. Let $q \in [1, +\infty)$, $\alpha \in \mathbb{R}$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The non-homgeneous version of variable exponent Herz space $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined as

$$K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p(\cdot)}(B(0,1))} + \left(\sum_{i=-\infty}^{+\infty} \|2^{\alpha i} f \chi_i\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}.$$

2.3. **Herz-Morrey spaces.** Now we will define variable Herz-Morrey spaces.

Definition 2.5. Let $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $0 < q < +\infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 \leq \beta < +\infty$. A variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\beta}_{q,p(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\sum_{i=-\infty}^{k_0} \|2^{\alpha(\cdot)i} f \chi_i\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}.$$

Definition 2.6. To define homogeneous version of GVHM spaces, let $p : \mathbb{R}^n \to [1, +\infty)$, $q \in [1, +\infty)$, $\theta > 0$, $0 \le \beta < +\infty$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The GVHM spaces are given by

$$M\dot{K}^{\alpha(\cdot),q),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),q),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)} < +\infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q),\theta}_{\beta,p(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{i=-\infty}^{k_0} \|2^{i\alpha(\cdot)} f \chi_i\|^{q(1+\epsilon))}_{L^{p(\cdot)}} \right)^{\frac{1}{q(1+\epsilon)}}.$$

The non-homogeneous version of GVHM spaces can be defined in a similar way.

As grand variable Herz-Morrey spaces are the generalization of grand variable Herz spaces, $\beta = 0$, grand variable Herz-Morrey spaces become grand variable Herz spaces.

Definition 2.7 (BMO space)**.** A BMO function is a locally integrable function u whose mean oscillation given by $\frac{1}{|B|}\int_B |u(y) - u_B| dy$ is bounded. Mathematically,

$$\|u\|_{BMO} = \sup_B \frac{1}{|B|} \int\limits_B |u(y) - u_B| dy < +\infty.$$

2.4. **Basic Lemmas.**

Lemma 2.1. ([17]) Let $D > 1$ and $q \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then,

$$\frac{1}{c_0} r^{\frac{n}{q(0)}} \le \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}} \le c_0 r^{\frac{n}{q(0)}}, \quad \text{for } 0 < r \le 1,$$

and

$$\frac{1}{c_\infty} r^{\frac{n}{q_\infty}} \le \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}} \le c_\infty r^{\frac{n}{q_\infty}}, \quad \text{for } r \ge 1,$$

respectively, where $c_0 \ge 1$ and $c_\infty \ge 1$ depend on D but do not depend on r.

Hölder's inequality in the variable exponent case has the form:

$$\int\limits_\Omega |f(x)g(x)|\, dx \le \kappa \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $\kappa = \frac{1}{p_-} + \frac{1}{(p')_-}$ and Ω is a measurable subset of $\mathbb{R}^n$.

The next statement is the generalized Hölder inequality for variable exponent Lebesgue spaces (see [7, 14]).

Lemma 2.2. *Let Q be a measurable subset of $\mathbb{R}^n$ and $p(\cdot)$ be an exponent such that $1 \le p_-(Q) \le p_+(Q) < +\infty$ and $q_-, r_- > 1$. Then,*

$$\|fg\|_{L^{r(\cdot)}(Q)} \le 2^{1/r_-} \|f\|_{L^{p(\cdot)}(Q)} \|g\|_{L^{q(\cdot)}(Q)}$$

holds, where $f \in L^{p(\cdot)}(Q)$, $g \in L^{q(\cdot)}(Q)$ and $\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$.

Lemma 2.3 ([13]). *Let $b \in BMO(\mathbb{R}^n)$, $m \in \mathbb{N}$, $r_0, t_0 \in \mathbb{Z}$, with $r_0 < t_0$. Then, we have*

$$C^{-1}\|b\|^m_{BMO(\mathbb{R}^n)} \leq \sup_{B:ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}} \|(b-b_B)^m \chi_B\|_{L^{q(\cdot)}} \leq C\|b\|^m_{BMO(\mathbb{R}^n)},$$
$$\|(b-b_{B_{r_0}})^m \chi_{B_{t_0}}\|_{L^{q(\cdot)}} \leq C(t_0-r_0)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_{B_{t_0}}\|_{L^{q(\cdot)}}.$$

Assuming that the order of the Riesz potential operator $\lambda(x)$ is not continuous but rather is a measurable function in $\mathbb{R}^n$ adds an interesting twist to the study of potential operators. We mentioned that the measurable function $\lambda(x)$ satisfies certain conditions:

(a) $\lambda_0 := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} \lambda(x) > 0$;
(b) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} r(x)\lambda(x) < n$;
(c) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} r(\infty)\lambda(x) < n$.

Proposition 2.1 ([16]). *Let $r(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}_{0,\infty}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, and assume*

$$1 < r(\infty) \leq r(x) \leq r_+ < +\infty.$$

Let $\lambda(x)$ be a measurable function in $\mathbb{R}^n$ satisfying the conditions (1), (2) *and* (3). *Under these conditions, we establish the following weighted Sobolev-type estimate for the fractional operator $I^{\lambda(\cdot)}$:*

$$\|(1+|x|)^{-\gamma(x)} I^{\lambda(\cdot)}(f)\|_{L^{r(\cdot)}} \leq C \|f\|_{L^{r'(\cdot)}},$$

where

$$\frac{1}{r(x)} = \frac{1}{r'(x)} - \frac{\lambda(x)}{n}$$

is the Sobolev exponent

$$\gamma(x) = C\lambda(x)\left(1 - \frac{\lambda(x)}{n}\right) \leq \frac{n}{4}C,$$

where C is the Dini-Lipschitz constant from the inequality (2.3) *in which $\alpha(\cdot)$ replaced by $r(\cdot)$.*

Remark 2.1. (i) Under the given condition (2.3): $|\lambda(x) - \lambda_\infty| \leq \frac{C_\infty}{\ln(e+|x|)}$ for $x \in \mathbb{R}^n$. Then, we can check that $(1+|z|)^{-\gamma(z)}$ is equivalent to the weight $(1+|z|)^{-\gamma_\infty}$.

(ii) The behavior of the Riesz potential operator over a bounded domain and how this behavior doesn't change significantly when replacing the variable order $\lambda(x)$ with $\lambda(y)$. The difference in potentials is unessential if the function $\lambda(x)$ satisfies the smoothness logarithmic condition as given in (2.2)

$$C_1 |x-y|^{n-\lambda(y)} | \leq |x-y|^{n-\lambda(x)} \leq C_2 |x-y|^{n-\lambda(y)}.$$

Let b be a locally integrable function, $0 < \lambda(x) < n$, and $m \in \mathbb{N}$; the higher order commutators of fractional integrable operator of variable order $I_m^{\lambda(\cdot),b}$ are defined by

$$I_m^{\lambda(\cdot),b} f(x) = \int_{\mathbb{R}^n} \frac{[b(x)-b(y)]^m}{|x-y|^{n-\lambda(x)}} f(y)dy. \tag{2.5}$$

When $m=0$, then $I_0^{\lambda(\cdot),b}=I^{\lambda(\cdot)}$ and for $m=1$, $I_1^{\lambda,b}=[b,I^{\lambda}]$. The Hardy-Littlewood-Sobolev theorem provides an important result regarding the boundedness of the fractional integral operator, denoted as I^{λ}, from one Lebesgue space to another under certain conditions. Specifically, the theorem asserts that this boundedness holds when the exponents of the Lebesgue spaces satisfy the equation:

$$\frac{1}{r_1}-\frac{1}{r_2}=\frac{\lambda}{n}, \tag{2.6}$$

where $0<r_1<r_2<+\infty$.

The next proposition is a generalization of variable exponents Herz spaces in [5]. We omit the proof of Proposition 2.2 since it is essentially similar to the proof given in [5] and with slight modification, we can obtain following result in grand variable Herz-Morrey spaces.

Proposition 2.2. *Let α,p,q are as defined in Definition 2.6, then*

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} &= \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{i=-\infty}^{k_0}\|2^{i\alpha(\cdot)}f\chi_i\|_{L^{p(\cdot)}}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\ &\approx \max\left\{\sup_{\epsilon>0}\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\|f\chi_i\|_{p(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},\right.\\ &\quad \sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\|f\chi_i\|_{L^{p(\cdot)}}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\ &\quad \left.+\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{i=0}^{k_0}2^{i\alpha_\infty q(1+\epsilon)}\|f\chi_i\|_{L^{p(\cdot)}}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\right\}.\end{aligned}$$

3. Boundedness Results for Grand Variable Herz-Morrey Spaces

Theorem 3.1. *Let $1<q<+\infty$, $b\in BMO(\mathbb{R}^n)$, $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\lambda(\cdot)}{n}$ and $0<\beta<+\infty$. If $0<\lambda(\cdot)<n$ and $\alpha,p_2\in\mathcal{P}_{0,\infty}(\mathbb{R}^n)$ such that*

$$\frac{-n}{p_{1\infty}}<a_\infty<\frac{n}{p'_{1\infty}},\quad \frac{-n}{p_1(0)}<\alpha(0)<\frac{n}{p'_1(0)},$$

then

$$\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f)\right\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)}\leq C\,\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f\in M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)$ and $f(x)=\sum_{t=-\infty}^{+\infty}f(x)\chi_t(x)=\sum_{t=-\infty}^{+\infty}f_t(x)$, we have

$$\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f)\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&=\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{i=-\infty}^{k_0}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f)\chi_i\right\|_{p_2(\cdot)}^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
&\le\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{i=-\infty}^{k_0}\left(\sum_{t=-\infty}^{\infty}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
&\le\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{i=-\infty}^{k_0}\left(\sum_{t=-\infty}^{i}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{i=-\infty}^{k_0}\left(\sum_{t=i+1}^{\infty}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
&=:E_1+E_2.
\end{aligned}
$$

Now we will find the estimate for E_1, for each $i\in\mathbb{Z}$ with $t\le i$ a.e. Let $x\in R_i$, $y\in R_t$. Then, $|x-y|\sim|x|\sim 2^i$, and we get

$$
\begin{aligned}
\left|I_m^{\lambda(\cdot),b}(f\chi_t)(x)\right| &\le C\int_{R_t}|x-y|^{\lambda(x)-n}|b(x)-b(y)|^m|f(y)|dy\\
&\le C2^{-in}\int_{R_t}|x|^{\lambda(x)}|b(x)-b(y)|^m|f(y)|dy\\
&\le C2^{-in}|x|^{\lambda(x)}\left(|b(x)-b_{B_t}|^m\int_{R_t}|f_t(y)|dy+\int_{R_t}|f_t(y)||b(y)-b_{B_t}|^m dy\right)\\
&\le C2^{-in}|x|^{\lambda(x)}\left\|f_t\right\|_{p_1(\cdot)}\left(|b(x)-b_{B_t}|^m\left\|\chi_t\right\|_{p_1'(\cdot)}+\left\|((b-b_{B_t})^m\chi_t\right\|_{p_1'(\cdot)}\right).
\end{aligned}
$$

It is known, see e.g. [30] that

$$
\begin{aligned}
I^{\lambda(\cdot)}((b(x)-b_{B_t})^m\chi_i)(x) &\ge I^{\lambda(\cdot)}(\chi_{B_i})(x)\cdot(\chi_i)(x)\\
&=\int_{B_i}\frac{|b(x)-b_{B_t}|^m}{|x-y|^{\lambda(x)-n}}dy\cdot\chi_i(x)\\
&\ge C|b(x)-b_{B_t}|^m|x|^{\lambda(x)}\cdot\chi_i(x)\\
&\ge C|b(x)-b_{B_t}|^m|x|^{\lambda(x)}\cdot\chi_i(x).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\\
\le& C2^{-in}\left\|f_t\right\|_{p_1(\cdot)}\left(\left\|(1+|x|)^{-\gamma(x)}|x|^{\lambda(x)}(b-b_{B_t})^m\chi_i\right\|_{p_2(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\right.\\
&\left.+\left\|(b-b_{B_t})^m\chi_t\right\|_{p_1'(\cdot)}\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}(\chi_{B_i})\right\|_{p_2(\cdot)}\right)\\
\le& C2^{-in}\left\|f_t\right\|_{p_1(\cdot)}\left(\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}((b-b_{B_t})^m\chi_i)\right\|_{p_2(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\right.
\end{aligned}
$$

$$+ \|(b-b_{B_t})^m\chi_t\|_{p_1'(\cdot)} \left\|(1+|x|)^{-\gamma(x)} I^{\lambda(\cdot)}(\chi_{B_i})\right\|_{p_2(\cdot)}\Big)$$
$$\le C 2^{-in} \|f_t\|_{p_1(\cdot)} \Big((i-t)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_i\|_{p_1(\cdot)} \|\chi_t\|_{p_1'(\cdot)}$$
$$+ \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_t\|_{p_1'(\cdot)} \|\chi_i\|_{p_1(\cdot)}\Big)$$
$$\le C 2^{-in}(i-t)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|f_t\|_{p_1(\cdot)} \|\chi_t\|_{p_1'(\cdot)} \|\chi_i\|_{p_1(\cdot)}.$$

By Lemma 2.8,

$$2^{-in}\|\chi_t\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)} \le C2^{-in}2^{\frac{in}{p_1(0)}}2^{\frac{tn}{p_1'(0)}} \le C2^{\frac{(t-i)n}{p_1'(0)}}. \tag{3.1}$$

Splitting E_1 by using Minkowski's inequality we have

$$E_1 \le \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^\theta \sum_{i=-\infty}^{k_0}\left(\sum_{t=-\infty}^{i}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\approx \max\Bigg\{\sup_{\epsilon>0}\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta \sum_{i=-\infty}^{-1} 2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},$$
$$\sup_{\epsilon>0}\sup_{k_0\ge0,k_0\in\mathbb{Z}} 2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta \sum_{i=-\infty}^{-1} 2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$+\sup_{\epsilon>0}\sup_{k_0\ge0,k_0\in\mathbb{Z}} 2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta \sum_{i=-0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\Bigg\}$$
$$:= \max\left\{E_1', E_{11}+E_{12}\right\}.$$

We will find the estimate for E_{11} and E_{12}, estimate for E_1' is obtained similarly. For E_{11} we get

$$E_{11} \le \sup_{\epsilon>0}\sup_{k_0\ge0,k_0\in\mathbb{Z}} 2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta \sum_{i=-\infty}^{-1} 2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}\left\|(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\ge0,k_0\in\mathbb{Z}} 2^{-k_0\beta}\Bigg(\epsilon^\theta \sum_{i=-\infty}^{-1} 2^{i\alpha(0)q(1+\epsilon)}$$

$$\times\left(\sum_{t=-\infty}^{i}(i-t)^m 2^{-in}\|f\chi_t\|_{p_1(\cdot)}\|\chi_t\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\Bigg)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=-\infty}^{i}(i-t)^m 2^{\frac{(t-i)n}{p_1'(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}.$$

Let $b_1 := \frac{n}{p_1'(0)}-\alpha(0)$, applying the fact $2^{-q(1+\epsilon)} < 2^{-q}$, Hölder's inequality and Fubini's theorem we get

$$E_{11}\leq C\|b\|_{BMO)(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}\left(\sum_{t=-\infty}^{i}2^{\alpha(0)t}\|f\chi_t\|_{p_1(\cdot)}(i-t)^m 2^{b_1(t-i)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}$$
$$\times\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^\theta\sum_{i=-\infty}^{-1}\left(\sum_{t=-\infty}^{i}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{b_1q(1+\epsilon)(t-i)/2}\right)\right.$$
$$\left.\times\left(\sum_{t=-\infty}^{i}(i-t)^{m(q(1+\epsilon))'/2}2^{b_1(q(1+\epsilon))'(t-i)/2}\right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}}\right]^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}$$
$$\times\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^\theta\sum_{i=-\infty}^{-1}\sum_{t=-\infty}^{i}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{b_1q(1+\epsilon)(t-i)/2}\right]^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}$$
$$\times\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^\theta\sum_{t=-\infty}^{-1}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\sum_{i=t}^{-1}2^{b_1q(1+\epsilon)(t-i)/2}\right]^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=-\infty}^{-1}2^{a(0)q(1+\epsilon)t}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$=C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=-\infty}^{k_0}\left\|2^{a(\cdot)t}f\chi_t\right\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}.$$

Now for E_{12}, by using Minkowski's inequality we have

$$\begin{aligned}E_{12} \leq & \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{i} \left\|\chi_i(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}} \\ \leq & \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{-1} \left\|\chi_i(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}} \\ & + \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=0}^{i} \left\|\chi_i(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}} \\ := & A_1 + A_2.\end{aligned}$$

The estimate for A_2 follows in a similar manner to E_{11} by replacing $p_1'(0)$ with $p_{1\infty}'$. For A_1 we have

$$2^{-in}\|\chi_i\|_{p_1(\cdot)}\|\chi_t\|_{p_1'(\cdot)(\mathbb{R}^n)} \leq C2^{-in}2^{\frac{in}{p_{1\infty}}}2^{\frac{tn}{p_1'(0)}} \leq C2^{\frac{-in}{p_{1\infty}}}2^{\frac{tn}{p_1'(0)}}. \tag{3.2}$$

As $\alpha_\infty - \frac{n}{p'_{1_\infty}} < 0$, we have

$$\begin{aligned}A_1 \leq & C \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{-1} \left\|\chi_i(1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}} \\ \leq & \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \Bigg(\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \\ & \times \left(\sum_{j=-\infty}^{-1} 2^{-in}\|f\chi_t\|_{p_1(\cdot)}\|b\|^m_{BMO(\mathbb{R}^n)}(i-t)^m\|\chi_t\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\Bigg)^{\frac{1}{q(1+\epsilon)}} \\ \leq & C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left[\epsilon^\theta \sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=-\infty}^{-1} (i-t)^m 2^{-in}2^{\frac{in}{p_{1\infty}}}2^{\frac{tn}{p_1'(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}} \\ \leq & C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\geq 0, k_0\in\mathbb{Z}} 2^{-k_0\beta}\end{aligned}$$

$$
\begin{aligned}
&\times \left[\epsilon^{\theta}\sum_{i=0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)}\left(\sum_{t=-\infty}^{-1}(i-t)^m 2^{\frac{-in}{p'_{1\infty}}}2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times \left[\epsilon^{\theta}\sum_{i=0}^{k_0} 2^{k\left(\alpha_\infty-\frac{n}{p'_{1\infty}}\right)q(1+\epsilon)}\left(\sum_{t=-\infty}^{-1}(i-t)^m 2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}(i-t)^m 2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times \left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}2^{t\alpha(0)}(i-t)^m 2^{-t\alpha(0)}2^{\frac{tn}{p'_1(0)}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times \left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}2^{t\alpha(0)q(1+\epsilon)}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{t\left(\frac{n}{p'_1(0)}-\alpha(0)\right)q(1+\epsilon)/2}\right)\right.\\
&\times \left.\left(\sum_{t=-\infty}^{-1}(i-t)^{m(q(1+\epsilon))'/2}2^{j\left(\frac{n}{p'_1(0)}-\alpha(0)\right)q(1+\epsilon)'/2}\right)^{q(1+\epsilon)/(q(1+\epsilon))'}\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times \left[\epsilon^{\theta}\sum_{i=0}^{k_0}\left(\sum_{t=-\infty}^{-1}2^{t\alpha(0)q(1+\epsilon)}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}2^{t\left(\frac{n}{p'_1(0)}-\alpha(0)\right)q(1+\epsilon)/2}\right)\right]^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{t=-\infty}^{-1}2^{a(0)q(1+\epsilon)t}\,\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}\\
=& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{t=-\infty}^{k_0}\left\|2^{a(\cdot)t}f\chi_t\right\|^{q(1+\epsilon)}_{p_1(\cdot)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\leq& C\,\|b\|^m_{BMO(\mathbb{R}^n)}\,\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now we will find the estimate for E_2, for every $i\in\mathbb{Z}$ with $t\geq i+1$ a.e. Let $x\in R_i$ and $y\in R_t$, we known that $|x-y|\approx|y|\approx 2^t$, we consider

$$\left|I^{\lambda(\cdot),b}_m(f\chi_t)(x)\right|\leq C\int_{R_t}|x-y|^{\lambda(x)-n}\,|b(x)-b(y)|^m|f(y)|dy$$

$$\leq C2^{-tn}\int_{R_t}|x|^{\lambda(x)}|b(x)-b(y)|^m|f(y)|dy$$
$$\leq C2^{-tn}|x|^{\lambda(x)}\left(|b(x)-b_{B_t}|^m\int\limits_{R_t}|f_t(y)|dy+\int\limits_{R_t}|f_t(y)||b(y)-b_{B_t}|^m dy\right)$$
$$\leq C2^{-tn}|x|^{\lambda(x)}\left\|f_t\right\|_{p_1(\cdot)}\left(|b(x)-b_{B_t}|^m\left\|\chi_t\right\|_{p_1'(\cdot)}+\|((b-b_{B_t})^m\chi_t\|_{p_1'(\cdot)}\right).$$

Similarly to E_1, for E_2, we have

$$\begin{aligned}
&\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\|_{p_2(\cdot)}\\
\leq& C2^{-tn}\left\|f_t\right\|_{p_1(\cdot)}\left(\left\|(1+|x|)^{-\gamma(x)}|x|^{\lambda(x)}(b-b_{B_t})^m\chi_i\right\|_{p_2(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\right.\\
&\left.+\left\|(b-b_{B_t})^m\chi_t\right\|_{p_1'(\cdot)}\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}(\chi_{B_i})\right\|_{p_2(\cdot)}\right)\\
\leq& C2^{-tn}\left\|f_t\right\|_{p_1(\cdot)}\left(\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}((b-b_{B_t})^m\chi_i)\right\|_{p_2(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\right.\\
&\left.+\left\|(b-b_{B_t})^m\chi_t\right\|_{p_1'(\cdot)}\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot)}(\chi_{B_i})\right\|_{p_2(\cdot)}\right)\\
\leq& C2^{-tn}\left\|f_t\right\|_{p_1(\cdot)}\left((i-t)^m\left\|b\right\|^m_{BMO(\mathbb{R}^n)}\left\|\chi_i\right\|_{p_1(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\right.\\
&\left.+\left\|b\right\|^m_{BMO(\mathbb{R}^n)}\left\|\chi_t\right\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}\right)\\
\leq& C2^{-tn}(i-t)^m\left\|b\right\|^m_{BMO(\mathbb{R}^n)}\left\|f_t\right\|_{p_1(\cdot)}\left\|\chi_t\right\|_{p_1'(\cdot)}\|\chi_i\|_{p_1(\cdot)}.
\end{aligned}$$

Next we find estimate of E_2

$$\begin{aligned}
E_2\leq&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{i=-\infty}^{k_0}\left(\sum_{t=i+1}^{\infty}\left\|2^{i\alpha(\cdot)}(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
\approx&\max\left\{\sup_{\epsilon>0}\sup_{k_0<0,k_0\in\mathbb{Z}}2^{-k_0\beta}\right.\\
&\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},\\
&\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\geq0,k_0\in\mathbb{Z}}2^{-k_0\beta}
\end{aligned}$$

$$\times \left(\epsilon^{\theta} \sum_{i=-0}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=i+1}^{\infty} \left\| (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t)\chi_i \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \Bigg\}$$

$$:= \max \left\{ E_2', E_{21} + E_{22} \right\}.$$

We will find the estimates for E_{21} and E_{22}, the estimate for E_2' is obtained similarly. First, we will find an estimate for E_{22},

$$\begin{aligned} E_{22} \le & C \sup_{\epsilon>0} \sup_{k_0\ge 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left(\epsilon^{\theta} \sum_{i=o}^{k_0} 2^{i\alpha_\infty q(1+\epsilon)} \left(\sum_{t=i+1}^{\infty} \left\| \chi_i (1+|x|)^{-\gamma(x)} I_m^{\lambda(\cdot),b}(f\chi_t) \right\|_{p_2(\cdot)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}} \\ \le & C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\ge 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left(\epsilon^{\theta} \sum_{i=0}^{k_0} \left(\sum_{t=i+1}^{\infty} 2^{\alpha_\infty t} \|f\chi_t\|_{p_1(\cdot)} (i-t)^m 2^{d(i-t)} \right)^{q(1+\epsilon)} \right)^{\frac{1}{q(1+\epsilon)}}, \end{aligned}$$

where $d = \frac{n}{p_{1\infty}} + \alpha_\infty > 0$. Then, we use Hölder's theorem for series and $2^{-q(1+\epsilon)} < 2^{-q}$ to obtain

$$\begin{aligned} E_{22} \le & C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\ge 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \Bigg[\epsilon^{\theta} \sum_{i=0}^{k_0} \left(\sum_{t=i+1}^{\infty} 2^{\alpha_\infty q(1+\epsilon)t} \|f\chi_t\|_{p_1(\cdot)}^{q(1+\epsilon)} 2^{dq(1+\epsilon)(i-t)/2} \right) \\ & \times \left(\sum_{t=i+1}^{\infty} (i-t)^{m(q(1+\epsilon))'/2} 2^{d(q(1+\epsilon))'(i-t)/2} \right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}} \Bigg]^{\frac{1}{q(1+\epsilon)}} \\ \le & C \|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\epsilon>0} \sup_{k_0\ge 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left[\epsilon^{\theta} \sum_{i=0}^{k_0} \sum_{t=i+1}^{\infty} 2^{\alpha_\infty q(1+\epsilon)t} \|f\chi_t\|_{p_1(\cdot)}^{q(1+\epsilon)} 2^{dq(1+\epsilon)(i-t)/2} \right]^{\frac{1}{q(1+\epsilon)}} \\ \le & C \|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\epsilon>0} \sup_{k_0\ge 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \\ & \times \left(\epsilon^{\theta} \sum_{t=0}^{\infty} 2^{\alpha_\infty q(1+\epsilon)t} \|f\chi_t\|_{p_1(\cdot)}^{q(1+\epsilon)} \sum_{i=0}^{t-1} 2^{dq(1+\epsilon)(i-t)/2} \right)^{\frac{1}{q(1+\epsilon)}} \\ \le & C \|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\epsilon>0} \sup_{k_0\ge 0, k_0\in\mathbb{Z}} 2^{-k_0\beta} \end{aligned}$$

$$\times\left(\epsilon^{\theta}\sum_{t=0}^{\infty}\sum_{j=-\infty}^{t}2^{\alpha_\infty q(1+\epsilon)j}\|f\chi_t\|_{p_1(\cdot)}^{q(1+\epsilon)}\sum_{i=0}^{t-1}2^{dq(1+\epsilon)(i-t)/2}\right)^{\frac{1}{q(1+\epsilon)}}$$

$$\leq C\,\|b\|_{BMO(\mathbb{R}^n)}^{m}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{t=0}^{\infty}\sum_{i=0}^{t-2}2^{dq(1+\epsilon)(i-t)/2}\right)^{\frac{1}{q(1+\epsilon)}}\|f\|_{M\dot{K}_{\beta,p_1(\cdot)}^{a(\cdot),q),\theta}(\mathbb{R}^n)}$$

$$\leq C\,\|b\|_{BMO(\mathbb{R}^n)}^{m}\|f\|_{M\dot{K}_{\beta,p_1(\cdot)}^{a(\cdot),q),\theta}(\mathbb{R}^n)}.$$

For E_{21}, by using Minkowski's inequality,

$$E_{21}\leq\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=i+1}^{\infty}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}},$$

$$\begin{aligned}E_{21}\leq&\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=k+2}^{-1}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\&+\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=0}^{\infty}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\:=&B_1+B_2.\end{aligned}$$

The estimate for B_1 can be obtained in a similar way to that of E_{22} by replacing $p_{1\infty}$ with $p_1(0)$ and using the inequality $\frac{n}{p_1(0)}+a(0)>0$, $\frac{n}{p_{1\infty}}+\alpha_\infty>0$. For B_2 we have

$$2^{-tn}\|\chi_i\|_{p_1(\cdot)}\|\chi_t\|_{p_1'(\cdot)}\leq C2^{-tn}2^{\frac{in}{p_1(0)}}2^{\frac{tn}{q'1\infty}}\leq C2^{\frac{in}{p_1(0)}}2^{\frac{-tn}{p_{1\infty}}},\tag{3.3}$$

and

$$\begin{aligned}B_2\leq&C\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=0}^{\infty}\left\|\chi_i(1+|x|)^{-\gamma(x)}I_m^{\lambda(\cdot),b}(f\chi_t)\right\|_{p_2(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\\\leq&C\,\|b\|_{BMO(\mathbb{R}^n)}^{m}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\\&\times\left(\epsilon^{\theta}\sum_{i=-\infty}^{-1}2^{i\alpha(0)q(1+\epsilon)}\left(\sum_{t=0}^{\infty}(i-t)^m2^{-tn}2^{\frac{in}{p_1(0)}}2^{\frac{tn}{p'_{1\infty}}}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}\end{aligned}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}\left(\sum_{t=0}^{\infty}2^{t\alpha(0)}(i-t)^m 2^{\frac{in}{p_1(0)}+ia(0)}2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)}\|f\chi_t\|_{p_1(\cdot)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\epsilon^\theta\sum_{i=-\infty}^{-1}\left(\sum_{t=0}^{\infty}2^{t\alpha(0)(q(1+\epsilon))}\left(2^{\frac{in}{p_1(0)}+ia(0)}2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)}\right)^{q(1+\epsilon)/2}\right)\right.$$
$$\left.\times\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\left(\sum_{t=0}^{k_0}\left((i-t)^m 2^{\frac{in}{p_1(0)}+ia(0)}2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)}\right)^{(q(1+\epsilon))'/2}\right)^{\frac{q(1+\epsilon)}{(q(1+\epsilon))'}}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{i=-\infty}^{-1}\sum_{t=0}^{\infty}2^{t\alpha(0)(q(1+\epsilon))}\|f\chi_t\|^{q(1+\epsilon)/2}_{p_1(\cdot)}\right.$$
$$\left.\times\left(2^{\frac{in}{p_1(0)}+ia(0)}2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)}\right)^{q(1+\epsilon)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=0}^{\infty}2^{t\alpha(0)(q(1+\epsilon))}\|f\chi_t\|^{q(1+\epsilon)}_{p_1(\cdot)}\right.$$
$$\left.\times\sum_{i=-\infty}^{t}\left(2^{\frac{in}{p_1(0)}+ia(0)}2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)}\right)^{q(1+\epsilon)/2}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=0}^{\infty}\sum_{j=-\infty}^{t}2^{ja(0)(q(1+\epsilon))}\|f\chi_j\|^{q(1+\epsilon)}_{p_1(\cdot)}\right.$$
$$\left.\times\sum_{i=-\infty}^{t}2^{\frac{in}{p_1(0)}+ia(0)}2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\geq 0,k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=0}^{\infty}\sum_{i=-\infty}^{t}2^{\frac{in}{p_1(0)}+ia(0)}2^{\frac{-tn}{p_{1\infty}}-t\alpha(0)}\right)^{\frac{1}{q(1+\epsilon)}}$$
$$\times\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_1 and E_2 yields

$$\left\|(1+|x|)^{-\gamma(x)}I^{\lambda(\cdot),b}_m(f)\right\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_2(\cdot)}(\mathbb{R}^n)}\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{M\dot{K}^{a(\cdot),q),\theta}_{\beta,p_1(\cdot)}(\mathbb{R}^n)},$$

which completes the proof. □

References

[1] A. Abdalmonem and A. Scapellato, *Fractional operators with homogeneous kernels in weighted Herz spaces with variable exponent*, Appl. Anal. **101**(6) (2022), 1953–1962.

[2] A. Abdalmonem, A. Scapellato, *Intrinsic square functions and commutators on Morrey-Herz spaces with variable exponents*, Math. Methods Appl. Sci. **44**(2021), 12408–12425.

[3] A. Abdalmonem, O. Abdalrhman and S. Tao, *Boundedness of fractional integral with variable kernel and their commutators on variable exponent Herz spaces*, Applied Math. **7**(10) (2016).

[4] A. Ajaib and A. Hussain, *Weighted CBMO estimates for commutators of matrix Hausdorff operator on the Heisenberg group*, Open Math. **18**(1) (2020), 496–511.

[5] A. Almeida and D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl. **394**(2) (2012), 781–795.

[6] R. E. Castillo and H. Rafeiro, *An Introductory Course in Lebesgue Spaces*, CMS Books, Mathematics/Ouvrages de Mathematiques de ta SMC, Springer, Cham, 2016.

[7] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*, Birkhäuser, Heidelberg, 2013.

[8] L.Grafakos, X.Li and D.Yang, *Bilinear operator on Herz-type Hardy spaces*, Trans. Amer. Math. Soc. **350**(3) (1998), 1249–1275.

[9] A. Hussain, I. Khan and A. Mohamed, *Variable Herz-Morrey estimates for rough fractional Hausdorff operator*, J. Inequal. Appl. **33** (2024).

[10] A. Hussain and A. Ajaib, *Some results for the commutators of generalized Hausdorff operator*, J. Math. Inequal. **13**(4) (2019), 1129–1146.

[11] M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math. **36**(1) (2010), 33–50.

[12] M. Izuki, *Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponents*, Math. Sci. J. **13**(10) (2009), 243–253.

[13] M. Izuki, *Boundedness of commutators on Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo. **59** (2010), 199–213.

[14] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Vol. 1, Birkhuser/Springer, Cham, 2016.

[15] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Vol. 2, Birkhuser/Springer, Cham, 2016.

[16] V. Kokilashvili and S. Samko, *On Sobolev theorem for Riesz-type potentials in the Lebesgue spaces with variable exponent*, Z. Anal. Anwend. **22** (2003), 899–910. https://www.doi.org/10.4171/ZAA/1178

[17] S. Samko, *Variable exponent Herz spaces*, Mediterr. J. Math. **10**(4) (2013), 2007–2025.

[18] M. Sultan, B. Sultan and A. Hussain, *Grand Herz-Morrey spaces with variable exponent*, Math. Notes **114**(5) (2023), 957–977.

[19] M. Sultan, B. Sultan, A. Aloqaily, N. Mlaiki, *Boundedness of some operators on grand Herz spaces with variable exponent*, AIMS Math. **8**(6) (2023), 12964–12985.

[20] B. Sultan, F. Azmi, M. Sultan, M. Mehmood and N. Mlaiki, *Boundedness of Riesz potential operator on grand Herz-Morrey spaces*, Axioms **11**(11) (2022), Article ID 583.

[21] M. Sultan, B. Sultan, A. Khan and T. Abdeljawad, *Boundedness of Marcinkiewicz integral operator of variable order in grand Herz-Morrey spaces*, AIMS Math. **8**(9) (2023), 22338–22353.

[22] B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M. A. Alghafli and N. Mlaik, *Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent*, AIMS Math. **8**(1) (2023), 752–764.

[23] B Sultan, F. M. Azmi, M. Sultan, T. Mahmood, N. Mlaiki and N. Souayah, *Boundedness of fractional integrals on grand weighted Herz-Morrey spaces with variable exponent*, Fractal fract. **6**(11) (2022), Article ID 660.

[24] B. Sultan and M. Sultan, *Boundedness of higher order commutators of Hardy operators on grand Herz-Morrey spaces*, Bull. Sci. Math. **190** (2024).

[25] B. Sultan, M. Sultan, A. Khan and T. Abdeljawad, *Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces*, J. Inequal. Appl. **2024** (2024), Article ID 93.

[26] B. Sultan and M. Sultan, *Boundedness of commutators of rough Hardy operators on grand variable Herz spaces*, Forum Math. (2023). https://doi.org/10.1515/forum-2023-0152

[27] B. Sultan, M. Sultan and I. Khan, *On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces*, Commun. Nonlinear Sci. Numer. Simul. **126** (2023). https://doi.org/10.1016/j.cnsns.2023.107464

[28] J. L. Wu, *Boundedness for fractional Hardy-type operator on Herz-Morrey spaces with variable exponent*, Bull. Korean Math. Soc. **51** (2014), 423–435.

[29] H. Wang, *Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent*, Czech. Math. J. **66** (2016), 251–269.

[30] J. L. Wu and W. J. Zhao, *Boundedness for fractional Hardy-type operator on variable-exponent Herz-Morrey spaces*, Kyoto J. Math. **56** (2016), 831–845.

[1]Department of Mathematics,
Capital University of Science and Technology,
Islamabad, Pakistan
Email address: mehvishsultanbaz@gmail.com
ORCID iD: https://orcid.org/0009-0005-2379-0476

[2]Department of Mathematics,
Quaid-I-Azam University,
Islamabad, Pakistan
Email address: babarsultan40@yahoo.com
ORCID iD: https://orcid.org/0000-0003-2833-4101