KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 43(2) (2019), PAGES 239–246.

(σ, τ)-DERIVATIONS OF SEMIPRIME RINGS

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ABSTRACT. In this paper we investigate some results about semiprime rings $\mathbb{R}$ with a 2-torsion-free and σ and τ being automorphisms mappings of $\mathbb{R}$. Suppose that there exists a (σ, τ)-derivation d of $\mathbb{R}$. If $\mathbb{R}$ admits d to satisfied some conditions, then d is a commuting mapping of $\mathbb{R}$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{R}$ be an associative ring. A map d: $\mathbb{R} \to \mathbb{R}$ is a *derivation* of $\mathbb{R}$ if d is additive and satisfies the *Leibnitz' rule*: $d(ab) = d(a)b + ad(b)$, for all $a, b \in \mathbb{R}$. A simple example is of course the usual derivative on various algebras consisting of differentiable functions. Basic examples in noncommutative rings are quite different.

Note that $[a, xy] = [a, x]y + x[a, y]$, for all $a, x, y \in \mathbb{R}$. For a fixed $a \in \mathbb{R}$, define d: $\mathbb{R} \to \mathbb{R}$ by $d(x) = [x, a]$, for all $x \in \mathbb{R}$. The function d so defined can be easily checked to be additive and $d(xy) = [xy, a] = x[y, a] + [x, a]y = xd(y) + d(x)y$, for all $x, y \in \mathbb{R}$. Thus, d is a derivation which is called *inner derivation* of $\mathbb{R}$ associated with a and is generally denoted by I_a. It is obvious to see that every inner derivation on a ring $\mathbb{R}$ is a derivation. But one can find plenty of examples of derivations which are not inner.

Some people ask why study derivations. At first, we can say that derivations on rings help us to understand rings better and can tell us about the structure of them. For instance a ring is *commutative* if and only if the only *inner derivation* on the ring is zero. Also derivations can be helpful for relating a ring with the set of matrices with entries in the ring (see [13]). Besides, derivations have a significant role in determining whether a ring is commutative or not, (see [1, 3–5, 8, 12]).

Key words and phrases. Semiprime rings, prime rings, (σ, τ)-derivations, torsion-free rings, commuting mappings.

2010 *Mathematics Subject Classification.* Primary: $16W25$. Secondary: $16N60$, $16U80$.

Received: July 14, 2017.

Accepted: October 31, 2017.

Derivations can be useful in other fields. For example, they play a great role in the calculation of the eigenvalues of matrices (see [7]), which is important in mathematics and other sciences, business and engineering. They are using in quantum physics (see [9]). Derivations can be added and subtracted and we still get a derivation, but when we compose a derivation with itself we do not necessarily get a derivation.

Throughout the present paper, $\mathbb{R}$ will denote an associative ring with *center* $Z(\mathbb{R})$. For any $x, y \in \mathbb{R}$ the bracket symbol $[x, y]$ represents the *commutator* $xy - yx$ and for a non-empty subset S of $\mathbb{R}$, the set of *all commutators* of elements of S will be written as $[S, S]$. Recall that a ring $\mathbb{R}$ is *prime* if $x\mathbb{R}y = 0$ implies $x = 0$ or $y = 0$, and *semiprime* if $x\mathbb{R}x = 0$ implies $x = 0$. In fact, a prime ring is semiprime, but the converse is not true in general. A ring $\mathbb{R}$ is named 2*-torsion-free* in case $2x = 0$ implies that $x = 0$, for any $x \in \mathbb{R}$. An additive mapping $d : \mathbb{R} \to \mathbb{R}$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathbb{R}$. Let σ and τ be two automorphisms of $\mathbb{R}$. An additive mapping $d : \mathbb{R} \to \mathbb{R}$ is called a (σ, τ)*-derivation* if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \mathbb{R}$. Of course, a $(1, 1)$-derivation where 1 is the identity map on $\mathbb{R}$ is a derivation.

A mapping $d : \mathbb{R} \to \mathbb{R}$ is said to be *centralizing* if $[d(x), x] \in Z(\mathbb{R})$, for all $x \in \mathbb{R}$. In the special case when $[d(x), x] = 0$, the mapping d is said to be *commuting* on $\mathbb{R}$. Furthermore, a mapping $d : \mathbb{R} \to \mathbb{R}$ is said to be (σ, τ)*-centralizing* (resp. (σ, τ)*-commuting*) if $[d(x), x]_{\sigma,\tau} \in Z(\mathbb{R})$ (resp. $[d(x), x]_{\sigma,\tau} = 0$) holds for all $x \in \mathbb{R}$. Of course, a $(1, 1)$-centralizing (resp. $(1, 1)$-commuting) mapping is a *centralizing* (resp. *commuting*) on R. There are several results in the existing literature dealing with centralizing and commuting mappings in rings. The study of centralizing mappings was initiated by E.C. Posner [14] which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (named Posner's Second Theorem). In an attempt to generalize the above result J. Vukman [15] confirms that if $\mathbb{R}$ is a 2-torsion-free prime ring and $d : \mathbb{R} \to \mathbb{R}$ a nonzero derivation such that the map $x \rightsquigarrow [d(x), x]$ is commuting on $\mathbb{R}$, then $\mathbb{R}$ is commutative.

In the present paper it is shown that the conclusion of the above theorem holds if for a (σ, τ)-derivation d the mapping $x \rightsquigarrow d(x)$ is a (σ, τ)-commuting. In fact, we have proved the following: M.J. Atteya has published a paper [6] which contains a proof of the result, $\mathbb{R}$ be a 2-torsion-free semiprime ring and $d : \mathbb{R} \to \mathbb{R}$ be a derivation on $\mathbb{R}$ such that $d^n(x \circ y) \pm (x \circ y) \in Z(\mathbb{R})$, for all $x, y \in \mathbb{R}$, then there exists C and an additive mapping ξ: $\mathbb{R} \to C$ such that $d(x) = \lambda x + \xi(x)$, for all $x \in \mathbb{R}$, where n is a fixed positive integer. Ö. Gölbaşı and E. Koç [11] were supply that (f, d) be a generalized (σ, τ)-derivation of a prime ring $\mathbb{R}$ with char$(\mathbb{R}) \neq 2$. If $af(x) = 0$, for all $x \in \mathbb{R}$, then $a = 0$ or $d = 0$. M. Ashraf, A. Khan and C. Haetinger [2] showed that under certain conditions on the prime ring $\mathbb{R}$, every Jordan (σ, τ)-higher derivation on $\mathbb{R}$ is a (σ, τ)-higher derivation on $\mathbb{R}$. B. Dhara and A. Pattanayak [10] proved that $\mathbb{R}$ be a semiprime ring, I a nonzero ideal of $\mathbb{R}$, and σ, τ two epimorphisms of $\mathbb{R}$, an additive mapping $F : \mathbb{R} \to \mathbb{R}$ is a *generalized* (σ, τ)*-derivation* on $\mathbb{R}$ if there exists a (σ, τ)-derivation $d : \mathbb{R} \to \mathbb{R}$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \mathbb{R}$. If $\tau(I)d(I) \neq 0$, then $\mathbb{R}$ contains a nonzero central ideal of $\mathbb{R}$, if the following condition holds: $F[x, y] = \pm(xoy)_{\sigma,\tau}$.

In this paper we will investigate about the commuting of additive mappings (σ, τ)-derivations on a semiprime ring $\mathbb{R}$.

2. The Main Results

We start with the following lemma.

Lemma 2.1. *Let $\mathbb{R}$ be a 2-torsion-free semiprime ring and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists a (σ, τ)-derivation d such that $d^2(\mathbb{R}) = 0$ and d commutes with both σ, τ. Then $d = 0$.*

Proof. For any $x \in \mathbb{R}$, we have $d^2(x) = 0$.

Replacing x by xy, we obtain

$$d^2(x)\sigma^2(y) + \tau(d(x))d(\sigma(y)) + d(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) = 0, \tag{2.1}$$

for all $x, y \in \mathbb{R}$. According to our hypothesis that $d^2(\mathbb{R}) = 0$ and d commutes with both σ and τ, the relation (2.1) yields that $\tau(d(x))d(\sigma(y)) = 0$.

Since σ and τ are two automorphism mappings of $\mathbb{R}$, we get

$$d(x)d(y) = 0, \quad \text{for all } x, y \in \mathbb{R}. \tag{2.2}$$

Replacing y by yr and using (2.2), we get $d(x)\tau(r)d(y) = 0$, for all $x, y, r \in \mathbb{R}$.

Replacing now y by x and using τ to be an automorphism mapping of $\mathbb{R}$, we obtain $d(x)\mathbb{R}d(x) = (0)$.

By semiprimeness of $\mathbb{R}$, we complete the proof. □

The proof of the following theorem is adapted from J. Vukman [16, Theorem 1].

Theorem 2.1. *Let $\mathbb{R}$ be a 2-torsion-free semiprime ring, σ and τ be automorphism mappings of $\mathbb{R}$. Suppose that there exists a (σ, τ)-derivation d such that $[d(x), x]_{\sigma,\tau} = 0$, for all $x \in \mathbb{R}$. Then d is a commuting mapping of $\mathbb{R}$.*

Proof. Let us introduce a mapping B: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for all $x, y \in \mathbb{R}$, by the relation $B(x, y) = [d(x), y]_{\sigma,\tau} + [d(y), x]_{\sigma,\tau}$.

Obviously B is symmetric (that is, $B(x, y) = B(y, x)$, for all $x, y \in \mathbb{R}$) and additive in both the arguments.

Notice that, for all $x, y, z \in \mathbb{R}$, $B(xy, z) = [d(xy), z]_{\sigma,\tau} + [d(z), xy]_{\sigma,\tau}$. Then

$$B(xy, z) = B(x, y)\sigma(y) + \tau(x)B(y, z) + d(x)\sigma([y, z]) + \tau([x, z])d(y). \tag{2.3}$$

Again, we introduce a mapping f from $\mathbb{R}$ into itself by $f(x) = B(x, x)$. Then $f(x) = 2[d(x), x]_{\sigma,\tau}$. The mapping f satisfies the relation

$$\begin{aligned} f(x + y) &= 2[d(x + y), x + y]_{\sigma,\tau} \\ &= 2[d(x), x]_{\sigma,\tau} + 2[d(y), x]_{\sigma,\tau} + 2[d(x), y]_{\sigma,\tau} + 2[d(y), y]_{\sigma,\tau}. \end{aligned} \tag{2.4}$$

Since $f(x) = 2[d(x), x]_{\sigma,\tau}$, then the relation (2.4) becomes

$$f(x + y) = f(x) + f(y) + 2B(x, y). \tag{2.5}$$

The assumption of the Theorem 2.1 can now be rewritten as

$$f(x) = 0, \quad \text{for all } x \in \mathbb{R}, \tag{2.6}$$

where we shall use the mappings f and B as well as the relations (2.3) and (2.5) without specific references.

Now, linearization of (2.6) (i.e., putting $x = x+y$), gives $f(x)+f(y)+2B(x,y) = 0$, and hence $2B(x,y) = 0$. Since $\mathbb{R}$ is 2-torsion-free, we get

$$B(x,y) = 0. \tag{2.7}$$

In the relation (2.7), replacing y by xy, we obtain:

$$B(x,xy) = f(x)\sigma(x) + \tau(x)B(x,y) + d(x)\sigma([x,y]) = 0.$$

By using (2.6) and (2.7), the above relation becomes $d(x)\sigma([x,y]) = 0$. Since σ is an automorphism of $\mathbb{R}$, then

$$d(x)[x,y] = 0. \tag{2.8}$$

Replacing y by ry in the relation (2.8), gives

$$d(x)r[x,y] + d(x)[x,r]y = 0, \quad \text{for all } x,y,r \in \mathbb{R}. \tag{2.9}$$

In (2.8), replacing y by r and using the result in (2.9), we obtain

$$d(x)r[x,y] = 0. \tag{2.10}$$

Left-multiplying (2.10) by x, we get

$$xd(x)r[x,y] = 0. \tag{2.11}$$

Again, in (2.10) replacing r by xr, we get

$$d(x)xr[x,y] = 0. \tag{2.12}$$

Subtracting (2.12) of (2.11), replacing y by $d(x)$ and using the semiprimeness of $\mathbb{R}$, we obtain that d is a commuting mapping in R as required. ☐

Theorem 2.2. *Let $\mathbb{R}$ be a 2-torsion-free semiprime ring and σ and τ be automorphism mappings of $\mathbb{R}$. If $\mathbb{R}$ admits a non-zero (σ,τ)-derivation d such that $d(xy) = d(yx)$, for all $x,y \in \mathbb{R}$ and d commutes with τ, then d is a commuting mapping of $\mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ be a constant, i.e., an element such that $d(c) = 0$, and let c be an arbitrary element of $\mathbb{R}$.

In our relation $d(xy) = d(yx)$, for all $x,y \in \mathbb{R}$, replacing x by c and y by z, we obtain $d(cz) = d(zc)$, for all $z,c \in \mathbb{R}$. Then

$$\tau(c)d(z) = d(z)\sigma(c), \quad \text{for all } z,c \in \mathbb{R}. \tag{2.13}$$

Now, for all $x,y \in R$, $[x,y]$ is a constant and hence from (2.13), we get

$$\tau([x,y])d(z) = d(z)\sigma([x,y]), \quad \text{for all } x,y,z \in \mathbb{R}. \tag{2.14}$$

We have $d(xy) = d(yx)$, for all $x,y \in \mathbb{R}$. This can be rewritten as

$$[d(x),y]_{\sigma,\tau} = [d(y),x]_{\sigma,\tau}. \tag{2.15}$$

Replacing x by x^2 in (2.15) and using it, we gain

$$d(x)\sigma([x,y]) + \tau([x,y])d(x) = 0, \quad \text{for all } x,y \in \mathbb{R}. \tag{2.16}$$

According to (2.14), the above (2.16) yields that $2\tau([x,y])d(x) = 0$. Since $\mathbb{R}$ is 2-torsion-free and τ is an automorphism mapping of $\mathbb{R}$, this relation yields $[x,y]d(x) = 0$, for all $x,y \in \mathbb{R}$.

By same technique in Theorem 2.1, we complete the proof. □

Theorem 2.3. *Let $\mathbb{R}$ be a 2-torsion-free semiprime ring and σ and τ be automorphism mappings of $\mathbb{R}$. If $\mathbb{R}$ admits a non-zero (σ,τ)-derivation d such that $[d(x),d(y)] = 0$, for all $x,y \in \mathbb{R}$ and d commutes with both σ,τ, then d is a commuting (resp. centralizing mapping) of $\mathbb{R}$.*

Proof. For all $x,y \in \mathbb{R}$, we have the following relation

$$[d(x),d(y)] = 0. \tag{2.17}$$

Replacing y by xy and using the relation (2.17), we get

$$d(x)[d(x),\sigma(y)] + [d(x),\tau(x)]d(y) = 0. \tag{2.18}$$

Now, in (2.18), replacing y by yr, we obtain

$$d(x)\sigma(y)[d(x),\sigma(r)] + [d(x),\tau(x)]\tau(y)d(r) = 0, \quad \text{for all } x,y,r \in \mathbb{R}. \tag{2.19}$$

Replacing r by $\sigma^{-1}(d(z))$ and using (2.17), we obtain

$$[d(x),\tau(x)]\tau(y)\sigma^{-1}(d^2(z)) = 0, \quad \text{for all } x,y,z \in \mathbb{R}. \tag{2.20}$$

Since τ is an automorphism mapping of $\mathbb{R}$, then the relation (2.20) reduces to

$$[d(x),x]y\sigma^{-1}(d^2(z)) = 0, \quad \text{for all } x,y,z \in \mathbb{R}. \tag{2.21}$$

Replacing y by $\sigma^{-1}(d^2(z))r[d(x),x]$, $r \in \mathbb{R}$, we obtain

$$[d(x),x]\sigma^{-1}(d^2(z))R[d(x),x]\sigma^{-1}(d^2(z)) = (0).$$

Since $\mathbb{R}$ is semiprime, we get $[d(x),x]\sigma^{-1}(d^2(z)) = 0$. Then $\sigma([d(x),x])(d^2(z)) = 0$, and since σ acts as an automorphism mapping of $\mathbb{R}$, we obtain

$$[d(x),x]d^2(z) = 0. \tag{2.22}$$

After replacing z by xy in (2.22), we get

$$[d(x),x](d^2(x)\sigma(y) + \tau(d(x))d(\sigma(y)) + d(\tau(x))\sigma(d(y)) + \tau(\tau(x)d^2(y)) = 0.$$

Since σ and τ are automorphism mappings of $\mathbb{R}$, above relation reduces to

$$[d(x),x](d^2(x)y + d(x)d(y) + d(x)d(y) + xd^2(y) = 0. \tag{2.23}$$

According to (2.22), the relation (2.23) reduces to

$$[d(x),x](2d(x)d(y) + xd^2(y)) = 0. \tag{2.24}$$

From (2.21), we get $\sigma([d(x), x]y)d^2(z) = 0$, for all $x, z \in \mathbb{R}$. Since σ is an automorphism mapping of $\mathbb{R}$, we obtain

$$[d(x), x]yd^2(z) = 0. \tag{2.25}$$

Replacing y by x and using the result in (2.24), we get $2[d(x), x]d(x)d(y) = 0$. According to the hypothesis that $\mathbb{R}$ is a 2-torsion-free semiprime ring, we get

$$[d(x), x]d(x)d(y) = 0. \tag{2.26}$$

Replacing y by xy, using (2.26) and since σ and τ are automorphism mappings of $\mathbb{R}$, we obtain $[d(x), x]d(x)yd(x) = 0$. Replacing y by $r[d(x), x], r \in \mathbb{R}$, and using that $\mathbb{R}$ is a semiprime ring, we get

$$[d(x), x]d(x) = 0. \tag{2.27}$$

Since $d(x)d(y) = d(y)d(x)$, for all $x, y \in \mathbb{R}$, the relation (2.26) becomes

$$[d(x), x]d(y)d(x) = 0. \tag{2.28}$$

In (2.28), replacing y by xy and using (2.27), we get

$$[d(x), x]yd(x) = 0. \tag{2.29}$$

After replacing y by $rx, r \in \mathbb{R}$, from the relation (2.29), we obtain

$$[d(x), x]rxd(x) = 0. \tag{2.30}$$

Once again, right multiplying the relation (2.29) by x and replacing y by r, where $r \in \mathbb{R}$, we gain

$$[d(x), x]rd(x)x = 0. \tag{2.31}$$

Subtracting the relations (2.30) from (2.31), and using the semiprimeness of $\mathbb{R}$, we completed the proof as required. $\square$

Remark 2.1. In the results of this paper, we cannot exclude the condition that the mappings σ and τ should be automorphism mappings of $\mathbb{R}$, as is showed below.

Example 2.1. Let $\mathbb{R}$=$\mathcal{M}_2(\mathbb{F})$ be a ring of 2×2 matrices over a field $\mathbb{F}$, that is: $\mathbb{R} = \mathcal{M}_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{F} \right\}$.

Let d be the inner derivation of $\mathbb{R}$, given by $d(x) = x \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$, for all $x \in \mathbb{R}$.

Now, let $a, b, g, h \in F$. We suppose that $x = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix}$. Then $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \mathbb{R}$, where

$$d\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix}\right) = d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\sigma\begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix} + \tau\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}d\begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix}.$$

The acts of σ and τ as automorphisms leads to the both sides of the above equation give us $\begin{pmatrix} 0 & ag \\ 0 & 0 \end{pmatrix}$.

Acknowledgements. The authors are greatly indebted to the referee for her/his careful reading the paper.

References

[1] S. Andima and H. Pajoohesh, *Commutativity of prime rings with derivations*, Acta Math. Hungar. **128**(1-2) (2010), 1–14.

[2] M. Ashraf, A. Khan and C. Haetinger, *On (σ, τ)-higher derivations in prime rings*, Int. Electron. J. Algebra **8** (2010), 65–79.

[3] M. J. Atteya, *Commutativity results with derivations on semiprime rings*, Journal of Mathematical and Computational Science **4**(2) (2012), 853–865.

[4] M. J. Atteya, *Derivations of semiprime rings with left cancellation property*, Cayley Journal of Mathematics **1**(1) (2012), 71–75.

[5] M. J. Atteya, *Generalized Derivations of Semiprime Rings*, LAP Lambert Academic Publishing, Saarbrucken, Germany, 2012.

[6] M. J. Atteya, *Some results on the identity $d(x) = \lambda x + \xi(x)$*, Journal of Mathematical Sciences and Applications, Serie VII, **2**(3) (2014), 28–32.

[7] G. A. Baker, *A new derivation of Newton's identities and their application to the calculation of the eigenvalues of a matrix*, Journal of the Society for Industrial and Applied Mathematics **7** (1959), 143–148.

[8] H. E. Bell and M. N. Daif, *On derivations and commutativity in prime rings*, Acta Math. Hungar. **66** (1995), 337–343.

[9] J. da Providencia, *The numerical ranges of derivations and quantum physics*, Special Issue, The numerical range and numerical radius, Linear Multilinear Algebra **37**(1-3) (1994), 213–220.

[10] B. Dhara and A. Pattanayak, *On generalized (σ, τ)-derivations in semiprime rings*, International Scholarly Research Network, ISRN Algebra, (2012), Article ID 120251, 7 pages, DOI 10.5402/2012/120251.

[11] Ö. Gölbaşi and E. Koç, *Some commutativity theorems of prime rings with generalized (σ, τ)-derivation*, Commun. Korean Math. Soc. **26**(3) (2011), 445–454, DOI 0.4134/CKMS.2011.26.3.445.

[12] I. N. Herstein, *A note on derivations*, Canad. Math. Bull. **21** (1978), 369–370.

[13] H. Pajoohesh, *Positive derivations on lattice ordered rings of matrices*, Quaestiones Mathematicae **30** (2007), 275–284.

[14] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[15] J. Vukman, *Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **109** (1990), 47–52.

[16] J. Vukman, *Derivations on semiprime rings*, Bull. Aust. Math. Soc. **53** (1995), 353–359.

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