

## NEW STRONG DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF MEROMORPHIC MULTIVALENT QUASI-CONVEX FUNCTIONS

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ABSTRACT. New strong differential subordination and superordination results are obtained for meromorphic multivalent quasi-convex functions in the punctured unit disk by investigating appropriate classes of admissible functions. Strong differential sandwich results are also obtained.

### 1. INTRODUCTION AND PRELIMINARIES

Let  $\Sigma_p$  denote the class of all functions  $f$  of the form:

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic in the punctured unit disk  $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ .

A function  $f \in \Sigma_p$  is meromorphic multivalent starlike if  $f(z) \neq 0$  and

$$-\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U^*).$$

Similarly,  $f \in \Sigma_p$  is meromorphic multivalent convex if  $f'(z) \neq 0$  and

$$-\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U^*).$$

Moreover, a function  $f \in \Sigma_p$  is called meromorphic multivalent quasi-convex function if there exists a meromorphic multivalent convex function  $g$  such that  $g'(z) \neq 0$

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and

$$-\operatorname{Re} \left\{ \frac{(zf'(z))'}{g'(z)} \right\} > 0 \quad (z \in U^*).$$

Let  $\mathcal{H}(U)$  be the class of analytic functions in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ . For a positive integer  $n$  and  $a \in \mathbb{C}$ , let  $\mathcal{H}[a, n]$  be the subclass of  $\mathcal{H}(U)$  consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,$$

with  $\mathcal{H} = \mathcal{H}[1, 1]$ .

Let  $f$  and  $g$  be members of  $\mathcal{H}(U)$ . The function  $f$  is said to be subordinate to  $g$ , or (equivalently)  $g$  is said to be superordinate to  $f$ , if there exists a Schwarz function  $w$  which is analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in U$ ) such that  $f(z) = g(w(z))$ . In such a case, we write  $f \prec g$  or  $f(z) \prec g(z)$ ,  $z \in U$ . Furthermore, if the function  $g$  is univalent in  $U$ , then we have the following equivalent (see [5])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let  $G(z, \zeta)$  be analytic in  $U \times \bar{U}$  and let  $f(z)$  be analytic and univalent in  $U$ . Then the function  $G(z, \zeta)$  is said to be strongly subordinate to  $f(z)$  or  $f(z)$  is said to be strongly superordinate to  $G(z, \zeta)$ , written as  $G(z, \zeta) \prec\prec f(z)$ , if for  $\zeta \in \bar{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ ,  $G(z, \zeta)$  as a function of  $z$  is subordinate to  $f(z)$ . We note that

$$G(z, \zeta) \prec\prec f(z) \Leftrightarrow G(0, \zeta) = f(0) \text{ and } G(U \times \bar{U}) \subset f(U).$$

**Definition 1.1.** [6] Let  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  and let  $h$  be a univalent function in  $U$ . If  $F$  is analytic in  $U$  and satisfies the following (second-order) strong differential subordination:

$$(1.1) \quad \phi \left( F(z), zF'(z), z^2F''(z); z, \zeta \right) \prec\prec h(z),$$

then  $F$  is called a solution of the strong differential subordination (1.1). The univalent function  $q$  is called a dominant of the solutions of the strong differential subordination or more simply a dominant if  $F(z) \prec q(z)$  for all  $F$  satisfying (1.1). A dominant  $\check{q}$  that satisfies  $\check{q}(z) \prec q(z)$  for all dominants  $q$  of (1.1) is said to be the best dominant.

**Definition 1.2.** [7] Let  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  and let  $h$  be analytic function in  $U$ . If  $F$  and  $\phi(F(z), zF'(z), z^2F''(z); z, \zeta)$  are univalent in  $U$  for  $\zeta \in \bar{U}$  and satisfy the following (second-order) strong differential superordination:

$$(1.2) \quad h(z) \prec\prec \phi \left( F(z), zF'(z), z^2F''(z); z, \zeta \right),$$

then  $F$  is called a solution of the strong differential superordination (1.2). An analytic function  $q$  is called a subordinant of the solutions of the strong differential superordination or more simply a subordinant if  $q(z) \prec F(z)$  for all  $F$  satisfying (1.2). A univalent subordinant  $\check{q}$  that satisfies  $q(z) \prec \check{q}(z)$  for all subordinants  $q$  of (1.2) is said to be the best subordinant.

**Definition 1.3.** [6] Denote by  $Q$  the set consisting of all functions  $q$  that are analytic and injective on  $\bar{U} \setminus E(q)$ , where

$$E(q) = \left\{ \xi \in \partial U : \lim_{z \rightarrow \xi} q(z) = \infty \right\},$$

and are such that  $q'(\xi) \neq 0$  for  $\xi \in \partial U \setminus E(q)$ .

Furthermore, let the subclass of  $Q$  for which  $q(0) = a$  be denoted by  $Q(a)$ ,  $Q(0) \equiv Q_0$ , and  $Q(1) \equiv Q_1$ .

**Definition 1.4.** [9] Let  $\Omega$  be a set in  $\mathbb{C}$ ,  $q \in Q$ , and  $n \in \mathbb{N}$ . The class of admissible functions  $\Psi_n[\Omega, q]$  consists of those functions  $\psi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  that satisfy the following admissibility condition:  $\psi(r, s, t; z, \zeta) \notin \Omega$ , whenever

$$r = q(\xi), \quad s = k\xi q'(\xi) \quad \text{and} \quad \operatorname{Re} \left\{ \frac{t}{s} + 1 \right\} \geq k \operatorname{Re} \left\{ \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right\},$$

$z \in U$ ,  $\xi \in \partial U \setminus E(q)$ ,  $\zeta \in \bar{U}$ , and  $k \geq n$ .

We simply write  $\Psi_1[\Omega, q] = \Psi[\Omega, q]$ .

**Definition 1.5.** [8] Let  $\Omega$  be a set in  $\mathbb{C}$  and  $q \in \mathcal{H}[a, n]$  with  $q'(z) \neq 0$ . The class of admissible functions  $\Psi'_n[\Omega, q]$  consists of those functions  $\psi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  that satisfy the following admissibility condition:  $\psi(r, s, t; \xi, \zeta) \in \Omega$ , whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \operatorname{Re} \left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \operatorname{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U$ ,  $\xi \in \partial U$ ,  $\zeta \in \bar{U}$ , and  $m \geq n \geq 1$ .

In particular, we write  $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$ .

In our investigations, we will need the following lemmas.

**Lemma 1.1.** [9] Let  $\psi \in \Psi_n[\Omega, q]$  with  $q(0) = a$ . If  $F \in \mathcal{H}[a, n]$  satisfies

$$\psi \left( F(z), zF'(z), z^2F''(z); z, \zeta \right) \in \Omega,$$

then  $F(z) \prec q(z)$ .

**Lemma 1.2.** [8] Let  $\psi \in \Psi'_n[\Omega, q]$  with  $q(0) = a$ . If  $F \in Q(a)$  and  $\psi \left( F(z), zF'(z), z^2F''(z); z, \zeta \right)$  is univalent in  $U$  for  $\zeta \in \bar{U}$ , then

$$\Omega \subset \left\{ \psi \left( F(z), zF'(z), z^2F''(z); z, \zeta \right) : z \in U, \zeta \in \bar{U} \right\}$$

implies  $q(z) \prec F(z)$ .

In recent years, several authors obtained many interesting results in strong differential subordination and superordination [1–4]. In this present investigation, by making use of the strong differential subordination results and strong differential superordination results of Oros and Oros [8, 9], we consider certain suitable classes of admissible functions and investigate some strong differential subordination and superordination properties of meromorphic multivalent quasi-convex functions.

## 2. STRONG SUBORDINATION RESULTS

**Definition 2.1.** Let  $\Omega$  be a set in  $\mathbb{C}$  and  $q \in Q_1 \cap \mathcal{H}$ . The class of admissible functions  $\Phi_{\mathcal{H}}[\Omega, q]$  consists of those functions  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  that satisfy the admissibility condition:  $\phi(u, v, w; z, \zeta) \notin \Omega$ , whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi)}{q(\xi)}, \quad q(\xi) \neq 0 \quad \text{and} \quad \operatorname{Re} \left\{ \frac{w + v^2}{v} \right\} \geq k \operatorname{Re} \left\{ \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right\},$$

where  $z \in U$ ,  $\zeta \in \bar{U}$ ,  $\xi \in \partial U \setminus E(q)$ , and  $k \geq 1$ .

**Theorem 2.1.** Let  $\phi \in \Phi_{\mathcal{H}}[\Omega, q]$ . If  $f \in \Sigma_p$  satisfies

$$(2.1) \quad \left\{ \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right. \right. \\ \left. \left. \times \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) : z \in U, \zeta \in \bar{U} \right\} \subset \Omega,$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

*Proof.* Let the analytic function  $F$  in  $U$  be defined by

$$(2.2) \quad F(z) = -\frac{(z^p f'(z))'}{g'(z)}.$$

After some calculation, we have

$$(2.3) \quad \frac{zF'(z)}{F(z)} = \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}.$$

Further computations show that

$$(2.4) \quad \frac{z^2 F''(z)}{F(z)} + \frac{zF'(z)}{F(z)} - \left( \frac{zF'(z)}{F(z)} \right)^2 = z \left[ \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)} \right]' \\ = \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} \\ + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right).$$

Define the transforms from  $\mathbb{C}^3$  to  $\mathbb{C}$  by

$$u = r, \quad v = \frac{s}{r}, \quad w = \frac{r(t+s) - s^2}{r^2}.$$

Let

$$(2.5) \quad \psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi \left( r, \frac{s}{r}, \frac{r(t+s) - s^2}{r^2}; z, \zeta \right).$$

The proof will make use of Lemma 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.5) that

$$(2.6) \quad \begin{aligned} & \psi \left( F(z), zF'(z), z^2F''(z); z, \zeta \right) \\ &= \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right. \\ & \quad \left. \times \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right). \end{aligned}$$

Therefore, (2.1) becomes  $\psi(F(z), zF'(z), z^2F''(z); z, \zeta) \in \Omega$ .

To complete the proof, we next show that the admissibility condition for  $\phi \in \Phi_{\mathcal{H}}[\Omega, q]$  is equivalent to the admissibility condition for  $\psi$  as given in Definition 1.4. Note that

$$\frac{t}{s} + 1 = \frac{w + v^2}{v}.$$

Hence  $\psi \in \Psi[\Omega, q]$ . By Lemma 1.1,  $F(z) \prec q(z)$  or equivalently

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z). \quad \square$$

We consider the special situation when  $\Omega \neq \mathbb{C}$  is a simply connected domain. In this case  $\Omega = h(U)$ , for some conformal mapping  $h$  of  $U$  onto  $\Omega$  and the class  $\Phi_{\mathcal{H}}[h(U), q]$  is written as  $\Phi_{\mathcal{H}}[h, q]$ . The following result is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** *Let  $\phi \in \Phi_{\mathcal{H}}[h, q]$ . If  $f \in \Sigma_p$  satisfies*

$$(2.7) \quad \begin{aligned} & \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right. \\ & \quad \left. \times \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) \prec\prec h(z), \end{aligned}$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

By taking  $\phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}$ ,  $\beta, \gamma \in \mathbb{C}$ , in Theorem 2.2, we state the following corollary.

**Corollary 2.1.** *Let  $\beta, \gamma \in \mathbb{C}$  and let  $h$  be convex in  $U$  with  $h(0) = 1$  and  $\operatorname{Re}\{\beta h(z) + \gamma\} > 0$ . If  $f \in \Sigma_p$  satisfies*

$$-\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - zg''(z)}{\gamma g'(z) - \beta (z^p f'(z))'} \prec\prec h(z),$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of  $q$  on  $\partial U$  is not known.

**Corollary 2.2.** *Let  $\Omega \in \mathbb{C}$  and  $q$  be univalent in  $U$  with  $q(0) = 1$ . Let  $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$  for some  $\rho \in (0, 1)$ , where  $q_\rho(z) = q(\rho z)$ . If  $f \in \Sigma_p$  satisfies*

$$\begin{aligned} & \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right. \\ & \left. \times \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) \in \Omega, \end{aligned}$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

*Proof.* Theorem 2.1 yields  $-\frac{(z^p f'(z))'}{g'(z)} \prec q_\rho(z)$ . The result is now deduced from the fact that  $q_\rho(z) \prec q(z)$ .  $\square$

**Theorem 2.3.** *Let  $h$  and  $q$  be univalent in  $U$  with  $q(0) = 1$  and set  $q_\rho(z) = q(\rho z)$  and  $h_\rho(z) = h(\rho z)$ . Let  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  satisfy one of the following conditions:*

- (1)  $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$  for some  $\rho \in (0, 1)$ ;
- (2) there exists  $\rho_0 \in (0, 1)$  such that  $\phi \in \Phi_{\mathcal{H}}[h_\rho, q_\rho]$  for all  $\rho \in (\rho_0, 1)$ .

If  $f \in \Sigma_p$  satisfies (2.7), then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

*Proof.* (1) By applying Theorem 2.1, we obtain  $-\frac{(z^p f'(z))'}{g'(z)} \prec q_\rho(z)$ , since  $q_\rho(z) \prec q(z)$ , we deduce

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

- (2) Let  $F(z) = -\frac{(z^p f'(z))'}{g'(z)}$  and  $F_\rho(z) = F(\rho z)$ . Then

$$\phi \left( F_\rho(z), zF'_\rho(z), z^2 F''_\rho(z); \rho z, \zeta \right) = \phi \left( F(\rho z), zF'(\rho z), z^2 F''(\rho z); \rho z, \zeta \right) \in h_\rho(U).$$

By using Theorem 2.1 and the comment associated with

$$\phi \left( F(z), zF'(z), z^2 F''(z); w(z), \zeta \right) \in \Omega,$$

where  $w$  is any function mapping  $U$  into  $U$ , with  $w(z) = \rho z$ , we obtain  $F_\rho(z) \prec q_\rho(z)$  for  $\rho \in (\rho_0, 1)$ . By letting  $\rho \rightarrow 1^-$ , we get  $F(z) \prec q(z)$ . Therefore,

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z). \quad \square$$

The next result gives the best dominant of the strong differential subordination (2.7).

**Theorem 2.4.** Let  $h$  be univalent in  $U$  and  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$ . Suppose that the differential equation

$$(2.8) \quad \phi \left( q(z), \frac{zq'(z)}{q(z)}, \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left( \frac{zq'(z)}{q(z)} \right)^2; z, \zeta \right) = h(z)$$

has a solution  $q$  with  $q(0) = 1$  and satisfies one of the following conditions:

- (1)  $q \in Q_1$  and  $\phi \in \Phi_{\mathcal{H}}[h, q]$ ;
- (2)  $q$  is univalent in  $U$  and  $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$  for some  $\rho \in (0, 1)$ ;
- (3)  $q$  is univalent in  $U$  and there exists  $\rho_0 \in (0, 1)$  such that  $\phi \in \Phi_{\mathcal{H}}[h_\rho, q_\rho]$  for all  $\rho \in (\rho_0, 1)$ .

If  $f \in \Sigma_p$  satisfies (2.7), then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z)$$

and  $q$  is the best dominant.

*Proof.* By applying Theorem 2.2 and Theorem 2.3, we deduce that  $q$  is a dominant of (2.7). Since  $q$  satisfies (2.8), it is also a solution of (2.7) and therefore  $q$  will be dominated by all dominants. Hence,  $q$  is the best dominant of (2.7).  $\square$

In the particular case  $q(z) = 1 + Mz$ ,  $M > 0$  and in view of Definition 2.1, the class of admissible functions  $\Phi_{\mathcal{H}}[\Omega, q]$  denoted by  $\Phi_{\mathcal{H}}[\Omega, M]$  can be expressed in the following form.

**Definition 2.2.** Let  $\Omega$  be a set in  $\mathbb{C}$  and  $M > 0$ . The class of admissible function  $\Phi_{\mathcal{H}}[\Omega, M]$  consists of those functions  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  such that

$$(2.9) \quad \phi \left( 1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left( \frac{kM}{M + e^{-i\theta}} \right)^2; z, \zeta \right) \notin \Omega,$$

whenever  $z \in U$ ,  $\zeta \in \bar{U}$ ,  $\theta \in \mathbb{R}$ ,  $\operatorname{Re}\{Le^{-i\theta}\} \geq k(k-1)M$ , for all  $\theta$  and  $k \geq 1$ .

**Corollary 2.3.** Let  $\phi \in \Phi_{\mathcal{H}}[\Omega, M]$ . If  $f \in \Sigma_p$  satisfies

$$\begin{aligned} & \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'}, \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right. \\ & \left. \times \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) \in \Omega, \end{aligned}$$

then

$$\left| \frac{(z^p f'(z))'}{g'(z)} + 1 \right| < M.$$

When  $\Omega = q(U) = \{w : |w - 1| < M\}$ , the class  $\Phi_{\mathcal{H}}[\Omega, M]$  is simply denoted by  $\Phi_{\mathcal{H}}[M]$ , then Corollary 2.3 takes the following form.

**Corollary 2.4.** Let  $\phi \in \Phi_{\mathcal{H}}[M]$ . If  $f \in \Sigma_p$  satisfies

$$\left| \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) \times \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) - 1 \right| < M,$$

then

$$\left| \frac{(z^p f'(z))'}{g'(z)} + 1 \right| < M.$$

*Example 2.1.* If  $M > 0$  and  $f \in \Sigma_p$  satisfies

$$\left| \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} - \left( \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right)^2 - \frac{z^2 g'''(z)}{g'(z)} + \left( \frac{zg''(z)}{g'(z)} \right)^2 \right| < M,$$

then

$$\left| \frac{(z^p f'(z))'}{g'(z)} + 1 \right| < M.$$

This implication follows from Corollary 2.4 by taking  $\phi(u, v, w; z, \zeta) = w - v + 1$ .

*Example 2.2.* If  $M > 0$  and  $f \in \Sigma_p$  satisfies

$$\left| \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \left( \frac{zg''(z)}{g'(z)} + 1 \right) \right| < \frac{M}{M+1},$$

then

$$\left| \frac{(z^p f'(z))'}{g'(z)} + 1 \right| < M.$$

This implication follows from Corollary 2.3 by taking  $\phi(u, v, w; z, \zeta) = v$  and  $\Omega = h(U)$ , where  $h(z) = \frac{M}{M+1}z$ ,  $M > 0$ . To apply Corollary 2.3, we need to show that  $\phi \in \Phi_{\mathcal{H}}[\Omega, M]$ , that is the admissibility condition (2.9) is satisfied follows from

$$\left| \phi \left( 1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left( \frac{kM}{M + e^{-i\theta}} \right)^2; z, \zeta \right) \right| = \frac{kM}{M+1} \geq \frac{M}{M+1},$$

for  $z \in U$ ,  $\zeta \in \bar{U}$ ,  $\theta \in \mathbb{R}$ , and  $k \geq 1$ .

### 3. STRONG SUPERORDINATION RESULTS

In this section, we obtain strong differential superordination. For this purpose the class of admissible functions given in the following definition will be required.

**Definition 3.1.** Let  $\Omega$  be a set in  $\mathbb{C}$  and  $q \in \mathcal{H}$ . The class of admissible functions  $\Phi'_{\mathcal{H}}[\Omega, q]$  consists of those functions  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$  that satisfy the admissibility condition:  $\phi(u, v, w; \xi, \zeta) \in \Omega$ , whenever

$$u = q(z), \quad v = \frac{zq'(z)}{mq(z)}, \quad q(z) \neq 0 \quad \text{and} \quad \operatorname{Re} \left\{ \frac{w + v^2}{v} \right\} \leq \frac{1}{m} \operatorname{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$



where  $z \in U$ ,  $\zeta \in \bar{U}$ ,  $\xi \in \partial U$ , and  $m \geq 1$ .

**Theorem 3.1.** *Let  $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$ . If  $f \in \Sigma_p$ ,  $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$ , and*

$$\begin{aligned} & \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} \right. \\ & \left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) \end{aligned}$$

is univalent in  $U$ , then

$$(3.1) \quad \Omega \subset \left\{ \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right. \right. \\ \left. \left. \times \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) : z \in U, \zeta \in \bar{U} \right\}$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

*Proof.* Let  $F$  defined by (2.2) and  $\psi(F(z), zF'(z), z^2F''(z); z, \zeta)$  defined by (2.6). Since  $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$ , from (2.6) and (3.1), we have

$$\Omega \subset \left\{ \psi \left( F(z), zF'(z), z^2F''(z); z, \zeta \right) : z \in U, \zeta \in \bar{U} \right\}.$$

From (2.5), we see that the admissibility condition for  $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$  is equivalent to the admissibility condition for  $\psi$  as given in Definition 1.5. Hence  $\psi \in \Psi'[\Omega, q]$  and by Lemma 1.2,  $q(z) \prec F(z)$  or equivalently

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}. \quad \square$$

We consider the special situation when  $\Omega \neq \mathbb{C}$  is a simply connected domain. In this case  $\Omega = h(U)$ , for some conformal mapping  $h$  of  $U$  onto  $\Omega$  and the class  $\Phi'_{\mathcal{H}}[h(U), q]$  is written as  $\Phi'_{\mathcal{H}}[h, q]$ . The following result is an immediate consequence of Theorem 3.1.

**Theorem 3.2.** *Let  $\phi \in \Phi'_{\mathcal{H}}[h, q]$ ,  $q \in \mathcal{H}$ , and  $h$  be analytic in  $U$ . If  $f \in \Sigma_p$ ,  $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$ , and*

$$\begin{aligned} & \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} \right. \\ & \left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) \end{aligned}$$

is univalent in  $U$ , then

$$(3.2) \quad h(z) \prec\prec \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} \right. \\ \left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right)$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

By taking  $\phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}$ ,  $\beta, \gamma \in \mathbb{C}$ , in Theorem 3.2, we state the following corollary.

**Corollary 3.1.** *Let  $\beta, \gamma \in \mathbb{C}$  and let  $h$  be convex in  $U$  with  $h(0) = 1$ . Suppose that the differential equation  $q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$  has a univalent solution  $q$  that satisfies  $q(0) = 1$  and  $q(z) \prec h(z)$ . If  $f \in \Sigma_p$ ,  $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$ , and*

$$-\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - zg''(z)}{\gamma g'(z) - \beta (z^p f'(z))'}$$

is univalent in  $U$ , then

$$h(z) \prec\prec -\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - zg''(z)}{\gamma g'(z) - \beta (z^p f'(z))'}$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

The next result gives the best subordinant of the strong differential superordination (3.2).

**Theorem 3.3.** *Let  $h$  be analytic in  $U$  and  $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$ . Suppose that the differential equation*

$$\phi \left( q(z), \frac{zq'(z)}{q(z)}, \frac{z^2 q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left( \frac{zq'(z)}{q(z)} \right)^2; z, \zeta \right) = h(z)$$

has a solution  $q \in Q_1$ . If  $\phi \in \Phi'_{\mathcal{H}}[h, q]$ ,  $f \in \Sigma_p$ ,  $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$ , and

$$\phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} \right. \\ \left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right)$$

is univalent in  $U$ , then

$$h(z) \prec \prec \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} \right. \\ \left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right)$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

and  $q$  is the best subordinant.

*Proof.* The proof is similar to that of Theorem 2.4 and is omitted.  $\square$

#### 4. SANDWICH RESULTS

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich theorem.

**Theorem 4.1.** Let  $h_1$  and  $q_1$  be analytic functions in  $U$ ,  $h_2$  be univalent in  $U$ ,  $q_2 \in Q_1$  with  $q_1(0) = q_2(0) = 1$  and  $\phi \in \Phi_{\mathcal{H}}[h_2, q_2] \cap \Phi'_{\mathcal{H}}[h_1, q_1]$ . If  $f \in \Sigma_p$ ,  $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$  and

$$\phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} \right. \\ \left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right)$$

is univalent in  $U$ , then

$$h_1(z) \prec \prec \phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} \right. \\ \left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)} \left( \frac{zg''(z)}{g'(z)} - 1 \right); z, \zeta \right) \\ \prec \prec h_2(z)$$

implies

$$q_1(z) \prec -\frac{(z^p f'(z))'}{g'(z)} \prec q_2(z).$$

By combining Corollary 2.1 and Corollary 3.1, we obtain the following sandwich corollary.

**Corollary 4.1.** Let  $\beta, \gamma \in \mathbb{C}$  and let  $h_1, h_2$  be convex in  $U$  with  $h_1(0) = h_2(0) = 1$ . Suppose that the differential equations  $q_1(z) + \frac{zq_1'(z)}{\beta q_1(z) + \gamma} = h_1(z)$ ,  $q_2(z) + \frac{zq_2'(z)}{\beta q_2(z) + \gamma} = h_2(z)$

have a univalent solutions  $q_1$  and  $q_2$ , respectively, that satisfy  $q_1(0) = q_2(0) = 1$  and  $q_1(z) \prec h_1(z)$ ,  $q_2(z) \prec h_2(z)$ . If  $f \in \Sigma_p$ ,  $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$ , and

$$-\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta (z^p f'(z))'}$$

is univalent in  $U$ , then

$$h_1(z) \prec\prec -\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta (z^p f'(z))'} \prec\prec h_2(z)$$

implies

$$q_1(z) \prec -\frac{(z^p f'(z))'}{g'(z)} \prec q_2(z).$$

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