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SOLVING THE FRACTIONAL SCHRÖDINGER EQUATION WITH SINGULAR POTENTIAL BY MEANS OF THE FOURIER TRANSFORM

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Abstract. The focus of this paper is on the study of fractional Schrödinger's equations with δ-like potential and initial data, which have both time-fractional and space-fractional components. We employ the Fourier transform to prove the existence-uniqueness theorems. Additionally, we give the association with the classical solution.

1. Introduction

The main focus of this paper is on the investigation of the fractional Schrödinger equation that involves distributional potentials. Specifically, we consider the Cauchy problem defined as follows:

$$\begin{cases} i\partial_t^\alpha u(t,x) + (-\Delta)^s u(t,\ x) + q(x)u(t,x) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ u(0,\ x) = u_0(x). \end{cases} \tag{1.1}$$

Here, α lies in the interval $(0,1)$, ∂_t^α represents the time-fractional Caputo derivative, $(-\Delta)^s$ denotes the space-fractional Laplacian, and $q(x)$ denotes the singular potential. The value of s is assumed to be greater than 0.

Colombeau algebra, also known as generalized functions or nonlinear generalized functions, is a mathematical concept developed by French mathematician Jean-Francois Colombeau in the 1980's [12]. The idea behind Colombeau algebra is to create a space of functions that is larger than the space of distributions but still

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contains it as a subset. Initially, Colombeau introduced his algebra as a tool to study the singularities of solutions to nonlinear partial differential equations. Later on, the theory was extended to include global analysis and differential geometry. Colombeau algebra has applications in a variety of fields, including mathematical physics, theoretical mechanics, and fluid mechanics (see [14]). The development of Colombeau algebra was motivated by the need to extend the theory of distributions, introduced by Laurent Schwartz in the 1950's [16], to include functions that are not distributions but still have some properties of distributions. Colombeau's approach was to define generalized functions as equivalence classes of smooth functions that are equal up to a set of measure zero. This allowed him to extend the algebraic and topological properties of the space of smooth functions to the space of generalized functions. Since its inception, Colombeau algebra has been the subject of extensive research, leading to numerous generalizations and applications. One of the main challenges in the development of the theory has been to find a suitable notion of convergence for sequences of generalized functions. This led to the introduction of the concept of the natural topology, which has been extensively studied and used in applications. Overall, Colombeau algebra has become an important tool in the study of singularities and nonlinear partial differential equations. It has also led to new insights in the theory of distributions and has opened up new avenues for research in other areas of mathematics. One of Stojanović's notable contributions is her work on extending the notion of Colombeau algebra of generalized functions to fractional derivatives [17]. In [18] Stojanović studied the fractional Schrödinger equation was first introduced by Laskin in quantum mechanics. Motivated by the previous paper, and also [5, 13, 19] and reference therein, we will studied the existence and uniqness of fractional Schrödinger equation in a suitable spaces.

The paper is structured as follows. Section 2 provides a review of fundamental spaces and their inclusion into Colombeau algebras type. The main result is presented in Section 3. Finally, we conclude the paper by discussing the implications and offering perspectives for future research.

2. Basic Spaces

In this section, we will discuss various concepts related to the Colombeau algebra type and its properties.

2.1. **Generalized Fractional Spaces.** Let $r > 0$, the fractional Sobolev space H^r is defined as:

$$H^r = \Big\{u \in L^2(\mathbb{R}) \mid \|u\|_r = \|u\|_{L^2} + \|(-\Delta)^{\frac{r}{2}}\|_{L^2} < +\infty\Big\}.$$

We denote by $\|\cdot\|_\alpha$ the norm defined by

$$\|u\|_\alpha = \|u(t,\cdot)\|_{L^2} + \|\partial_t^\alpha u(t,\cdot)\|_{L^2} + \|(-\Delta)^{\frac{r}{2}}(t,\cdot)\|_{L^2}.$$

Let Ω be an open subset of $\mathbb{R}^n$. We denote $\mathcal{E}(\Omega)$ the set of all $\mathcal{C}^\infty$ maps from Ω into $\mathbb{R}$. The set of moderate functions is defined as

$$\mathcal{E}_s(\Omega) = \left\{ R \in \mathcal{E}(\Omega) \mid (\forall D)(\exists N) \|R\|_{H^r} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right) \right\}.$$

The set of negligible functions is defined by

$$\mathcal{N}_s(\Omega) = \left\{ R \in \mathcal{E}_s(\Omega) \mid (\forall D)(\forall q \in \mathbb{N}) \|R\|_{H^r} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{q}\right) \right\}.$$

Then, our space $\mathcal{G}_{H^r}(\Omega) = \mathcal{E}_s(\Omega)/\mathcal{N}_s(\Omega)$ of simplified generalized functions on Ω is the quotient algebra. In the same we define the algebra type Colombeau $\mathcal{G}_{L^\infty}$ as a factor algebra given by the quotient $\mathcal{E}_\infty/\mathcal{N}_\infty$, where

$$\mathcal{E}_\infty = \left\{ (R_\epsilon) \mid (\forall D)(\exists N) \sup_{x\in\Omega} \|DR_\epsilon(x)\|_{L^\infty} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right) \right\}$$

and

$$\mathcal{N}_\infty = \left\{ (R_\epsilon) \mid (\forall D)(\forall q) \sup_{x\in\Omega} \|DR_\epsilon(x)\|_{L^\infty} = \mathcal{O}_{\epsilon\to 0}(\epsilon^{q}) \right\}.$$

Now, let $D_{L^\infty}(\Omega)$ the set of all $\mathcal{C}^\infty$ functions on Ω, globally bounded on Ω as well as all its derivatives, then to f associate $f_\epsilon = f$. This given the following inclusion $D_{L^\infty} \subset \mathcal{G}_\infty$. Let f be a function in the space $L^\infty(R^d)$, then to f associate $f_\epsilon = f * \rho_\epsilon$ with a chosen $\rho_\epsilon(t) = \epsilon^{-d}\rho\left(\frac{t}{\epsilon}\right)$, where $\rho \in \mathcal{D}(\mathbb{R}^d)$ and $\int \rho = 1$. For any given mollifier ρ this gives an inclusion $L^\infty(\mathbb{R}^d) \subset \mathcal{G}_\infty$. More generally let T be a distribution in D'_{L^∞}, i.e., T is a finite sum of derivatives of functions in $L^\infty(\mathbb{R}^d)$. To T associate $T_\epsilon = T * \rho_\epsilon$ as above. For given ρ as above this gives an inclusion of $D'_{L^\infty} \subset \mathcal{G}_{H^r}$. Similarly, one has an inclusion of $\mathcal{E}'$ space of all distributions with compact support into $\mathcal{G}_{H^r}$.

2.2. **Regularized Laplace-fractional operator.** In this section, we regularize the fractional Laplace operator as described in reference [1], but this time we use a scaling function. The Laplace-Fractional operator is given by

$$(-\triangle)^{\frac{r}{2}} f(x) = \frac{-\Gamma[\frac{r-1}{2}]}{\pi^{\frac{2-r}{2}} 2^{2-r}\Gamma[\frac{2-r}{2}]} \int \frac{\triangle f(\xi)}{|x-\xi|^{r-1}} d\xi.$$

Note that

$$(-\triangle)^{\frac{r}{2}} f(x) = \frac{\eta}{t^{r-1}} * \Delta f(t),$$

where $\eta = \frac{-\Gamma[(r-1)/2)]}{\pi^{(2-r)/2} 2^{2-r}\Gamma[(2-r)/2]}$. Now using the following regularization

$$(-\tilde{\Delta})^{\frac{r}{2}} f(t) = \frac{\eta}{t^{r-1}} * \Delta f(t) * \rho_{h(\epsilon)}(t),$$

where $h : [0,1] \to [0,1]$ is a scaling function, for more information see [11].

Proposition 2.1. *For each* $(u_\epsilon) \in \mathcal{E}_s$, $\left((-\tilde{\Delta})^{\frac{r}{2}} u_\epsilon\right) \in \mathcal{E}_s$.

Proof. Through the beginning of the section, for all derivative D we have

$$D\Big((-\tilde{\Delta})^{\frac{r}{2}} u_\epsilon\Big) = \frac{\eta}{t^{r-1}} * \Delta f(t) * D\rho_{h(\epsilon)}(t),$$

which proves the result. □

Proposition 2.2. *We have the following result*

$$(-\tilde{\Delta})^{\frac{r}{2}} f \approx_{L^2} (-\Delta)^{\frac{r}{2}} f.$$

Proof. Since $\rho_{h(\epsilon)} \to \delta$, by applying The dominated convergence theorem, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left|(-\tilde{\triangle})^{\frac{r}{2}} f_\epsilon(x) - (-\triangle)^{\frac{r}{2}} f_\epsilon(x)\right| &= \eta \left|(-\tilde{\triangle})^{\frac{r}{2}} f_\epsilon(x) - (-\triangle)^{\frac{r}{2}} f_\epsilon(x)\right| \\ &= \eta \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} \triangle f_\epsilon(x) t^{1-r} |\rho_{h(\epsilon)}(t) - \delta(t)| \to 0,\end{aligned}$$

but, f_ϵ is a function with compact support, then

$$\left\|\triangle f_\epsilon(x) t^{1-r} \left(\rho_{h(\epsilon)}(t) - \delta(t)\right)\right\|_{L^2} \to 0.$$ □

We regularize the Caputo fractional derivative in the same way, we put

$$\tilde{D^\alpha} u_\epsilon(t) = D^\alpha u_\epsilon * \rho_{h(\epsilon)}(t).$$

In the same we can prove that $\tilde{D^\alpha} u_\epsilon \approx D^\alpha u_\epsilon$.

In what remains we note D^α and $(-\triangle)^r$ in the place of $\tilde{D^\alpha}$ and $\tilde{\triangle}^\alpha$.

3. Main Results

The objective of this section is to establish the existence, uniqueness, and continuity of the problem (1.1). We begin by formulating our problem for each representative solution u of equation (1.1).

Now let's consider the approximate problem.

$$\begin{cases} i\partial_t^\alpha u_\epsilon(t,x) + (-\Delta)_\epsilon^s u_\epsilon(t,x) + q_\epsilon(x) u_\epsilon(t,x) = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ u_\epsilon(0,x) = u_{0\epsilon}(x). \end{cases} \tag{3.1}$$

3.1. **Existence and uniqueness.** We provide the following definition for the concept of a generalized solution.

Definition 3.1. A solution (3.2) to the problem is a generalized function u which belongs to the $\mathcal{G}_{H^r}$ such that for each representant u_ϵ of u satisfy the problem (1.1).

Proposition 3.1 ([14])**.** *A moderate function (u_ϵ), is negligible if and only if the following condition is satisfied:*

$$\|u_\epsilon\|_{L^\infty} = \mathcal{O}_{\epsilon\to 0}(\epsilon^m), \quad \text{for all } n \in \mathbb{N}.$$

The main results are presented in the following theorem.

Theorem 3.1. *If $u_0 \in \mathcal{G}_{H_r}$ and $q \in G_{L^\infty}$, then for all $T > 0$ the problem (1.1) has a unique solution in $\mathcal{G}([0,T] \times H_r)$.*

Proof. **Existence.** We have

$$i\partial_t^{\alpha} u_\epsilon(t,\ x) + (-\tilde{\Delta}_\epsilon)^r u_\epsilon(t,\ x) + q_\epsilon(x) u_\epsilon(t,\ x) = 0.$$

After application of the Fourier transformation in this equation we get

$$i\partial_t \hat{u}_\epsilon(t,\xi) + |\xi|^{2r} \hat{u}_\epsilon(t,\xi) = \hat{f}_\epsilon(t,\xi),$$

u_ϵ and f_ϵ with respect to the spatial variable x and $f_\epsilon(t,x) = -p_\epsilon(x)u_\epsilon(t,x)$, where $\widehat{u}_\epsilon, \widehat{f}_\epsilon$, denote the Fourier transforms.

Now using Duhamel's principle, we get the following representation of the solution to the Cauchy problem

$$\begin{aligned} \hat{u}_\epsilon(t,\xi) =& \hat{u}_{0\epsilon} e^{-i|\xi|^{2r} e^{\frac{i\alpha\pi}{2}} t} e^{i|\xi|^{2r} t^{1-\alpha}} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{i|\xi|^{2r} e^{\frac{i\alpha\pi}{2}}(t-s)} e^{i|\xi|^{2r}(t-s)^{1-\alpha}} \hat{f}_\epsilon(s) ds. \end{aligned} \tag{3.2}$$

Further, we can find two numbers a and b such that

$$\|(-\Delta)^r u_\epsilon\|_{L^2} \leq a\|u_\epsilon\|_{L^2} \quad \text{and} \quad \|\partial^\alpha u_\epsilon\|_{L^2} \leq b\|u_\epsilon\|_{L^2},$$

which implies the estimate

$$\|\hat{u}_\epsilon\|_{L^2} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right),$$

for some $N \in \mathbb{N}$.

Using the Plancherel-Parseval formula, we can write

$$\|u_\epsilon\|_{L^2} = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

Then,

$$\|u_\epsilon\| = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

As we know the Fractional Laplace $(-\Delta)^s$ can be written as a convolution of

$$\frac{-\Gamma[(r-1)/2)]}{*^{d-2+2s}\pi^{(2-r)/2}2^{2-r}\Gamma[(2-r)/2]}$$

and $\Delta u(t,\cdot)$, which is permutable with any integer derivative D. Thus,

$$i\partial_t \hat{Du}_\epsilon(t,\xi) + |\xi|^{2r} \hat{Du}_\epsilon(t,\xi) = \hat{Df}_\epsilon(t,\xi).$$

By the same method, we can prove that for each derivative D

$$\|Du_\epsilon\| = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

Then, for some $N \in \mathbb{N}$, that is (u_ϵ) is moderate, it follows that the classe u is a solution of the problem.

Uniqueness. Let u and v be two solutions of the problem (1.1). Put $U = u - v$, it is clear that $u_0 = v_0$.

Let's go to the Fourier transform,

$$\hat{U}_\epsilon = \left(\hat{U}_{0\epsilon}\right) e^{-i|\xi|^{2r} e^{\frac{i\alpha\pi}{2}} e^{i|\xi|^{2r} t^{1-\alpha}}} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{i|\xi|^{2r} e^{\frac{i\alpha\pi}{2}}(t-s)} e^{i|\xi|^{2r}(t-s)^{1-\alpha}} q_\epsilon \hat{U}_\epsilon(s) ds, \tag{3.3}$$

which implies that

$$\|\hat{U}_\epsilon\|_{L^2} \leq \|\hat{U}_{0\epsilon}\|_{L^2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|q_\epsilon\|_{L^\infty} \|\hat{U}_\epsilon\|_{L^2} ds.$$

Gronwall's lemma and Plancherel-Parseval ensure that U_ϵ has bound negligible functions. We use [14, (1.2.3) page 11], and we find that $(U_\epsilon) \in \mathcal{N}_s\left([0,T] \times H^r\right)$. □

3.2. **Association.** In this section we will prove the association with the classical solution to the problem (1.1). First, we will define the meaning of association.

Definition 3.2. A function $f \in \mathcal{G}(\mathbb{R})$ is considered to have an "associated distribution", denoted as $f \approx u$, if for every representative $f(\varphi_\epsilon, y)$ of f and $\psi(y) \in \mathcal{D}(\mathbb{R})$, there exists a natural number q such that for any $\varphi(y) \in \mathcal{A}_q(\mathbb{R})$, we have:

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} f(\varphi_\epsilon, y)\psi(y) dy = \langle u, \psi \rangle.$$

Then, we give the following result.

Theorem 3.2. *Let $q \in L^\infty(\mathbb{R}^d)$. Assume that $u_0 \in H^r(\mathbb{R}^d)$ the Cauchy problem*

$$\begin{cases} i\partial^\alpha u_t(t,x) + (-\Delta)^r u(t,x) + q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3.4}$$

has a unique solution $u \in \mathcal{C}^1\left([0,T] : L^2(\mathbb{R}^d)\right) \cap \mathcal{C}\left([0,T] : H^r(\mathbb{R}^d)\right)$.

Proof. To prove the existence and uniqueness of a solution, we can use the theory of linear evolution equations.

We consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}u = -(-\Delta)^r u - q(x)u. \tag{3.5}$$

The fractional Sobolev space $H^r(\mathbb{R}^d)$ is the natural domain of the operator $(-\Delta)^r$. It is a reflexive Banach space, and we can prove that $\mathcal{L}$ is a closed operator from $H^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. By applying theorem of Lumer-Phillips [15], the operator $\mathcal{L}$ generates a strongly continuous semigroup on $L^2(\mathbb{R}^d)$. Moreover, the semigroup satisfies the properties of positivity, contractivity and boundedness. This means that for each $t \geq 0$, there exists a linear operator $S(t)$ such that $\int_0^\infty S(\left(\frac{t^\alpha}{\theta^\alpha}\right) t) u_0 d\theta$ is the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(t,x) = \mathcal{L}u(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{3.6}$$

for the argument see [21]. □

Theorem 3.3. *The classical solution given by Theorem* 3.2 *is associated with the solution of the problem* (3.1).

Proof. Let u be the classical solution to

$$\begin{cases} i\partial^\alpha u_t(t,x) + (-\Delta)^r u(t,x) + q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases}$$

We have $u(t,\cdot) \in H^r(\mathbb{R}^d)$ for all $t \in [0,T]$, and let $[(u_\epsilon)_\epsilon]$ be the solution of (3.2). It satisfies

$$\begin{cases} i\partial^\alpha u_\epsilon(t,x) + (-\Delta)^r u_\epsilon(t,x) + q_\epsilon(x)u_\epsilon(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u_\epsilon(0,x) = u_{0,\epsilon}(x). \end{cases}$$

Let us denote by $U_\epsilon(t,x) := u(t,x) - u_\epsilon(t,x)$. It solves

$$\begin{cases} i\partial^\alpha U_\epsilon(t,x) + (-\Delta)^r U_\epsilon(t,x) + q_\epsilon(x)U_\epsilon(t,x) = p_\epsilon(t,x), & (t,x) \in (0,\ T) \times \mathbb{R}^d, \\ U_\epsilon(0,x) = (u_0 - u_{0,\epsilon})(x), \end{cases}$$

where $p_\epsilon(t,x) = (q_\epsilon(x) - q(x))u(t,x)$.

Using Duhamel's principle and similar arguments as in Theorem 3.2, we get the estimate

$$\|U_\epsilon(t,\cdot)\|_{L^2} \le \|u_0 - u_{0,\epsilon}\|_{L^2} + \frac{1}{\gamma(\alpha)} \int_0^T T^{\alpha-1} \|g_\epsilon(s,\cdot)\|_{L^2} ds,$$

where $g_\epsilon = p_\epsilon - q_\epsilon u$, which implies that

$$\|U_\epsilon(t,\cdot)\|_{L^2} \le \|u_0 - u_{0,\epsilon}\|_{L^2} + \frac{T^\alpha}{\Gamma(\alpha+1)} \|p_\epsilon\|_{L^\infty} + \frac{T^\alpha}{\Gamma(\alpha)} \int_0^T \|U_\epsilon(s,\cdot)\|_{L^2} ds.$$

Now, use the Gronwal's lemma, we obtain

$$\|U_\epsilon(t,\cdot)\|_{L^2} \le \left(\|U_{0,\epsilon}\|_{L^2} + \frac{T^\alpha}{\Gamma(\alpha+1)} \|p_\epsilon\|_{L^\infty} \right) \exp \frac{T^{\alpha+1}}{\Gamma(\alpha)}.$$

When $\epsilon \to 0$, the right hand side of the last inequality tends to 0, since $\|p_\epsilon\|_{L^\infty} \to 0$ and $\|U_{0,\epsilon}\|_{L^2} \to 0$. Hence, $U \approx 0$. ☐

4. Conclusion

In this paper, we utilize the Fourier transform on an arbitrary representative to establish the existence and uniqueness of a generalized fractional Schrödinger equation. The utilization of Gronwall's lemma and the Plancherel-Parseval formula plays a crucial role in achieving this objective. In the future, we plan to further investigate this type of equation through numerical simulations.

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