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SYMMETRIC N-ADDITIVE MAPPINGS ADMITTING SEMIPRIME RING

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Abstract. Let $\mathcal{R}$ be a ring with centre $Z(\mathcal{R})$. An n-additive map $D : \mathcal{R}^n \to \mathcal{R}$ is called symmetric n-additive if $D(x_1, \dots, x_n) = D(x_{\pi(1)}, \dots, x_{\pi(n)})$ for all $x_i \in \mathcal{R}$ and for every permutation $(\pi(1), \pi(2), \dots, \pi(n))$. A mapping $\triangle : \mathcal{R} \to \mathcal{R}$ defined by $\triangle(x) = D(x, x, \dots, x)$ is called the trace of D. In this paper, we prove that a nonzero Lie ideal L of a semiprime ring $\mathcal{R}$ of characteristic different from $(2^n - 2)$ is central, if it satisfies any one of the following properties: (i) $\triangle([x, y]) \mp xy \in Z(\mathcal{R})$; (ii) $\triangle([x, y]) \mp [y, x] \in Z(\mathcal{R})$; (iii) $\triangle(xy) \mp \triangle(x) \mp [x, y] \in Z(\mathcal{R})$; (iv) $\triangle([x, y]) \mp yx \in Z(\mathcal{R})$; (v) $\triangle(xy) \mp \triangle(y) \mp [x, y] \in Z(\mathcal{R})$.

1. Introduction

Throughout the paper, $\mathcal{R}$ always represents an associative ring, $Z(\mathcal{R})$ is its centre. Let $x, y, z \in \mathcal{R}$. We write the notation $[y, x]$ for the commutator $yx - xy$ and make use of the identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that $\mathcal{R}$ is prime if $a\mathcal{R}b = \{0\}$ implies that either $a = 0$ or $b = 0$ semiprime if $a\mathcal{R}a = \{0\}$ implies that $a = 0$. Let $\mathcal{R}$ and $\mathcal{S}$ be abelian groups. A map $q : \mathcal{R} \to \mathcal{S}$ is called the trace of a biadditive map if there exists a biadditive map $B : \mathcal{R} \times \mathcal{R} \to \mathcal{S}$ such that $q(x) = B(x, x)$ for all $x \in \mathcal{R}$. Assuming further that $\mathcal{R} \subseteq \mathcal{R}'$ are rings, we say that q is commuting if $[q(x), x] = q(x)x - xq(x) = 0$ for all $x \in \mathcal{R}$. An example is a map of the form $q(x) = \lambda x^2 + \mu(x)x + \nu(x)$ where $\lambda \in C$, the centre of $\mathcal{S}$ and $\mu, \nu : \mathcal{R} \to C$, μ is additive and ν is the trace of a biadditive map. Quite often it turns out that this obvious example is in fact the only possible example of a commuting trace of a biadditive map of $\mathcal{R}$ into $\mathcal{S}$. The basic result of this type states that this is true in the

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case when $\mathcal{R}$ is a prime ring with $char(\mathcal{R}) \neq 2$ and $\mathcal{S}$ is its central closure provided, however that $\mathcal{R}$ does not satisfy s_4, the standard polynomial identity of degree 4 ([11], Theorem 1). This theorem has turned out to be the key for solving different problems and to a great extent it initiated the development of the theory of functional identities. We studies about bidervations and the traces of mapping in articles [1, 9, 10, 12] for details. A map $f : \mathcal{R} \to \mathcal{R}$ is centralizing on $\mathcal{R}$ if $[f(x), x] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. An additive map $D : \mathcal{R} \to \mathcal{R}$ is called a derivation if it satisfies the Leibnitz rule $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{R}$. Let $n \geq 2$ be a fixed positive integer. A map $D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n-\text{times}} \to \mathcal{R}$ is said to be symmetric (or permuting), if the equation $D(x_1, x_2, \ldots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for all $x_i \in \mathcal{R}$ and for every permutation $(\pi(1), \pi(2), \ldots, \pi(n))$. Let us consider the following maps Let $n \geq 2$ be a fixed positive integer. An n-additive map

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n-\text{times}} \to \mathcal{R}$$

will be called an n-derivation if the relations

$$\begin{aligned} D(x_1 x_1', x_2, \ldots, x_n) =& D(x_1, x_2, \ldots, x_n) x_1' + x_1 D(x_1', x_2, \ldots, x_n), \\ D(x_1, x_2 x_2', \ldots, x_n) =& D(x_1, x_2, \ldots, x_n) x_2' + x_2 D(x_1, x_2', \ldots, x_n), \\ &\vdots \\ D(x_1, x_2, \ldots, x_n') =& D(x_1, x_2, \ldots, x_n) x_n' + x_n D(x_1, x_2, \ldots, x_n'), \end{aligned}$$

are valid for all $x_i, x_i' \in \mathcal{R}$. Of course, an 1-derivation is a derivation and a 2-derivation is called a bi-derivation. If D is symmetric, then the above inequalities are equivalent to each other. Let $n \geq 2$ be a fixed positive integer. If $\mathcal{R}$ is commutative, then a map

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n-\text{times}} \to \mathcal{R},$$

defined by

$$(x_1, x_2, \ldots, x_n) \to D(x_1) D(x_2) \cdots D(x_n), \quad \text{for all } x_i \in \mathcal{R}, i = 1, 2, \ldots, n,$$

is a symmetric n-derivation, where D is a derivation on $\mathcal{R}$. Let $n \geq 2$ be a fixed positive integer and let a map $\triangle : \mathcal{R} \to \mathcal{R}$ defined by $\triangle(x) = D(x, x, \ldots, x)$ for all $x \in \mathcal{R}$, where

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n-\text{times}} \to \mathcal{R}$$

is a symmetric map, be the trace of D. It is obvious that, in case when

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n-\text{times}} \to \mathcal{R}$$

is a symmetric map which is also n-additive, the trace $\triangle$ of D satisfies the relation

$$\triangle(x+y)=\triangle(x)+\triangle(y)+\sum_{k=1}^{n-1}\binom{n}{k}h_k(x,y), \quad \text{for all } x,y\in\mathcal{R},$$

and

$$h_k(x,y)=D(\underbrace{x,x,\ldots,x}_{(n-k)-\text{times}},\underbrace{y,y,\ldots,y}_{k-\text{times}}).$$

Gy. Maksa [3] introduced the concept of a symmetric biderivation (see also [2], where an example can be found). It was shown in [3] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [12] and [5]. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f:\mathcal{R}\to\mathcal{R}$ gives rise to a biderivation on $\mathcal{R}$. Namely linearizing $[x,f(x)]=0$ for all $x\in\mathcal{R}$, we get

$$[f(x),y]=[x,f(y)], \quad \text{for all } x\in\mathcal{R},$$

and hence we note that the mapping $(x,y)\to[f(x),y]$ is a biderivation (moreover, all derivations appearing are inner). Motivated by the aforementioned results we prove that a nonzero Lie ideal L of a semiprime ring $\mathcal{R}$ of characteristic different from (2^n-2) is central, if it satisfies any one of the following properties: (i) $\triangle([x,y])\mp xy\in Z(\mathcal{R})$; (ii) $\triangle([x,y])\mp[y,x]\in Z(\mathcal{R})$; (iii) $\triangle(xy)\mp\triangle(x)\mp[x,y]\in Z(\mathcal{R})$; (iv) $\triangle([x,y])\mp yx\in Z(\mathcal{R})$; (v) $\triangle(xy)\mp\triangle(y)\mp[x,y]\in Z(\mathcal{R})$.

2. Preliminary Results

We make extensive use of basic commutator identities

$$[xy,z]=[x,z]y+x[y,z], \quad [x,yz]=[x,y]z+y[x,z].$$

Moreover, we shall require the following lemmas.

Lemma 2.1 ([5], Lemma 1.1.5)**.** *If $\mathcal{R}$ is a semiprime ring, then the center of a nonzero one sided ideal is contained in the center of $\mathcal{R}$. As an immediate consequence, any commutative one sided ideal is contained in the center of $\mathcal{R}$.*

Lemma 2.2. *Let $\mathcal{R}$ be a semiprime ring and L be a nonzero Lie ideal of $\mathcal{R}$. If $[L,L]\subseteq Z(\mathcal{R})$, then $L\subseteq Z(\mathcal{R})$.*

Proof. Since $xy\in Z(\mathcal{R})$ for all $x,y\in L$, $xy-yx=[x,y]\in Z(\mathcal{R})$ for all $x,y\in L$. Using Lemma 2.1 we get the required result. □

3. Main Results

Theorem 3.1. *Let $\mathcal{R}$ be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D:\mathcal{R}^n \to \mathcal{R}$ be a symmetric n-additive mapping and $\triangle$ be the trace of D. If $\triangle([x,y]) \mp xy \in Z(\mathcal{R})$ for all $x,y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Let

$$\triangle([x,y]) - xy \in Z(\mathcal{R}), \quad \text{for all } x,y \in L. \tag{3.1}$$

Replacing y by $y+z$ in (3.1), we have

$$\triangle([x,y]+[x,z]) - xy - xz \in Z(\mathcal{R}), \quad \text{for all } x,y,z \in L.$$

This implies that

$$\triangle([x,y]) + \triangle([x,z]) + \sum_{k=1}^{n-1} \binom{n}{k} h_k([x,y],[x,z]) - xy - xz \in Z(\mathcal{R}).$$

By using (3.1), we obtain

$$\sum_{k=1}^{n-1} \binom{n}{k} h_k([x,y],[x,z]) \in Z(\mathcal{R}), \quad \text{for all } x,y,z \in L.$$

This gives that

$$\begin{aligned}&\binom{n}{1} h_1([x,y],[x,z]) + \binom{n}{2} h_2([x,y],[x,z]) + \binom{n}{3} h_3([x,y],[x,z]) \\ &+ \cdots + \binom{n}{n-1} h_{n-1}([x,y],[x,z]) \in Z(\mathcal{R}).\end{aligned} \tag{3.2}$$

Substituting y for z in (3.2), we obtain

$$\begin{aligned}&\binom{n}{1} h_1([x,y],[x,y]) + \binom{n}{2} h_2([x,y],[x,y]) + \binom{n}{3} h_3([x,y],[x,y]) \\ &+ \cdots + \binom{n}{n-1} h_{n-1}([x,y],[x,y]) \in Z(\mathcal{R}).\end{aligned}$$

This implies that

$$\begin{aligned}&\binom{n}{1} D(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-1)-\text{times}}, \underbrace{[x,y]}_{1-\text{times}}) + \binom{n}{2} D(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-2)-\text{times}}, \underbrace{[x,y]}_{2-\text{times}}) \\ &+ \cdots + \binom{n}{n-1} D(\underbrace{[x,y]}_{1-\text{times}}, \underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-1)-\text{times}}) \in Z(\mathcal{R}).\end{aligned}$$

This shows that

$$\left(\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1}\right) D([x,y],[x,y],\ldots,[x,y]) \in Z(\mathcal{R}).$$

We obtain

$$(2^n-2)D([x,y],[x,y],\dots,[x,y]) \in Z(\mathcal{R}), \quad \text{for all } x,y \in L. \tag{3.3}$$

Since $\mathcal{R}$ is not of characteristic (2^n-2), we get

$$D([x,y],[x,y],\dots,[x,y]) \in Z(\mathcal{R}), \quad \text{for all } x,y \in L.$$

Applying the definition of the trace

$$\triangle([x,y]) \in Z(\mathcal{R}), \quad \text{for all } x,y \in L. \tag{3.4}$$

Using (3.1), we get $xy \in Z(\mathcal{R})$ for all $x,y \in L$. This implies that $[x,y] \in Z(\mathcal{R})$. By using Lemma 2.2, we get $L \subseteq Z(\mathcal{R})$.

Similarly, we can prove the result if $f([x,y]) + xy \in Z(\mathcal{R})$ for all $x,y \in L$. ☐

Theorem 3.2. *Let $\mathcal{R}$ be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D : \mathcal{R}^n \to \mathcal{R}$ be a symmetric n-additive mapping and $\triangle$ be the trace of D. If $\triangle([x,y]) \mp [y,x] \in Z(\mathcal{R})$ for all $x,y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Using the same argument as in Theorem 3.1. ☐

Theorem 3.3. *Let $\mathcal{R}$ be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D : \mathcal{R}^n \to \mathcal{R}$ be a symmetric n-additive mapping and $\triangle$ be the trace of D. If $\triangle(xy) \mp \triangle(x) \mp [x,y] \in Z(\mathcal{R})$ for all $x,y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Suppose

$$\triangle(xy) - \triangle(x) - [x,y] \in Z(\mathcal{R}), \quad \text{for all } x,y \in L. \tag{3.5}$$

Replacing x by $x+z$ in (3.5), we have

$$\triangle((x+z)y) + \triangle(x+z) - [x+z,y] \in Z(\mathcal{R}), \quad \text{for all } x,y,z \in L.$$

This implies that

$$\triangle(xy+zy) - \triangle(x+z) - [x,y] - [z,y] \in Z(\mathcal{R}), \quad \text{for all } x,y,z \in L.$$

This gives that

$$\begin{aligned}\triangle(xy) + \triangle(zy) + \sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,zy) - \triangle(x) - \triangle(z)\\ - \sum_{k=1}^{n-1}\binom{n}{k} h_k(x,z) - [x,y] - [z,y] \in Z(\mathcal{R}).\end{aligned}$$

This implies that

$$\begin{aligned}&\triangle(xy) - \triangle(x) - [x,y] + \triangle(zy) - \triangle(z) - [z,y]\\ &+ \sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,zy) - \sum_{k=1}^{n-1}\binom{n}{k} h_k(x,z) \in Z(\mathcal{R}).\end{aligned}$$

Using (3.5), we get

$$\sum_{k=1}^{n-1}\binom{n}{k}h_k(xy,zy)-\sum_{k=1}^{n-1}\binom{n}{k}h_k(x,z)\in Z(\mathcal{R}),\quad \text{for all } x,y,z\in L.$$

This shows that

$$\begin{aligned}&\binom{n}{1}h_1(xy,zy)+\binom{n}{2}h_2(xy,zy)+\cdots+\binom{n}{n-1}h_{n-1}(xy,zy)\\ &-\binom{n}{1}h_1(x,z)-\binom{n}{2}h_2(x,z)-\cdots-\binom{n}{n-1}h_{n-1}(x,z)\in Z(\mathcal{R}).\end{aligned}\tag{3.6}$$

Substituting x for z in (3.6), we have

$$\begin{aligned}&\binom{n}{1}h_1(xy,xy)+\binom{n}{2}h_2(xy,xy)+\cdots+\binom{n}{n-1}h_{n-1}(xy,xy)\\ &-\binom{n}{1}h_1(x,x)-\binom{n}{2}h_2(x,x)-\cdots-\binom{n}{n-1}h_{n-1}(x,x)\in Z(\mathcal{R}).\end{aligned}$$

We find that

$$\begin{aligned}&\binom{n}{1}D(\underbrace{xy,xy,\ldots,xy}_{(n-1)-\text{times}},\underbrace{xy}_{1-\text{times}})+\binom{n}{2}D(\underbrace{xy,xy,\ldots,xy}_{(n-2)-\text{times}},\underbrace{xy}_{2-\text{times}})\\ &+\cdots+\binom{n}{n-1}D(\underbrace{xy}_{1-\text{times}},\underbrace{xy,xy,\ldots,xy}_{(n-1)-\text{times}})\in Z(\mathcal{R}).\end{aligned}$$

This implies that

$$(2^n-2)\left(D(xy,xy,\ldots,xy)-D(x,x,\ldots,x)\right)\in Z(\mathcal{R}),\quad \text{for all } x,y\in L.$$

Since $\mathcal{R}$ is not of characteristic (2^n-2),

$$D(xy,xy,\ldots,xy)-D(x,x,\ldots,x)\in Z(\mathcal{R}),\quad \text{for all } x,y,z\in L.$$

By definition of the trace, we get

$$\triangle(xy)-\triangle(x)\in Z(\mathcal{R}),\quad \text{for all } x,y\in L.\tag{3.7}$$

Using (3.5), $[x,y]\in Z(\mathcal{R})$ for all $x,y\in L$. Arguing similar manner as in the Theorem 3.1, we get the result. Similarly, we can prove the result if $\triangle(xy)+\triangle(x)+[x,y]\in Z(\mathcal{R})$ for all $x,y\in L$. □

Theorem 3.4. *Let $\mathcal{R}$ be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D:\mathcal{R}^n\to\mathcal{R}$ be a symmetric n-additive mapping and $\triangle$ be the trace of D. If $\triangle([x,y])\mp yx\in Z(\mathcal{R})$ for all $x,y\in L$, then $L\subseteq Z(\mathcal{R})$.*

Proof. Using the same argument as in Theorem 3.3. □

Theorem 3.5. *Let $\mathcal{R}$ be a semiprime ring of characteristic not (2^n-2) and L be a nonzero left ideal of $\mathcal{R}$. Let $D:\mathcal{R}^n\to\mathcal{R}$ be a symmetric n-additive mapping and $\triangle$ be the trace of D. If $\triangle(xy)\mp\triangle(y)\mp[x,y]\in Z(\mathcal{R})$ for all $x,y\in L$, then $L\subseteq Z(\mathcal{R})$.*

Proof. Suppose

$$\triangle(xy) - \triangle(y) - [x,y] \in Z(\mathcal{R}), \quad \text{for all } x,y \in L. \tag{3.8}$$

Replacing y by $y+z$ in (3.8), we obtain

$$\triangle(x(y+z)) - \triangle(y+z) - [x,y+z] \in Z(\mathcal{R}), \quad \text{for all } x,y,z \in L.$$

This shows that

$$\begin{aligned}&\triangle(xy) + \triangle(xz) + \sum_{k=1}^{n-1}\binom{n}{k}h_k(xy,xz) - \triangle(y)\\ &- \triangle(z) - \sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z) - [x,y] - [x,z] \in Z(\mathcal{R}).\end{aligned}$$

We find that

$$\begin{aligned}&\triangle(xy) - \triangle(y) - [x,y] + \sum_{k=1}^{n-1}\binom{n}{k}h_k(xy,xz) + \triangle(xz)\\ &- \triangle(z) - [x,z] - \sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z) \in Z(\mathcal{R}).\end{aligned}$$

Using (3.8), we have

$$\sum_{k=1}^{n-1}\binom{n}{k}h_k(xy,xz) - \sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z) \in Z(\mathcal{R}).$$

On simplifying,

$$\begin{aligned}&\binom{n}{1}h_1(xy,xz) + \binom{n}{2}h_2(xy,xz) + \cdots + \binom{n}{n-1}h_{n-1}(xy,xz)\\ &- \binom{n}{1}h_1(y,z) - \binom{n}{2}h_2(y,z) - \cdots - \binom{n}{n-1}h_{n-1}(y,z) \in Z(\mathcal{R}).\end{aligned} \tag{3.9}$$

Substituting y for z in (3.9), we get

$$\begin{aligned}&\binom{n}{1}h_1(xy,xy) + \binom{n}{2}h_2(xy,xy) + \cdots + \binom{n}{n-1}h_{n-1}(xy,xy)\\ &- \binom{n}{1}h_1(y,y) - \binom{n}{2}h_2(y,y) - \cdots - \binom{n}{n-1}h_{n-1}(y,y) \in Z(\mathcal{R}).\end{aligned}$$

This implies that

$$\binom{n}{1}D(\underbrace{xy,\dots,xy}_{(n-1)-\text{times}},\underbrace{xy}_{1-\text{times}})+\binom{n}{2}D(\underbrace{xy,xy,\dots,xy}_{(n-2)-\text{times}},\underbrace{xy}_{3-\text{times}})$$
$$+\cdots+\binom{n}{n-1}D(\underbrace{xy}_{1-\text{times}},\underbrace{xy,xy,\dots,xy}_{(n-1)-\text{times}})-\binom{n}{1}D(\underbrace{y,y,\dots,y}_{(n-1)-\text{times}},\underbrace{y}_{1-\text{times}})$$
$$-\binom{n}{2}D(\underbrace{y,y,\dots,y}_{(n-2)-\text{times}},\underbrace{y}_{2-\text{times}})-\cdots-\binom{n}{n-1}D(\underbrace{y}_{1-\text{times}},\underbrace{y,y,\dots,y}_{(n-1)-\text{times}})\in Z(\mathcal{R}).$$

Now solving the above equation, we get

$$\left(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right)D(xy,xy,\dots,xy)$$
$$-\left(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right)D(y,y,\dots,y)\in Z(\mathcal{R}).$$

This gives that

$$(2^n-2)\left(D(xy,xy,\dots,xy)-D(y,y,\dots,y)\right)\in Z(\mathcal{R}),\quad \text{for all } x,y\in L.$$

Since $\mathcal{R}$ is not characteristic (2^n-2), we find

$$D(xy,xy,\dots,xy)-D(y,y,\dots,y)\in Z(\mathcal{R}),\quad \text{for all } x,y\in L.$$

This shows that

$$\triangle(xy)-\triangle(y)\in Z(\mathcal{R}),\quad \text{for all } x,y\in L. \tag{3.10}$$

Using (3.8) and (3.10), we have $[x,y]\in Z(\mathcal{R})$ for all $x,y\in L$. Arguing in similar manner as in Theorem 3.1, we get the result. Similarly, we can prove the result if $\triangle([x,y])+\triangle(y)+[x,y]\in Z(\mathcal{R})$ for all $x,y\in L$. □

4. Examples

The following examples illustrate that $\mathcal{R}$ to be semiprime and characteristic not (2^n-2) for $n>1$ is essential in the hypothesis of the above theorem.

Example 4.1. Let $\mathcal{R}=\left\{\begin{pmatrix} p & q\\ 0 & r\end{pmatrix} \mid p,q,r\in\mathbb{Z}, \text{ ring of integers}\right\}$ and the Lie ideal $L=\left\{\begin{pmatrix} 0 & q\\ 0 & 0\end{pmatrix} \mid q\in\mathbb{Z}\right\}$. Then $Z(\mathcal{R})=\left\{\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix} \mid p\in\mathbb{Z}\right\}$. Define a map $D:\underbrace{\mathcal{R}\times\mathcal{R}\times\cdots\times\mathcal{R}}_{n-\text{times}}\to\mathcal{R}$ by

$$D\left(\begin{pmatrix} p_1 & q_1\\ 0 & r_1\end{pmatrix},\begin{pmatrix} p_2 & q_2\\ 0 & r_2\end{pmatrix},\dots,\begin{pmatrix} p_n & q_n\\ 0 & r_n\end{pmatrix}\right)=\begin{pmatrix} p_1p_2p_3\cdots p_n & 0\\ 0 & 0\end{pmatrix}.$$

Then D is symmetric n-additive with trace $\triangle$ defined by $\triangle : \mathcal{R} \to \mathcal{R}$ such that $\triangle\left(\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\right) = D\left(\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}, \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}, \dots, \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\right)$ satisfying hypothesis of the above theorems. However, $L \nsubseteq Z(\mathcal{R})$.

Example 4.2. Let $\mathcal{R} = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mid x, y, z \in \mathbb{Z}, \text{ ring of integers} \right\}$ and the Lie ideal $L = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid y \in \mathbb{Z} \right\}$. Then $Z(\mathcal{R}) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z} \right\}$. Define a map $D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n-\text{times}} \to \mathcal{R}$ by

$$D\left(\begin{pmatrix} x_1 & 0 \\ y_1 & z_1 \end{pmatrix}, \begin{pmatrix} x_2 & 0 \\ y_2 & z_2 \end{pmatrix}, \dots, \begin{pmatrix} x_n & 0 \\ y_n & z_n \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & z_1 z_2 z_3 \cdots z_n \end{pmatrix}.$$

Then D is symmetric n-additive with trace $\triangle$ defined by $\triangle : \mathcal{R} \to \mathcal{R}$ such that $\triangle\left(\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}\right) = D\left(\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, \dots, \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}\right)$ satisfying hypothesis of the above theorems. However, $L \nsubseteq Z(\mathcal{R})$.

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