Kragujevac Journal of Mathematics
Volume 45(4) (2021), Pages 531–542.

OSCILLATION CRITERIA FOR SECOND ORDER IMPULSIVE DELAY DYNAMIC EQUATIONS ON TIME SCALE

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Abstract. In this work, we study the oscillation of a kind of second order impulsive delay dynamic equations on time scale by using impulsive inequality and Riccati transformation technique. Some examples are given to illustrate our main results.

1. Introduction

Consider a class of second order impulsive nonlinear dynamic equations of the form:

$$(E)\begin{cases}[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta}+q(t)x(\sigma(t)-\delta)=0, \quad t\in\mathbb{J}_{\mathbb{T}}:=[0,\infty)\cap\mathbb{T},\ t\neq\tau_k,\ t\geq t_0,\\ x(\tau_k^+)=M_k(x(\tau_k)), \quad x^{\Delta}(\tau_k^+)=N_k(x^{\Delta}(\tau_k)), \quad k\in\mathbb{N},\\ x(t_0^+)=x_0, \quad x^{\Delta}(t_0^+)=x_0^{\Delta}, \quad t_0-\delta\leq t\leq t_0,\end{cases}$$

under the following hypotheses.

(A_1) $\gamma\geq 1$ is the quotient of odd positive integers, $\mathbb{T}$ is an unbouned above time scale with $0\in\mathbb{T}$ and $\tau_k\in\mathbb{T}$ satisfying the properties $0\leq t_0<\tau_1<\tau_2<\cdots<\tau_k$, $\lim_{k\to\infty}\tau_k=\infty$,

$$x(\tau_k^+)=\lim_{h\to 0^+}x(\tau_k+h), \quad x^{\Delta}(\tau_k^+)=\lim_{h\to 0^+}x^{\Delta}(\tau_k+h),$$

which represent the right limit of $x(t)$ at $t=\tau_k$ in the sense of time scale. If τ_k is right scattered, then $x(\tau_k^+)=x(\tau_k)$, $x^{\Delta}(\tau_k^+)=x^{\Delta}(\tau_k)$. Similarly, we can define $x(\tau_k^-)$, $x^{\Delta}(\tau_k^-)$.

(A_2) $\delta\in\mathbb{R}_+$, $\sigma(t)-\delta\in\mathbb{T}$, $r(t)>0$, $q(t)\in C_{rd}(\mathbb{T},[t_0,\infty)_{\mathbb{T}})$.

Key words and phrases. Oscillation, delay dynamic equation, impulse, time scales.
2010 *Mathematics Subject Classification.* Primary: 34A37. Secondary: 34A60, 39A12.
Received: January 29, 2019.
Accepted: Match 12, 2019.

(A_3) $M_k, N_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $M_k(0) = 0 = N_k(0)$ and there exist numbers a_k, a_k^*, b_k, b_k^* such that $a_k^* \le \frac{M_k(u)}{u} \le a_k$, $b_k^* \le \frac{N_k(u)}{u} \le b_k$, $u \neq 0$, $k \in \mathbb{N}$.

In this work, our objective is to extend the work of [15] to the second order impulsive delay dynamic equations (E). About the time scale concept and fundamentals of time scale calculus we refer the monographs [6] and [7].

Oscillation theory of impulsive differential/difference equation has brought the attention of many researchers, as it provides a more adequate mathematical model for numerous process and phenomena studied in physics, biology, engineering and to mention a few. In the literature, most of the results obtained for difference equations is the continuous analogues of differential equations and vice versa. Hence it was an immediate question to find a way for which one can unify the qualitative properties of both equations. In 1988 Stefen Hilger introduced the concept of time scales calculus, which unify the continuous and discrete calculus in his Ph.D. thesis [12]. The study of impulsive dynamic equations on time scales has been initiated by Benchora et al. [4].

In [15], Huang has considered the second order impulsive dynamic equation of the form

$$\begin{cases} [r(t)(y^{\Delta}(t))^{\gamma}]^{\Delta} + f(t, y^{\sigma}(t)) = 0, \quad t \in \mathbb{J}_{\mathbb{T}} := [0,\infty) \cap \mathbb{T},\ t \neq \tau_k,\ t \ge t_0, \\ y(\tau_k^+) = g_k(y(\tau_k)), \quad y^{\Delta}(\tau_k^+) = h_k(y^{\Delta}(\tau_k)), \quad k \in \mathbb{N}, \\ y(t_0^+) = y_0, \quad y^{\Delta}(t_0^+) = y_0^{\Delta}, \end{cases}$$

and improved the results of [13] and [14].

To the best of the author's knowledge, there is no such results for the impulsive delay dynamic equations on time scales. Hence, in this work an attempt is made to study the impulsive dynamic equations (E) and from which we can find the corresponding results for impulsive differential/difference equation. In this direction, we refer the reader to some works ([2], [13]-[19]) and the references cited there in.

AC^i=$\{x : \mathbb{J}_{\mathbb{T}} \to \mathbb{R}$ is i-times Δ-differentiable, whose ith delta derivative $x^{\Delta^{(i)}}$ is absolutely continuous$\}$, $PC = \{x : \mathbb{J}_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous at the points $\tau_k, k \in \mathbb{N}$ for which $x(\tau_k^-)$, $x(\tau_k^+)$, $x^{\Delta}(\tau_k^-)$ and $x^{\Delta}(\tau_k^+)$ exist, with $x(\tau_k^-)$=$x(\tau_k)$, $x^{\Delta}(\tau_k^-)$=$x^{\Delta}(\tau_k)\}$.

Definition 1.1. A solution of $x(t)$ of (E) is said to be regular if it is defined on some half line $[\tau_x, \infty)_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$ and $\sup\{|x(t)| : t \ge t_x\} > 0$. A regular solution $x(t)$ of (E) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $x(t) > 0$ $(x(t) < 0)$ for $t \ge t_1$.

Definition 1.2. A function $x(t) \in PC \cap AC^2(\mathbb{J}_{\mathbb{T}} \setminus \{\tau_1, \tau_2, \dots\}, \mathbb{R})$ is called a solution of (E) if:

(I) it satisfies (E) a.e. on $\mathbb{J}_{\mathbb{T}} \setminus \{\tau_k\}, k \in \mathbb{N}$;
(II) for $t = \tau_k, k \in \mathbb{N}$, $x(t)$ satisfies (E);
(III) for any $t \in [t_0 - \delta, t_0]$, $x(t) = \phi(t)$, $x(t_0^+) = x_0$, $x^{\Delta}(t_0^+) = x_0^{\Delta}$.

Definition 1.3. A nontrivial solution $x(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0 \geq 0$ such that $x(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $x(t)$ is said to be oscillatory.

For completeness in the paper, we give the time scale concept and some fundamentals of time scale calculus in Section 4.

2. Basic Lemmas

We need the time scale version of the following well known results for our use in the sequel.

Lemma 2.1 ([1]). *Let $y, f \in C_{rd}$ and $p \in \mathcal{R}$. Then $y^{\Delta}(t) \leq p(t)y(t) + f(t)$, implies that for all $t \in \mathbb{T}$*

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s.$$

Lemma 2.2 ([15]). *Assume that*

(*i*) $m \in PC \cap AC^1(\mathbb{J}_{\mathbb{T}} \setminus \{\tau_k\}, \mathbb{R})$;

(*ii*) *for $k \in \mathbb{N}$ and $t \geq t_0$, we have*

$$m^{\Delta}(t) \leq p(t)m(t) + v(t), \quad t \in \mathbb{J}_{\mathbb{T}} = [0, \infty) \cap \mathbb{T}, t \neq \tau_k,$$
$$m(\tau_k^+) \leq d_k m(\tau_k) + e_k.$$

Then the following inequality holds

$$\begin{aligned} m(t) \leq & m(t_0) \prod_{t_0<\tau_k<t} d_k e_p(t_0, t) + \int_{t_0}^{t} \prod_{s<\tau_k<t} d_k e_p(t, \sigma(s))v(s)\Delta s \\ & + \sum_{t_0<\tau_k<t} \Big(\prod_{\tau_k<\tau_j<t} d_j e_p(t, \tau_k) \Big) e_k, t \geq t_0. \end{aligned}$$

Lemma 2.3. *Suppose that (A_1)-(A_3), $a_k, b_k > 0$, $k \in \mathbb{N}$ hold. Furthermore, assume that there exists $T \geq t_0$ such that $x(t) > 0$ for $t \geq T$ and*

(A_4) $\int_T^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(s)} \prod_{T<\tau_k<s} \frac{b_k^*}{a_k} \Delta s = \infty.$

Then $x^{\Delta}(\tau_k^+) \geq 0$ and $x^{\Delta}(t) \geq 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$ and $\tau_k \geq T$.

Proof. Let $x(t)$ be an eventually positive solution of (E) for $t \geq t_0$. Without loss of generality we assume that $x(t) > 0$ and $x(t-\delta) > 0$ for $t \geq t_1 > t_0 + \delta$. From (E), we get $[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} = -q(t)f(x(t-\delta)) \leq 0$. Therefore, $r(t)(x^{\Delta}(t))^{\gamma}$ is monotonically decreasing on $[t_2, \infty)_{\mathbb{T}}, t_2 > t_1 + \delta$. Assume that $\tau_k > t_2$ for $k \in \mathbb{N}$. Consider the interval $(\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$. We assert that $x^{\Delta}(\tau_k) \geq 0$. If not, there exists $\tau_j \geq t_2$ such that $x^{\Delta}(\tau_j) < 0$ and hence $x^{\Delta}(\tau_j^+) = N_k(x^{\Delta}(\tau_k)) \leq b_k^* x^{\Delta}(\tau_k) < 0$. Let $x^{\Delta}(\tau_j^+) = -\alpha$, $\alpha > 0$. Now for $t \in (\tau_j, \tau_{j+1}]_{\mathbb{T}}$, we have $r(\tau_{j+1})(x^{\Delta}(\tau_{j+1}))^{\gamma} \leq r(\tau_j)(x^{\Delta}(\tau_j^+))^{\gamma}$, that is,

$$x^{\Delta}(\tau_{j+1}) \leq \left(\frac{r(\tau_j)}{r(\tau_{j+1})}\right)^{\frac{1}{\gamma}} x^{\Delta}(t_j^+) = -b_j^* \alpha \left(\frac{r(\tau_j)}{r(\tau_{j+1})}\right)^{\frac{1}{\gamma}} < 0.$$

If $t \in (\tau_{j+1}, \tau_{j+2}]_{\mathbb{T}}$, then

$$x^{\Delta}(\tau_{j+2}) \leq \left(\frac{r(\tau_{j+1})}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} x^{\Delta}(\tau_{j+1}^{+}) = \left(\frac{r(\tau_{j+1})}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} N_{j+1}(x^{\Delta}(\tau_{j+1}))$$
$$\leq b_{j+1}^{*} \left(\frac{r(\tau_{j+1})}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} x^{\Delta}(\tau_{j+1}),$$

that is,

$$x^{\Delta}(\tau_{j+2}) \leq -b_j^{*} b_{j+1}^{*} \alpha \left(\frac{r(\tau_j)}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} < 0.$$

Hence, by the method of induction

$$x^{\Delta}(\tau_{j+n}) \leq -b_j^{*} b_{j+1}^{*} b_{j+2}^{*} \cdots b_{j+n-1}^{*} \alpha \left(\frac{r(\tau_j)}{r(\tau_{j+n})}\right)^{\frac{1}{\gamma}}$$
$$= -\left(\frac{r(\tau_j)}{r(\tau_{j+n})}\right)^{\frac{1}{\gamma}} \left(\prod_{i=1}^{n-1} b_{j+i}^{*}\right) \alpha < 0,$$

for $t \in (\tau_{j+n-1}, \tau_{j+n}]_{\mathbb{T}}$.

Now, we consider the following impulsive dynamic inequalities

$$(E_1) \begin{cases} [r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} \leq 0, & t > \tau_j, t \neq \tau_k, k = j+1, j+2, \ldots, \\ x^{\Delta}(\tau_k^{+}) \leq b_k^{*} x^{\Delta}(\tau_k), & k = j+1, j+2, \ldots \end{cases}$$

Let $m(t) = r(t)(x^{\Delta}(t))^{\gamma}$, then (E_1) becomes

$$\begin{cases} m^{\Delta}(t) \leq 0, & t > \tau_j, t \neq \tau_k, k = j+1, j+2, \ldots, \\ m(\tau_k^{+}) \leq (b_k^{*})^{\gamma} m(\tau_k), & k = j+1, j+2, \ldots, \end{cases}$$

and, by Lemma 2.2, it follows that

$$m(t) \leq m(\tau_j^{+}) \prod_{\tau_j < \tau_k < t} (b_k^{*})^{\gamma},$$

that is,

$$x^{\Delta}(t) \leq \left(\frac{r(\tau_j)}{r(t)}\right)^{\frac{1}{\gamma}} x^{\Delta}(\tau_j^{+}) \prod_{\tau_j < \tau_k < t} b_k^{*} = -\alpha \left(\frac{r(\tau_j)}{r(t)}\right)^{\frac{1}{\gamma}} \prod_{\tau_j < \tau_k < t} b_k^{*}. \tag{2.1}$$

For $k = j+1, j+2, \ldots$, we also have $x(\tau_k^{+}) \leq a_k x(\tau_k)$. By (2.1) and since $x(\tau_k^{+}) \leq a_k x(\tau_k)$, $k = j+1, j+2, \ldots$, it follows from Lemma 2.2 that

$$x(t) \leq x(\tau_j^{+}) \prod_{\tau_j < \tau_k < t} a_k - \int_{\tau_j}^{t} \prod_{s < \tau_k < t} a_k \left[\alpha \left(\frac{r(\tau_j)}{r(t)}\right)^{\frac{1}{\gamma}} \prod_{\tau_j < \tau_k < s} b_k^{*}\right] \Delta s$$
$$\leq \prod_{\tau_j < \tau_k < t} a_k \left[x(\tau_j^{+}) - \alpha \, (r(\tau_j))^{\frac{1}{\gamma}} \int_{\tau_j}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \prod_{\tau_j < \tau_k < s} \frac{b_k^{*}}{a_k} \Delta s\right]$$

$\to -\infty$ as $t \to \infty$.

Due to (A_4), a contradiction to the fact that $x(t) > 0$ eventually. Hence, our assertation holds, that is, $x^{\Delta}(\tau_k) \geq 0$ for $\tau_k \geq T$ and hence $x^{\Delta}(t) > x^{\Delta}(\tau_k^+)$. Since $[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} \leq 0$ for any $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $\tau_k \geq T$, then

$$x^{\Delta}(t) \geq \left(\frac{r(\tau_{k+1})}{r(t)}\right)^{\frac{1}{\gamma}} x^{\Delta}(\tau_{k+1}) \geq 0, \quad t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}.$$

Therefore, $x^{\Delta}(\tau_k^+) > 0$ and $x^{\Delta}(t) > 0$ for $t \in (\tau_k, \tau_{k+i})]_{\mathbb{T}}$, $t \geq t_2$, and the lemma is proved. $\square$

Remark 2.1. If $x(t)$ is an eventually negative solution of (E). Then, using (A_1)-(A_3), it is easy to prove that $x^{\Delta}(\tau_k^+) \leq 0$ and $x^{\Delta}(t) \leq 0$, for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$ and $\tau_k \geq T \geq t_0$.

3. Sufficient Conditions for Oscillation

Theorem 3.1. *Let all conditions of Lemma* 2.3 *hold. Furthermore, assume that*

(A_5) $\int_{t_0}^{\infty} \prod_{t_0<\tau_k<s} \frac{1}{b_k^{\gamma}} q(s)\Delta s = \infty.$

Then every solution of (E) *oscillates.*

Proof. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (E). Without loss of generality, assume that $x(t) > 0$, $x(\sigma(t) - \delta) > 0$ for $t \geq t_1$. Hence, by Lemma 2.3, there exists $t_2 > t_1$ such that $x^{\Delta}(t) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$ and $\tau_k \geq t_2$. Indeed, $x^{\Delta}(t - \delta) > 0$ for $t \geq t_3 \geq t_2 + \delta$. Let

$$w(t) = \frac{r(t)(x^{\Delta}(t))^{\gamma}}{x(t-\delta)}. \tag{3.1}$$

Then $w(\tau_k^+) \geq 0$ and $w(t) \geq 0$ for $\tau_k \geq t_3$. From (3.1), for $t \neq \tau_k$ we have

$$\begin{aligned} w^{\Delta}(t) &= \frac{[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} x(t-\delta) - r(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma} x^{\Delta}(t-\delta)}{x(t-\delta)x(\sigma(t)-\delta)} \\ &\leq \frac{[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta}}{x(\sigma(t)-\delta)} - \frac{r(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma} x^{\Delta}(t-\delta)}{x(t-\delta)x(\sigma(t)-\delta)} \\ &\leq -q(t), \end{aligned}$$

that is,

$$w^{\Delta}(t) \leq -q(t), \quad t \neq \tau_k. \tag{3.2}$$

We note that

$$w(\tau_k^+) = \frac{r(\tau_k^+)(x^{\Delta}(\tau_k^+))^{\gamma}}{x(\tau_k^+ - \delta)} \leq \frac{b_k^{\gamma} r(\tau_k)(x^{\Delta}(\tau_k))^{\gamma}}{x(\tau_k - \delta)} = b_k^{\gamma} w(\tau_k).$$

Now, we have the following impulsive dynamics inequalities

$$\begin{aligned} w^{\Delta}(t) &\leq -\, q(t), \quad t \neq \tau_k \\ w(\tau_k^+) &\leq b_k^{\gamma} w(\tau_k), \quad k \in \mathbb{N}, \end{aligned}$$

and, by Lemma 2.2, it follows that

$$\begin{aligned}
w(t) &\leq w(t_3) \prod_{t_3<\tau_k<t} b_k^{\gamma} - \int_{t_3}^{t} \prod_{s<\tau_k<t} b_k^{\gamma} q(s)\Delta s \\
&\leq \prod_{t_3<\tau_k<t} b_k^{\gamma} \left[w(t_3) - \int_{t_3}^{t} \prod_{t_3<\tau_k<s} \frac{1}{b_k^{\gamma}} q(s)\Delta s \right] \\
&\to -\infty \text{ as } t \to \infty.
\end{aligned}$$

Due to (A_5), a contradiction to the fact that $w(t) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$. This completes the proof of the theorem. □

Theorem 3.2. *Let all conditions of Lemma* 2.3 *hold. Furthermore, assume that* $\tau_{k+1} - \tau_k = \delta$ *and*

(A_6) $\int_{t_0}^{\infty} \prod_{t_0<\tau_k<s} \frac{1}{d_k} q(s)\Delta s = \infty$,

where

$$d_k = \begin{cases} b_1^{\gamma}, & \text{if } k = 1, \\ d\dfrac{b_k^{\gamma}}{a_{k-1}^{*}}, & \text{if } k = 2, 3, \ldots, \end{cases}$$

hold. Then every solution of (E) *oscillates.*

Proof. Proceed as in the proof Theorem 3.1 to obtain that $x^{\Delta}(t) > 0$ and $x^{\Delta}(\tau_k^+) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$, $t \geq t_2$. Indeed, $x^{\Delta}(t - \delta) > 0$ for $t \geq t_3 \geq t_2 + \delta$. Define $w(t)$ as in (3.1), we get (3.2) holds for $\tau_k \geq t_3$ and $t \neq \tau_k$. Now, if $k = 1$ we have

$$w(\tau_1^+) = \frac{r(\tau_1^+)(x^{\Delta}(\tau_1^+))^{\gamma}}{x(\tau_1^+ - \delta)} \leq \frac{b_1^{\gamma} r(\tau_1)(x^{\Delta}(\tau_1))^{\gamma}}{x(\tau_1 - \delta)} = d_1 w(\tau_1).$$

If $k = 2, 3, \ldots$, then

$$\begin{aligned}
w(\tau_k^+) &= \frac{r(\tau_k^+)(x^{\Delta}(\tau_k^+))^{\gamma}}{x(\tau_k^+ - \delta)} \leq \frac{b_k^{\gamma} r(\tau_k)(x^{\Delta}(\tau_k))^{\gamma}}{x(\tau_{k-1}^+ - \delta)} \leq \frac{b_k^{\gamma} r(\tau_k)(x^{\Delta}(\tau_k))^{\gamma}}{a_{k-1}^{*} x(\tau_{k-1} - \delta)} \\
&\leq \frac{b_k^{\gamma} r(\tau_k)(x^{\Delta}(\tau_k))^{\gamma}}{a_{k-1}^{*} x(\tau_k - \delta)} = d_k w(\tau_k).
\end{aligned}$$

Consider the following impulsive dynamic inequality

$$\begin{cases} w^{\Delta}(t) \leq -q(t), & t \neq \tau_k, t \geq t_3 \\ w(\tau_k^+) \leq d_k w(\tau_k), & k \in \mathbb{N}. \end{cases}$$

Therefore, by Lemma 2.2, we get

$$w(t) \leq w(t_3) \prod_{t_3<\tau_k<t} d_k - \int_{t_3}^{t} \prod_{u<\tau_k<t} d_k q(u)\Delta u.$$

Then proceeding as in the proof of Theorem 3.1 and using (A_6), we get a contradiction to the fact that $w(t) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$. This completes the proof of the theorem. □

Corollary 3.1. *Let all conditions of Lemma* 2.3 *hold. Assume that there exists a positive integer* k_0 *such that* $a_k^* \geq 1$*,* $b_k \leq 1$ *for* $k \geq k_0$*. Furthermore, assume that*

(A_7) $\int_{t_0}^{\infty} q(s)\Delta s = \infty$

holds, then every solution of (E) *oscillates.*

Proof. Without loss of generality, we assume that $k_0 = 1$. Since $b_k \leq 1$, then $\frac{1}{b_k^\gamma} \geq 1$. Therefore,

$$\int_{t_0}^{t} \prod_{t_0 \leq \tau_k < s} \frac{1}{b_k^\gamma} q(s)\Delta s \geq \int_{t_0}^{t} q(s)\Delta s.$$

Letting $t \to \infty$ and in view of Theorem 3.1, We get every solution of (E) is oscillatory. This completes the proof. ☐

Corollary 3.2. *Let all conditions of Lemma* 2.3 *hold. Assume that there exists a positive integer* k_0 *and a positive constant* α *such that* $a_k^* \geq 1$ *and* $\frac{1}{b_k} \geq \left(\frac{\tau_{k+1}}{\tau_k}\right)^\alpha$ *for* $k \geq k_0$*. Furthermore, assume that*

(A_8) $\int_{t_0}^{\infty} s^\alpha q(s)\Delta s = \infty$

holds, then every solution of (E) *oscillates.*

Proof. Without loss of generality, we assume that $k_0 = 1$. Now

$$\begin{aligned}\int_{t_0}^{t} \prod_{t_0 < \tau_k < s} \frac{1}{b_k^\gamma} q(s)\Delta s &= \sum_{i=1}^{n} \prod_{t_0 < \tau_k < \tau_{i+1}} \frac{1}{b_k^\gamma} \int_{\tau_i}^{\tau_{i+1}} q(s)\Delta s \\ &\geq \frac{1}{\tau_1^\alpha} \sum_{i=1}^{n} \tau_{i+1}^\alpha \int_{\tau_i}^{\tau_{i+1}} q(s)\Delta s \\ &\geq \frac{1}{\tau_1^\alpha} \sum_{i=1}^{n} \int_{\tau_i}^{\tau_{i+1}} s^\alpha q(s)\Delta s \\ &= \frac{1}{\tau_1^\alpha} \int_{\tau_1}^{\tau_{n+1}} s^\alpha q(s)\Delta s.\end{aligned}$$

Letting $t \to \infty$ and in view of Theorem 3.1, we get every solution of (E) is oscillatory. This completes the proof. ☐

Corollary 3.3. *Let all conditions of Lemma* 2.3 *hold. Assume that there exists a positive integer* k_0 *and a positive constant* α *such that* $a_k^* \geq 1$ *and* $\frac{1}{d_k} \geq \left(\frac{\tau_{k+1}}{\tau_k}\right)^\alpha$ *for* $k \geq k_0$*. If* (A_8) *hold, then every solution of* (E) *oscillates.*

Proof. The proof of the corollary can be be follows from Corollary 3.2 and Theorem 3.2. Hence, details are omitted. ☐

Next, we present some new oscillation criteria for (E), by using an integral averaging condition of Kamenev type.

Theorem 3.3. *Let all the conditions of Lemma* 2.3 *and* $b_k \geq 1$ *hold. Furthermore, assume that*

(A_9) $\limsup_{k\to\infty} \frac{1}{t^m}\int_{t_0}^{\tau_{k+1}}(t-s)^m q(s)\Delta s = \infty$,
then every solution of (E) *oscillates.*

Proof. Proceeding as in the proof of Theorem 3.1, we get

$$w^{\Delta}(t) \le -q(t), \quad \text{for } t \ne \tau_k.$$

Multiplying $(t-s)^m$ to both side of the preceding inequality and integrating from τ_k to τ_{k+1}, we get

$$\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)ds \le -\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s.$$

Indeed,

$$\begin{aligned}&\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)\Delta s\\ =&(t-s)^m u(s)|_{\tau_k}^{\tau_{k+1}} - \int_{\tau_k}^{\tau_{k+1}}((t-s)^m)^{\Delta_s} w(s)\Delta s\\ =&\int_{\tau_k}^{\tau_{k+1}} m(t-s)^{m-1}w(s)\Delta s + (t-\tau_{k+1})^m w(\tau_{k+1}) - (t-\tau_k)^m w(\tau_k^+),\end{aligned}$$

because $((t-s)^m)^{\Delta_s} = -m(t-s)^{m-1}$. As a result,

$$\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)\Delta s \ge -(t-\tau_k)^m w(\tau_k^+).$$

Therefore,

$$\begin{aligned}\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s &\le -\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)\Delta s\\ &\le (t-\tau_k)^m w(\tau_k^+)\\ &\le b_k(t-\tau_k)^m w(\tau_k),\end{aligned}$$

that is,

$$\frac{1}{t^m}\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s \le b_k\left(\frac{t-\tau_k}{t}\right)^m w(\tau_k),$$

and hence,

$$\limsup_{k\to\infty}\frac{1}{t^m}\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s < \infty,$$

a contradiction to (A_9). This completes the proof of the theorem. □

4. Appendix: Time Scale Preliminaries

We will briefly recall some basic definitions and facts from the time scale calculus that we will use in the sequel. For more details see [2,3,19]. On any time scale $\mathcal{T}$, we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s\in\mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s\in\mathbb{T} : Vs < t\},$$

where $\inf \phi = \sup \mathbb{T}$, $\sup \phi = \inf \mathbb{T}$, and ϕ denotes the empty set. A nonmaximal element $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. A nonminimal element $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and left-scattered if $\rho(t) > t$. The graininess μ of the time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A mapping $f : \mathbb{T} \to \mathbb{X}$ is said to be differentiable at $t \in \mathbb{T}$, if there exists $f^{\Delta}(t) \in \mathbb{X}$ such that for any $\epsilon > 0$, there exists a neighborhood U of t satisfying

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and forward jump operator σ are related by the formula

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Let f be a differentiable function on $[a, b]_{\mathbb{T}}$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]_{\mathbb{T}}$ if $f^{\Delta} > t$, $f^{\Delta} < t$, $f^{\Delta} \geq t$, $f^{\Delta} \leq t$ for all $t \in [a, b)_{\mathbb{T}}$, respectively. We will make use of the following product fg and quotient $\frac{f}{g}$ rules for the derivative of two differentiable functions f and g

$$\begin{aligned}(fg)^{\Delta} =& f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \\ \left(\frac{f}{g}\right)^{\Delta} =& \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}},\end{aligned}$$

where $f^{\sigma} = fo\sigma$, $gg^{\sigma} \neq 0$. The integration by parts formula reads

$$\int_a^b f^{\Delta}(t) g(t) = f(t) g(t)|_a^b - \int_a^b f^{\sigma}(t) g^{\Delta}(t) \Delta t.$$

Chain Rule. Assume $g : \mathbb{T} \to \mathbb{R}$ is $\Delta-$ differentiable on $\mathbb{T}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $fog : \mathbb{T} \to \mathbb{R}$ is $\Delta-$ differentiable and satisfies

$$(fog)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t) g^{\Delta}(t)) dh \right\} g^{\Delta}(t).$$

Regressive. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if for all $t \in \mathbb{T}$, $1 + \mu(t) p(t) \neq 0$.

The set of all function $p : \mathbb{T} \to \mathbb{R}$, which are regressive and rd-continuous will be denoted by $\mathcal{R}$. We define the set $\mathcal{R}^{+}$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^{+} = \{p \in \mathcal{R} : 1 + \mu(t) p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Exponential Function. If $p \in \mathcal{R}$, then general exponential function e_p on $\mathbb{T}$ is defined as

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z) p(z)) \Delta z \right),$$

with $\mu(z) \neq 0$ and $s, t \in \mathbb{T}$.

5. Examples

Example 5.1. Consider the impulsive dynamic equation

$$(5.1)\qquad \begin{cases} x^{\Delta\Delta}(t) + \dfrac{1}{t}x(t-\dfrac{1}{2}) = 0, \quad t > \dfrac{1}{2}, t \neq \tau_k, \\ x(\tau_k^+) = \dfrac{k+1}{k}x(\tau_k), \quad x^{\Delta}(\tau_k^+) = x^{\Delta}(\tau_k), \quad k \in \mathbb{N}, \end{cases}$$

where $\gamma = 1$, $r(t) = 1$, $\delta = \frac{1}{2}$, $q(t) = \frac{1}{t} \geq 0$, $a_k^* = a_k = \frac{k+1}{k}$, $b_k^* = b_k = 1$, $\tau_k = 3k$, $\tau_{k+1} - \tau_k = 3 > 2$, $k \in \mathbb{N}$. Then, from (A_4)

$$\begin{aligned} &\int_T^{\infty} \prod_{T<\tau_k<s} \frac{b_k^*}{a_k}\, \Delta s \\ =&\int_2^{\infty} \prod_{2<\tau_k<s} \frac{k}{k+1} ds \\ =&\int_2^{\tau_1} \prod_{2<\tau_k<s} \frac{k}{k+1}\Delta s + \int_{\tau_1^+}^{\tau_2} \prod_{2<\tau_k<s} \frac{k}{k+1}\Delta s + \int_{\tau_2^+}^{\tau_3} \prod_{2<\tau_k<s} \frac{k}{k+1}\Delta s + \cdots \\ =&\frac{1}{2}(\tau_1 - 2) + \frac{1}{2} \times \frac{2}{3}(\tau_2 - \tau_1) + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}(\tau_3 - \tau_2) + \cdots \\ =&\frac{1}{2} \times 2 + \frac{1}{3} \times 3 + \frac{1}{4} \times 3 + \frac{1}{5} \times 3 + \cdots \\ \geq&\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{i=2}^{\infty} \frac{1}{i} = \infty, \end{aligned}$$

and from (A_5)

$$\int_2^{\infty} \prod_{\frac{1}{2}<\tau_k<s} \frac{1}{b_k^{\gamma}} \frac{1}{s}\Delta s = \int_2^{\infty} \frac{1}{s}\Delta s \to \infty.$$

Therefore, all conditions of Theorem 3.1 are satisfied and hence (5.1) has an oscillatory solution.

Example 5.2. Consider the impulsive dynamic equation

$$(5.2)\qquad \begin{cases} x^{\Delta\Delta}(t) + \dfrac{1}{t^3}x(t-1) = 0, \quad t > 1, t \neq \tau_k, \\ x(\tau_k^+) = \dfrac{k-1}{k}x(\tau_k), \quad k \in \mathbb{N}, k > k_0, \\ x^{\Delta}(\tau_k^+) = \dfrac{1}{k}x^{\Delta}(\tau_k), \quad k \in \mathbb{N}, k > k_0, \end{cases}$$

where $\gamma = 1$, $\delta = 1$, $r(t) = 1$, $q(t) = \frac{1}{t^3} \geq 0$, $a_k^* = a_k = \frac{k-1}{k}$, $b_k^* = b_k = \frac{1}{k}$, $\tau_k = 3k$, $\tau_{k+1} - \tau_k = 3 > 1$, $k \in \mathbb{N}$, $k > k_0 = 1$. Clearly, from (A_4) we have

$$\int_T^{\infty} \prod_{T<\tau_k<s} \frac{b_k^*}{a_k}\, \Delta s$$

$$
\begin{aligned}
&= \int_1^\infty \prod_{1<\tau_k<s} \frac{1}{k-1} \Delta s \\
&= \int_1^{\tau_2} \prod_{1<\tau_k<s} \frac{1}{k-1} \Delta s + \int_{\tau_2^+}^{\tau_3} \prod_{1<\tau_k<s} \frac{1}{k-1} \Delta s + \int_{\tau_3^+}^{\tau_4} \prod_{1<\tau_k<s} \frac{1}{k-1} \Delta s + \cdots \\
&= (\tau_2 - 1) + \frac{1}{2} \times (\tau_3 - \tau_2) + \frac{1}{2} \times \frac{1}{3} \times (\tau_4 - \tau_3) + \cdots \\
&= 2 + \frac{1}{2} \times 2^2 + \frac{1}{2} \times \frac{1}{3} \times 2^3 + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times 2^4 + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = 1 + \sum_{i=2}^{\infty} \frac{1}{i} = \infty.
\end{aligned}
$$

Let $\alpha = 1$. Then

$$\frac{1}{b_k} = k \geq \left(\frac{\tau_{k+1}}{\tau_k}\right)^\alpha = \frac{k+1}{k}.$$

Also, from (A_8) we have

$$\int_1^\infty s^\alpha q(s) \Delta s = \int_1^\infty s^3 \frac{1}{s^3} \Delta s = \int_1^\infty \Delta s = \infty.$$

All conditions of Corollary 3.2 are satisfied for (5.2) and hence, (5.2) has an oscillatory solution.

Acknowledgements. This work is supported by Rajiv Gandhi National fellowship(UGC), New Delhi, India, through the Letter No. F1-17.1/2017-18 RGNF-2017-18-SC-ORI-35849, dated 11th July, 2017.

We are very thankful to the anonymous reviewers and editor for their constructive comments and suggestions which help us to improve the manuscript

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