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A NOTE ON A NEW SHARP RESULT IN SPACES OF PLURIHARMONIC FUNCTIONS AND RELATED PROBLEMS

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Abstract. We provide a new sharp result in Bergman spaces of pluriharmonic functions related to the trace operator, extending previously known assertions. Related new estimates for other pluriharmonic spaces in product domains will be also discussed.

1. Introduction

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, \dots, z_n) \colon z_k \in \mathbb{C}, 1 \le k \le n\}$ be the n-dimensional space of complex coordinates. We denote the unit polydisk by

$$U^n = \{z \in \mathbb{C}^n \colon \ |z_k| < 1, 1 \le k \le n\}$$

and the distinguished boundary of U^n by

$$T^n = \{z \in \mathbb{C}^n \colon \ |z_k| = 1, 1 \le k \le n\}.$$

We define Lusin cone in a usual manner as follows

$$\Gamma_\alpha(\xi) = \{z \in U \colon \ |1 - z\xi| < \alpha(1 - |z|)\}, \quad \text{where } \alpha > 1, \xi \in T.$$

We use m_{2_n} to denote the volume measure on U^n and m_n to denote the normalized Lebesgue measure on T^n. Let $H(U^n)$ be the space of all holomorphic functions on U^n. When $n = 1$, we simply denote U^1 by U, T^1 by T, m_{2_n} by m_2, m_n by m. We refer to [4, 5, 13] for further details.

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The Hardy spaces, denoted by $H^p(U^n)$, $0<p<+\infty$, are defined by

$$H^p(U^n)=\left\{f\in H(U^n)\colon \sup_{0<r<1} M_p(f,r)<+\infty\right\},$$

where

$$M_p^p(f,r)=\int_{T^n}|f(r\xi)|^p\,dm_n(\xi),\quad M_\infty(f,r)=\max_{\xi\in T^n}|f(r\xi)|,$$

$r\in(0,1)$, $f\in H(U^n)$.

As usual, we denote by $\vec{\alpha}$ the vector $(\alpha_1,\dots,\alpha_n)$.

For $\alpha_j>-1$, $j=1,\dots,n$, $0<p<+\infty$, recall that the weighted Bergman space $A^p_{\vec{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk satisfying the condition (see [4, 5, 13])

$$\|f\|^p_{A^p_{\vec{\alpha}}}=\int_{U^n}|f(z)|^p\prod_{i=1}^n(1-|z_i|^2)^{\alpha_i}dm_{2_n}(z)<+\infty.$$

Let

$$\mathbb{Z}^n_+=\{(k_1,\dots,k_n)\colon k_j\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}\},$$
$$\mathbb{Z}^n_-=\{(k_1,\dots,k_n)\colon k_j\in\mathbb{Z}_-, j=1,\dots,n\}.$$

If u is n-harmonic (harmonic by each variable), then as usual

$$u(r_1e^{i\varphi_1},\dots,r_ne^{i\varphi_n})=\sum_{k_1,\dots,k_n=-\infty}^{+\infty}C_{k_1,\dots,k_n}\prod_{j=1}^n r_j^{|k_j|}e^{ik_j\varphi_j}.$$

We define a fractional derivative of order α of an n-harmonic function in a usual way as follows

$$\mathcal{D}^\alpha u(\vec{r}e^{i\vec{\varphi}})=\sum_{k_1,\dots,k_n=-\infty}^{+\infty}C_{k_1,\dots,k_n}\frac{\Gamma(\alpha+|k|+1)}{\Gamma(\alpha+1)\Gamma(|k|+1)}\prod_{j=1}^n r_j^{|k_j|}e^{ik_j\varphi_j},$$

where

$$\frac{\Gamma(\alpha+|k|+1)}{\Gamma(\alpha+1)\Gamma(|k|+1)}=\prod_{j=1}^n\frac{\Gamma(\alpha_j+|k_j|+1)}{\Gamma(\alpha_j+1)\Gamma(|k_j|+1)},\quad \alpha_j>-1.$$

Let further

$$h^p(\vec{\alpha})=\Big\{u \text{ is } n\text{-harmonic:} \int_{U^n}\prod_{k=1}^n(1-|z_k|)^{\alpha_k}|u(z_1,\dots,z_n)|^p\,dm_{2_n}(z)<+\infty,$$
$$0<p<+\infty, \alpha_k>-1, k=1,\dots,n\Big\}.$$

Note that it is easy to check that $\mathcal{D}^\alpha u$ is n-harmonic if u is n-harmonic.

We will call a function u pluriharmonic if $u=\mathrm{Re}(f)$, $f\in H(U^n)$, and we denote

$$\tilde{h}^p(\vec{\alpha})=\{u \text{ is pluriharmonic: } \|u\|_{h^p(\vec{\alpha})}<+\infty\}.$$

If $1\le p<+\infty$, $j=1,\dots,n$, both $h^p(\alpha)$ and $\tilde{h}^p(\vec{\alpha})$ are Banach spaces, for $0<p\le 1$, $j=1,\dots,n$, $h^p(\alpha)$ and $\tilde{h}^p(\vec{\alpha})$ classes are quasinormed spaces.

Some interesting results for pluriharmonic functions can be found in [8–13]. Some results on related plurisubharmonic functions can be seen in [6, 7].

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The notation $A \asymp B$ means that there is a positive constant C, such that $\frac{B}{C} \le A \le CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

2. Main Results

Let us remind the main definition.

Definition 2.1 (see [1–5]). Let $\mathcal{X} \subset H(U), \mathcal{Y} \subset H(U^n)$ be subspaces of $H(U)$ and $H(U^n)$. We say that the diagonal of $\mathcal{Y}$ coincides with $\mathcal{X}$ if for any function f, $f \in \mathcal{Y}$, $f(z, \dots, z) \in \mathcal{X}$, and the reverse is also true: for every function g from $\mathcal{X}$ there exists an expansion $f(z_1, \dots, z_n)$, $f \in \mathcal{Y}$ such that $f(z, \dots, z) = g(z)$.

Note that when $\operatorname{diag}(\mathcal{Y}) = \mathcal{X}$, then $\|f\|_{\mathcal{X}} \asymp \inf_{\Phi} \|\Phi(f)\|_{\mathcal{Y}}$, where $\Phi(f)$ is an arbitrary analytic extension of f from the diagonal of the polydisk to the polydisk. The study of the diagonal map and its applications was first suggested by W. Rudin in [1–5]. Later several papers appeared in which the complete solutions were given for classical holomorphic spaces such as the Hardy, Bergman classes see [1–5] and references there. Recently, a complete answer was given for so-called mixed norm spaces in [1–5]. For many other classes, the answer is unknown. The goal of this note is to add various new results in this research area.

Theorems on the diagonal map have numerous applications in the theory of holomorphic functions. Analogues of the diagonal map problem so-called trace problems in various functional spaces in $\mathbb{R}^n$ are well-known (see, for example, [1–5]).

Note that, very similarly, the definition of the diagonal map (Definition 2.1) can be extended to spaces of pluriharmonic functions in the unit polydisk. In this case, the restriction of such a function to the diagonal $(z, \dots, z)$ is an n-harmonic function. Moreover to the following sharp result is valid.

Theorem 2.1. *Let* $0 < p < +\infty$, $\alpha_j > -1$, $j = 1, \dots, n$. *Then,*

$$\operatorname{diag}(\widetilde{h}^p(\vec{\alpha})) = h^p\Big(\sum_{j=1}^{n} \alpha_j + 2n - 2\Big), \quad n \ge 1.$$

Remark 2.1. Since $\widetilde{h}^p(\vec{\alpha})$ classes contain holomorphic Bergman spaces in polydisk, Theorem 1 can be considered as an extension of the theorems on the diagonal map in $A^p_{\vec{\alpha}}(U^n)$ - Bergman classes in the polydisk (see [1–5]).

Proof. The proof of Theorem 2.1 almost repeats the corresponding proof for the classical Bergman classes in the polydisk (see, for example, [1–5]). We add few remarks that are needed. For every harmonic function v such that

$$\int_U |v(z)|^p (1-|z|)^\alpha dm_2(z) < +\infty, \quad \alpha > 1, 0 < p < +\infty,$$

where

$$v(z) = C(\beta)\int_U v(w)(1-|w|^2)^{n(\beta+2)-2}\mathrm{Re}\Big(\frac{1}{1-\langle \bar{w}, z\rangle}\Big)^{n(\beta+2)} dm_2(w),$$

we put

$$u(z_1,\dots,z_n) = C(\beta)\int_U v(w)(1-|w|^2)^{n(\beta+2)-2}\mathrm{Re}\left(\frac{1}{\prod_{k=1}^n (1-\langle z_k,\bar{w}\rangle)^{\beta+2}}\right) dm_2(w).$$

Note that $u(z,\dots,z) = v(z)$, $z \in U^n$, and u is a pluriharmonic function. Indeed to prove the last assertion we have

$$\begin{aligned}
v(w) = \sum_{k=-\infty}^{+\infty} C_k \rho^{|k|} e^{ik\theta}, \quad w = \rho e^{i\theta}, z = re^{i\varphi}, \\
u(z_1,\dots,z_n) =& C(\beta)\int_0^1\int_{-\pi}^{\pi}\sum_{k=-\infty}^{+\infty} C_k\rho^{|k|}e^{ik\theta}(1-\rho^2)^{n(\beta+2)-2} \\
&\times \sum_{(k_1,\dots,k_n)\in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n} \frac{\Gamma(\beta+|k|+2)}{\Gamma(\beta+2)\Gamma(|k|+1)} r_1^{|k_1|}\cdots r_n^{|k_n|} \\
&\times \prod_{j=1}^n e^{ik_j\varphi_j}\rho^{|k_j|}e^{-ik_j\theta}\rho\, d\rho\, d\theta \\
=& C(\beta)\sum_{(k_1,\dots,k_n)\in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n} C_k \frac{\Gamma(\beta+|k|+2)}{\Gamma(\beta+2)\Gamma(|k|+1)}\prod_{j=1}^n r_j^{|k_j|}e^{ik_j\varphi_j} \\
&\times \int_0^1 (1-\rho^2)^{n(\beta+2)-2}\rho^{2\left(\sum_{j=1}^n |k_j|\right)+1} d\rho \\
=& \sum_{(k_1,\dots,k_n)\in\mathbb{Z}_+^n\cup\mathbb{Z}_-^n} C_{k_1,\dots,k_n} r_1^{|k_1|}\cdots r_n^{|k_n|} e^{ik_1\varphi_1}\cdots e^{ik_n\varphi_n},
\end{aligned}$$

hence u is pluriharmonic (see, for example, [1–5]). □

The rest is a repetition of the arguments in the proof of the well-known theorem on traces in classical analytic Bergman spaces in the unit polydisk. We refer the reader to [1–5] for this

$$\int_T \sup_{z_1\in\Gamma_t(\xi)} \cdots \sup_{z_n\in\Gamma_t(\xi)} |f(z_1,\dots,z_n)|^p\, d\xi,$$

$$\int_T \int_{\Gamma_r(\xi)} \cdots \int_{\Gamma_r(\xi)} |f(w)|^q \prod_{k=1}^n (1-|w_k|)^{\alpha_k} dm_2(w_1)\dots dm_2(w_2)dm(\xi),$$

$$\int_0^1 \left(\int_{|z_1|<r} \cdots \int_{|z_n|<r} |f(z)|^p \cdot \prod_{j=1}^n (1-|z_j|)^{\alpha_j} dm_{2_n}(z) \right)^{\frac{q}{p}} (1-r)^\beta dr$$

(we consider only $q = p$ case for this quazinorm below).

The proofs of these assertions, as in the case of Bergman spaces which were considered above, are the similar to the proofs for analytic function spaces cases provided earlier in [3].

We denote last two spaces indicated in the polydisk by $\widetilde{M}^{p,\vec{\alpha}}$ and $\widetilde{N}^p_{\vec{\alpha},\beta}$ of pluriharmonic functions.

Theorem 2.2. 1) *We have*

$$h^p_{|\alpha|+2n-1}(U) \subset \operatorname{diag}(\widetilde{M}^{p,\vec{\alpha}})(U^n),$$

where $|\alpha| = \sum_{k=1}^n \alpha_k$, $0 < p < +\infty$, $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_j > -1$, $j = 1,\dots,n$.

2) *We have*

$$h^p_{\beta+|\alpha|+2n-1}(U) \subset \operatorname{diag}(\widetilde{N}^p_{\vec{\alpha},\beta})(U^n),$$

where $0 < p < +\infty$, $\alpha_j > -1$, $j = 1,\dots,n$, $\beta > -1$.

Related problems on the diagonal can be considered for Bergman-Sobolev type function classes of n-harmonic functions, that is, for spaces of n-harmonic functions with finite quazinorms $\left\|\mathcal{D}^\beta f\right\|_{A^p_\alpha}$ with some restrictions on the parameters p, α and β.

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