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A GENERALIZED CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $\mathcal{H}_\alpha^\phi(\beta)$ denote the class of functions f, analytic in the open unit disk $\mathbb{E}$ which satisfy the condition

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>\beta, \quad z\in\mathbb{E},$$

where $\alpha,\ \beta$ are pre-assigned real numbers and $\phi(z)$ is a starlike function. The special cases of the class $\mathcal{H}_\alpha^\phi(\beta)$ have been studied in literature by different authors. In 2007, Singh et al. [5] studied the class $\mathcal{H}_\alpha^z(\beta)$ and they established that functions in $\mathcal{H}_\alpha^z(\beta)$ are univalent for all real numbers $\alpha,\ \beta$ satisfying the condition $\alpha\le\beta<1$ and the result is sharp in the sense that constant β cannot be replaced by a real number smaller than α. Singh et al. [7] in 2005, proved that for $0<\alpha<1$ functions in class $\mathcal{H}_\alpha^z(\alpha)$ are univalent. In 1975, Al-Amiri and Reade [2] showed that functions in class $\mathcal{H}_\alpha^z(0)$ are univalent for all $\alpha\le 0$ and also for $\alpha=1$ in $\mathbb{E}$. In the present paper, we prove that members of the class $\mathcal{H}_\alpha^\phi(\beta)$ are close-to-convex and hence univalent for real numbers $\alpha,\ \beta$ and for a starlike function ϕ satisfying the condition $\beta+\alpha-1<\alpha\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right)\le\beta<1$.

1. Introduction and Preliminary

Let $\mathcal{A}$ be the class of functions f, analytic in the open unit disk $\mathbb{E}=\{z:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the classes of starlike and convex function respectively analytically defined as follows:

$$\mathcal{S}^*=\left\{f\in\mathcal{A}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E}\right\},$$

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and

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E} \right\}.$$

It is well-known that

$$f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}^*. \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number α, $-\pi/2 < \alpha < \pi/2$, and a convex function g (not necessarily normalized) such that

$$\text{Re}\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{E}.$$

In view of the relation (1.1), the above definition takes the following form in case g is normalized. A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number α, $-\pi/2 < \alpha < \pi/2$, and a starlike function ϕ such that

$$\text{Re}\left(e^{i\alpha}\frac{f'(z)}{\phi(z)}\right) > 0, \quad z \in \mathbb{E}.$$

It is well known that every close-to-convex function is univalent. In 1934/35, Noshiro [4] and Warchawski [8] independently obtained a simple but elegant criterion for univalence of analytic functions. They proved that if an analytic function f satisfies $\text{Re}f'(z) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.

For pre-assigned real numbers α, β and $\phi \in \mathcal{S}^*$, the class $\mathcal{H}_\alpha^\phi(\beta)$ is defined as the class of functions $f \in \mathcal{A}$ as follows:

$$\text{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathbb{E}.$$

The following special cases of the class $\mathcal{H}_\alpha^\phi(\beta)$ have been studied in literature by different authors. In fact, the class $\mathcal{H}_\alpha^z(0)$ was first studied in 1975 by Al-Amiri and Reade [2]. They proved that for $\alpha \leq 0$, each function in $\mathcal{H}_\alpha^z(0)$ satisfies $\text{Re}(f'(z)) > 0$ in $\mathbb{E}$ and hence univalent in $\mathbb{E}$. They left the problem of univalence open for $\alpha > 0$ (except for $\alpha = 1$, where f is convex, obviously). Ahuja and Silverman [1] observed that the convex function $f(z) = z/(1-z)$ is not in $\mathcal{H}_\alpha^z(0)$ for any real α, $\alpha \neq 1$. Further this problem pursued by Singh et al. [7] and they proved that for $0 < \alpha < 1$, the class $\mathcal{H}_\alpha^z(\alpha)$ consisting univalent functions. In 2007, Singh et al. [5] studied the class $\mathcal{H}_\alpha^z(\beta)$. They proved that if $f \in \mathcal{H}_\alpha^z(\beta)$, then $\text{Re}(f'(z)) > 0$ in $\mathbb{E}$ for all real numbers α, β satisfying $\alpha \leq \beta < 1$ and the result is best possible one in the sense that β cannot be replaced by a real number smaller than α. Their result contains the previous result of Singh et al. [7] and improves the result of Al-Amiri and Reade [2].

In the present paper, we study a more general class $\mathcal{H}_\alpha^\phi(\beta)$ and establish that the functions in $\mathcal{H}_\alpha^\phi(\beta)$ are close-to-convex and consequently univalent subject to the condition

$$\beta + \alpha - 1 < \alpha\text{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) \leq \beta < 1.$$

where α, β are pre-assigned real numbers and ϕ is a starlike function. We claim that our results generalize the previous known results in this direction.

To prove our result, we shall need the following lemma of Miller [3].

Lemma 1.1. *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$, where $\mathbb{C}$ is the complex plane and let $\Phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ ($u_1,\ u_2,\ v_1,\ v_2$ are reals), let Φ satisfies the following conditions:*

(i) $\Phi(u, v)$ *is continuous in* $\mathbb{D}$*;*
(ii) $(1, 0) \in \mathbb{D}$ *and* $\mathrm{Re}(\Phi(1, 0)) > 0$ *and*
(iii) $\mathrm{Re}\ \Phi(iu_2, v_1) \leq 0$ *for all* $((iu_2, v_1) \in \mathbb{D}$ *such that* $v_1 \leq -(1 + u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in the open unit disk $\mathbb{E}$, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If

$$\mathrm{Re}(\Phi(p(z), zp'(z))) > 0, \quad z \in \mathbb{E},$$

then $\mathrm{Re}(p(z)) > 0$*,* $z \in \mathbb{E}$*.*

2. Univalence of Functions in $\mathcal{H}_\alpha^\phi(\beta)$

Theorem 2.1. *Let ϕ be a starlike function and let $\alpha,\ \beta$ be real numbers such that*

$$\beta + \alpha - 1 < \alpha \mathrm{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) \leq \beta < 1. \tag{2.1}$$

If $f \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathbb{E}, \tag{2.2}$$

then $\mathrm{Re}\left(\dfrac{zf'(z)}{\phi(z)}\right) > 0$ *in $\mathbb{E}$. So f is close-to-convex and hence univalent in $\mathbb{E}$.*

Proof. Write $p(z) = \frac{zf'(z)}{\phi(z)}$, where p is analytic in $\mathbb{E}$ such that $p(0) = 1$ and ϕ is a starlike in $\mathbb{E}$. Then

$$(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = (1-\alpha)p(z) + \alpha\left(\frac{zp'(z)}{p(z)} + \frac{z\phi'(z)}{\phi(z)}\right).$$

Thus, condition (2.2) is equivalent to

$$\mathrm{Re}\left(\frac{1-\alpha}{1-\beta}p(z) + \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha\frac{z\phi'(z)}{\phi(z)} - \beta}{1-\beta}\right) > 0, \quad z \in \mathbb{E}. \tag{2.3}$$

Let $\mathbb{D} = \mathbb{C} \setminus \{0\} \times \mathbb{C}$ and define $\Phi(u, v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\Phi(u, v) = \frac{1-\alpha}{1-\beta}u + \frac{\alpha}{1-\beta}\frac{v}{u} + \frac{\alpha\frac{z\phi'(z)}{\phi(z)} - \beta}{1-\beta}.$$

Then $\Phi(u, v)$ is continuous in $\mathbb{D}$, $(1, 0) \in \mathbb{D}$ and in view of the given condition, we have

$$\operatorname{Re}(\Phi(1,0)) = \frac{1-\alpha\left(1-\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right)\right)-\beta}{1-\beta} > 0.$$

Further, from (2.3), we get $\operatorname{Re}[\Phi(p(z), zp'(z))] > 0$, $z \in \mathbb{E}$. Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ where u_1, u_2, v_1 *and* v_2 are all reals. Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leq -\frac{1+u_2^2}{2}$, we have

$$\begin{aligned}\operatorname{Re}(\Phi(iu_2, v_1)) =& \operatorname{Re}\left(\frac{1-\alpha}{1-\beta} iu_2 + \frac{\alpha}{1-\beta}\frac{v_1}{iu_2} + \frac{\alpha\left(\frac{z\phi'(z)}{\phi(z)}\right)-\beta}{1-\beta}\right)\\ =& \frac{\alpha\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right)-\beta}{1-\beta} \leq 0.\end{aligned}$$

The proof now follows from Lemma 1.1. □

To illustrate the above result, we consider the following example.

Example 2.1. On selecting $\phi(z) = ze^z$ and $f(z) = z + \frac{z^2}{2}$ in Theorem 2.1, we can easily check that for $\alpha = -0.1$ and $\beta = 0$, the condition (2.1) is satisfied as follows

$$-1.1 < -0.1\operatorname{Re}(1+z) \leq 0 < 1$$

and

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) = \operatorname{Re}\left(1.1e^{-z}(1+z) - \frac{0.1+0.2z}{1+z}\right) > 0.$$

Therefore,

$$\operatorname{Re}\left(\frac{zf'(z)}{\phi(z)}\right) = \operatorname{Re}(1+z)e^{-z} > 0,$$

thus f is close-to-convex and hence univalent in $\mathbb{E}$.

Theorem 2.2. *Suppose that ϕ is starlike in $\mathbb{E}$ and α, β are real numbers such that*

$$\beta + \alpha - 1 > \alpha\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) \geq \beta > 1.$$

If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) < \beta, \quad z \in \mathbb{E}, \tag{2.4}$$

then $\operatorname{Re}\left(\frac{zf'(z)}{\phi(z)}\right) > 0$ *in $\mathbb{E}$. So, f is close-to-convex and hence univalent in $\mathbb{E}$.*

Proof. Write $\frac{zf'(z)}{\phi(z)} = p(z)$, where p is analytic in $\mathbb{E}$ such that $p(0) = 1$ and ϕ is starlike in $\mathbb{E}$. Note that $1-\beta < 0$, thus the condition (2.4) reduces to

$$\operatorname{Re}\left(\frac{1-\alpha}{1-\beta}p(z) + \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha\frac{z\phi'(z)}{\phi(z)} - \beta}{1-\beta}\right) > 0, \quad z \in \mathbb{E}.$$

The proof can now be completed on the same lines as the proof of Theorem 2.1. □

In a special case when $\phi(z) = z$ in Theorem 2.1, we obtain the following result of Singh et al. [5].

Theorem 2.3. *Let α and β be real numbers such that $\alpha \leq \beta < 1$. Assume that an analytic function $f \in \mathcal{A}$ satisfies the condition*

$$\operatorname{Re}\left((1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathbb{E}. \tag{2.5}$$

Then $\operatorname{Re} f'(z) > 0$ *in* $\mathbb{E}$. *So, f is close-to-convex and hence univalent in* $\mathbb{E}$. *The result is sharp in the sense that the constant β on the right hand side of* (2.5) *cannot be replaced by a constant smaller than α.*

Selecting $\phi(z) = z$ in Theorem 2.2, we obtain the following result of Singh et al. [6].

Theorem 2.4. *For real numbers α and β such that $\alpha \geq \beta > 1$, if $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left((1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) < \beta, \quad z \in \mathbb{E}.$$

Then $\operatorname{Re} f'(z) > 0$ *in* $\mathbb{E}$. *So, f is close-to-convex and hence univalent in* $\mathbb{E}$.

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