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# POSITIVE SOLUTIONS FOR FIRST-ORDER NONLINEAR CAPUTO-HADAMARD FRACTIONAL RELAXATION DIFFERENTIAL EQUATIONS

ABDELOUAHEB ARDJOUNI[1] AND AHCENE DJOUDI[2]

Abstract. This article concerns the existence and uniqueness of positive solutions of the first-order nonlinear Caputo-Hadamard fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(x\left(t\right)-g\left(t,x\left(t\right)\right)\right)+wx\left(t\right)=f\left(t,x\left(t\right)\right), \ 1<t\leq e, \\ x\left(1\right)=x_0>g\left(1,x_0\right)>0, \end{cases}$$

where $0<\alpha\leq 1$. In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct appropriate mappings and employ the Krasnoselskii fixed point theorem and the method of upper and lower solutions to show the existence of a positive solution of this equation. We also use the Banach fixed point theorem to show the existence of a unique positive solution. Finally, an example is given to illustrate our results.

## 1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear

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fractional differential equations with and without delay have received the attention of many authors, see [1]–[13], [16] and the references therein.

Zhang in [16] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), & 0 < t \leq 1, \\ x(0) = 0, \end{cases}$$

where $D^{\alpha}$ is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)) + {}^{C}D^{\alpha-1}g(t, x(t)), & 0 < t \leq T, \\ x(0) = \theta_1 > 0, \quad x'(0) = \theta_2 > 0, \end{cases}$$

has been investigated in [4], where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $1 < \alpha \leq 2$, $g, f : [0, T] \times [0, \infty) \to [0, \infty)$ are given continuous functions, $g$ is non-decreasing on $x$ and $\theta_2 \geq g(0, \theta_1)$. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the authors obtained positivity results.

In [6], Chidouh, Guezane-Lakoud and Bebbouchi discussed the existence and uniqueness of the positive solution of the following nonlinear fractional relaxation differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) + wx(t) = f(t, x(t)), & 0 < t \leq 1, \\ x(0) = x_0 > 0, \end{cases}$$

where $0 < \alpha \leq 1$, $w > 0$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By using the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the existence and uniqueness of solutions has been established.

Ahmad and Ntouyas in [3] studied the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\mathfrak{D}_1^{\beta}u(t) - g(t, u_t)\right) = f(t, u_t), & t \in [1, b], \\ u(t) = \phi(t), \quad t \in [1 - r, 1], \\ \mathfrak{D}_1^{\beta}u(1) = \eta \in \mathbb{R}, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ and $\mathfrak{D}_1^{\beta}$ are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional differential equations. Inspired and motivated by the works mentioned above and the papers [1]–[13], [16] and the references therein, we concentrate on the positivity of solutions for the first-order nonlinear Caputo-Hadamard fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(x\left(t\right)-g\left(t,x\left(t\right)\right)\right)+wx\left(t\right)=f\left(t,x\left(t\right)\right), \quad 1<t\leq e, \\ x\left(1\right)=x_0>g\left(1,x_0\right)>0, \end{cases} \tag{1.1}$$

where $0<\alpha\leq 1$, $w>0$, $g,f:[1,e]\times[0,\infty)\to[0,\infty)$ are continuous. To show the existence and uniqueness of the positive solution, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use the Krasnoselskii and Banach fixed point theorems.

This paper is organized as follows. In Section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1.1) and the Banach and Krasnoselskii fixed point theorems. For details on the Banach and Krasnoselskii theorems we refer the reader to [15]. In Sections 3 and 4, we give and prove our main results on positivity and we provide an example to illustrate our results.

## 2. Preliminaries

Let $X=C\left([1,e]\right)$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1,e]$, endowed with the maximum norm. Define the cone

$$\mathcal{E}=\left\{x\in X: x(t)\geq 0 \text{ for all } t\in[1,e]\right\}.$$

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [9,13].

**Definition 2.1** ([9]). The Hadamard fractional integral of order $\alpha>0$ for a continuous function $x:[1,+\infty)\to\mathbb{R}$ is defined as

$$\mathfrak{I}_1^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}, \quad \alpha>0.$$

**Definition 2.2** ([9]). The Caputo-Hadamard fractional derivative of order $\alpha>0$ for a continuous function $x:[1,+\infty)\to\mathbb{R}$ is defined as

$$\mathfrak{D}_1^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n(x)(s)\frac{ds}{s}, \quad n-1<\alpha<n,$$

where $\delta^n=\left(t\frac{d}{dt}\right)^n$, $n\in\mathbb{N}$.

**Lemma 2.1** ([9]). *Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$ and $x\in C^n\left([1,T]\right)$. Then*

$$\left(\mathfrak{I}_1^{\alpha}\mathfrak{D}_1^{\alpha}x\right)(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(1)}{\Gamma(k+1)}(\log t)^k.$$

**Lemma 2.2** ([9]). *For all $\mu>0$ and $\nu>-1$*

$$\frac{1}{\Gamma(\mu)}\int_1^t\left(\log\frac{t}{s}\right)^{\mu-1}(\log s)^{\nu}\frac{ds}{s}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

**Definition 2.3** ([14]). The two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad \alpha>0,\ \beta\in\mathbb{C},\ z\in\mathbb{C}.$$

For $\beta=1$, we obtain the Mittag-Leffler function in one parameter

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha>0,\ z\in\mathbb{C}.$$

**Lemma 2.3** ([14]). *The generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ with $x\geq 0$ is completely monotonic if and only if $0<\alpha\leq 1$ and $\beta\geq\alpha$. In other words, it yields*

$$(-1)^n\frac{d^n}{dx^n}E_{\alpha,\beta}(-x)\geq 0,\quad \textit{for all } n\in\mathbb{N}.$$

*Obviously, $0\leq E_{\alpha,\beta}(-x)\leq\frac{1}{\Gamma(\beta)}$, where $x\geq 0$, $0\leq\alpha\leq 1$ and $\beta\geq\alpha$.*

The following lemma is fundamental to our results.

**Lemma 2.4.** *Let $x\in C([1,e])$, $x'$ and $\frac{\partial g}{\partial t}$ exist, then $x$ is a solution of* (1.1) *if and only if*

$$\begin{aligned}x(t)&=(x_0-g(1,x_0))E_\alpha(-w(\log t)^\alpha)+g(t,x(t))\\&\quad+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^{\alpha}\right)F(s,x(s))\frac{ds}{s},\quad 1\leq t\leq e,\end{aligned}\tag{2.1}$$

*where $F(t,x)=f(t,x)-wg(t,x)$.*

*Proof.* It is easy to prove by the Laplace transform. □

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a positive solution of (1.1).

**Definition 2.4.** Let $(X,\|\cdot\|)$ be a Banach space and $\mathcal{H}:X\to X$. The operator $\mathcal{H}$ is a contraction operator if there is an $\lambda\in(0,1)$ such that $x,y\in X$ imply

$$\|\mathcal{H}x-\mathcal{H}y\|\leq\lambda\|x-y\|.$$

**Theorem 2.1** (Banach [15]). *Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space $X$ and $\mathcal{H}:\mathcal{K}\to\mathcal{K}$ be a contraction operator. Then there is a unique $x\in\mathcal{K}$ with $\mathcal{H}x=x$.*

**Theorem 2.2** (Krasnoselskii fixed point theorem [15]). *If $\mathcal{K}$ is a nonempty bounded, closed and convex subset of a Banach space $X$, $\mathcal{A}$ and $\mathcal{B}$ two operators defined on $\mathcal{K}$ with values in $X$ such that*

i) *$\mathcal{A}x+\mathcal{B}y\in\mathcal{K}$ for all $x,y\in\mathcal{K}$;*
ii) *$\mathcal{A}$ is continuous and compact;*
iii) *$\mathcal{B}$ is a contraction.*

*Then there exists $z\in\mathcal{K}$ such that $z=\mathcal{A}z+\mathcal{B}z$.*

## 3. Existence of Positive Solutions

In this section, we consider the results of existence problem for many cases of (1.1). Moreover, we introduce the following conditions.

$(H1)$ $g, F : [1, e] \times [0, \infty) \to [0, \infty)$ are continuous functions and $g$ is nondecreasing on $x$.

$(H2)$ There exists $L_g \in (0, 1)$ such that

$$|g(t, x) - g(t, y)| \leq L_g \|x - y\| .$$

$(H3)$ There exists $L_F > 0$ such that

$$|F(t, x) - F(t, y)| \leq L_F \|x - y\| .$$

We note that to apply Theorem 2.2 we need to construct two mappings, one is contraction and the other is completely continuous. Therefore, we express (2.1) as

$$x(t) = (\mathcal{A}x)(t) + (\mathcal{B}x)(t) = (\mathcal{H}x)(t), \tag{3.1}$$

where the operators $\mathcal{A}, \mathcal{B} : \mathcal{E} \to X$ are defined by

$$(\mathcal{A}x)(t) = \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}$$

and

$$(\mathcal{B}x)(t) = (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, x(t)) .$$

We need the following lemmas to establish our results.

**Lemma 3.1.** *Assume that $(H1)$ holds. Then, the operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is completely continuous.*

*Proof.* By Lemma 2.3 and taking into account that $F$ is continuous nonnegative function, we get that $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is continuous. The function $F : [1, e] \times B_\eta \to [0, \infty)$ is bounded, then there exists $\rho > 0$ such that $0 \leq F(t, x(t)) \leq \rho$, where $B_\eta = \{x \in \mathcal{E},\ \|x\| \leq \eta\}$. We obtain

$$\begin{aligned}
|(\mathcal{A}x)(t)| &= \left|\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |F(s, x(s))| \frac{ds}{s} \\
&\leq \frac{\rho}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{\rho(\log t)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Thus,

$$\|\mathcal{A}x\| \leq \frac{\rho}{\Gamma(\alpha+1)}.$$

Hence, $\mathcal{A}(B_\eta)$ is uniformly bounded.

Now, we will prove that $\mathcal{A}(B_\eta)$ is equicontinuous. Let $x \in B_\eta$, then for any $t_1, t_2 \in [1, e]$, $t_2 > t_1$, we have

$$\begin{aligned}
&\left|(\mathcal{A}x)(t_1) - (\mathcal{A}x)(t_2)\right| \\
=&\left|\int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t_1}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}\right. \\
&\left. - \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t_2}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}\right| \\
\leq&\frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left|\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1}\right| |F(s, x(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |F(s, x(s))| \frac{ds}{s} \\
\leq&\frac{\rho}{\Gamma(\alpha)} \left(\int_1^{t_1} \left(\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1}\right) \frac{ds}{s} + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s}\right) \\
\leq&\frac{\rho}{\Gamma(\alpha+1)} \left((\log t_1)^{\alpha} - (\log t_2)^{\alpha} + 2\left(\log \frac{t_2}{t_1}\right)^{\alpha}\right) \\
\leq&\frac{2\rho}{\Gamma(\alpha+1)} \left(\log \frac{t_2}{t_1}\right)^{\alpha},
\end{aligned}$$

which is independent of $x$ and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{A}(B_\eta)$ is equicontinuous. So, the compactness of $\mathcal{A}$ follows by Ascoli Arzela's theorem. ☐

**Lemma 3.2.** *Assume that* $(H1)$ *and* $(H2)$ *hold. Then the operator* $\mathcal{B} : \mathcal{E} \to \mathcal{E}$ *is a contraction.*

*Proof.* By Lemma 2.3 and taking into account that $g$ is continuous nonnegative function and $x_0 > g(1, x_0)$, we get that $\mathcal{B} : \mathcal{E} \to \mathcal{E}$. For $x, y \in \mathcal{E}$ we have

$$|(\mathcal{B}x)(t) - (\mathcal{B}y)(t)| = |g(t, x(t)) - g(t, y(t))| \leq L_g \|x - y\|.$$

Thus, $\|\mathcal{B}x - \mathcal{B}y\| \leq L_g \|x - y\|$. Hence, $\mathcal{B}$ is a contraction. ☐

Now, for any $x \in [a, b] \subset \mathbb{R}^+$, we define respectively the upper and lower control functions as follows

$$H(t, x) = \sup_{a \leq y \leq x} F(t, y), \quad h(t, x) = \inf_{x \leq y \leq b} F(t, y).$$

It is clear that these functions are nondecreasing on $[a, b]$.

**Definition 3.1.** Let $\overline{x}, \underline{x} \in \mathcal{E}$, $a \leq \underline{x} \leq \overline{x} \leq b$, satisfying

$$\begin{aligned}
\overline{x}(t) \geq& (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, \overline{x}(t)) \\
&+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) H(s, \overline{x}(s)) \frac{ds}{s}, \quad 1 \leq t \leq e,
\end{aligned}$$

and

$$\begin{aligned}\underline{x}(t) \leq{}& (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, \underline{x}(t)) \\ &+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) h(s, \underline{x}(s)) \frac{ds}{s}, \quad 1 \leq t \leq e.\end{aligned}$$

Then the functions $\overline{x}$ and $\underline{x}$ are called a pair of upper and lower solutions for the equation (1.1).

**Theorem 3.1.** *Assume that* $(H1)$ *and* $(H2)$ *hold and* $\overline{x}$ *and* $\underline{x}$ *are respectively upper and lower solutions of* (1.1)*, then* (1.1) *has at least one positive solution.*

*Proof.* Let

$$\mathcal{K} = \{x \in \mathcal{E} : \underline{x}(t) \leq x(t) \leq \overline{x}(t),\ t \in [1, e]\}.$$

As $\mathcal{K} \subset E$ and $\mathcal{K}$ is a nonempty bounded, closed and convex subset. By Lemma 3.1, $\mathcal{A} : \mathcal{K} \to \mathcal{E}$ is completely continuous. Also, from Lemma 3.2, $\mathcal{B} : \mathcal{K} \to \mathcal{E}$ is a contraction. Next, we show that if $x, y \in \mathcal{K}$, we have $\mathcal{A}x + \mathcal{B}y \in \mathcal{K}$. For any $x, y \in \mathcal{K}$, we have $\underline{x} \leq x, y \leq \overline{x}$, then

$$\begin{aligned}&(\mathcal{A}x)(t) + (\mathcal{B}y)(t) \\ &= (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, y(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) F(s, x(s)) \frac{ds}{s} \\ &\leq (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, \overline{x}(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) H(s, \overline{x}(s)) \frac{ds}{s} \\ &\leq \overline{x}(t)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&(\mathcal{A}x)(t) + (\mathcal{B}y)(t) \\ &= (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, y(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) F(s, x(s)) \frac{ds}{s} \\ &\geq (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, \underline{x}(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) h(s, \underline{x}(s)) \frac{ds}{s} \\ &\geq \underline{x}(t).\end{aligned} \tag{3.3}$$

Thus, from (3.2) and (3.3), we obtain that $\mathcal{A}x + \mathcal{B}y \in \mathcal{K}$. We now see that all the conditions of the Krasnoselskii's fixed point theorem are satisfied. Thus there exists a fixed point $x$ in $\mathcal{K}$. Therefore, (1.1) has at least one positive solution $x$ in $\mathcal{K}$. $\square$

**Corollary 3.1.** *Assume that* $(H1)$ *and* $(H2)$ *hold and there exist* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ *such that*

$$\lambda_1 \leq g(t,x) \leq \lambda_2, \quad (t,x) \in [1,e] \times [0,+\infty), \tag{3.4}$$

*and*

$$\lambda_3 \leq F(t,x) \leq \lambda_4, \quad (t,x) \in [1,e] \times [0,+\infty). \tag{3.5}$$

*Then* (1.1) *has at least one positive solution* $x \in \mathcal{E}$*, moreover*

$$x(t) \geq (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_1 + \lambda_3 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha) \tag{3.6}$$

*and*

$$x(t) \leq (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 + \lambda_4 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha). \tag{3.7}$$

*Proof.* From (3.5) and the definition of control functions, we have

$$\lambda_3 \leq h(t,x) \leq H(t,x) \leq \lambda_4. \tag{3.8}$$

Now, let

$$\begin{aligned}
\overline{x}(t) &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 + \lambda_4 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha) \\
&= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 \\
&\quad + \lambda_4 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) \frac{ds}{s}.
\end{aligned}$$

Taking into account (3.4) and (3.8), we have

$$\begin{aligned}
\overline{x}(t) &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 \\
&\quad + \lambda_4 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) \frac{ds}{s} \\
&\geq (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + g(t,\overline{x}(t)) \\
&\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) H(s,\overline{x}(s)) \frac{ds}{s}.
\end{aligned}$$

It is clear that $\overline{x}$ is the upper solution of (1.1).

Now, let

$$\begin{aligned}
\underline{x}(t) &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_1 + \lambda_3 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha) \\
&= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_1 \\
&\quad + \lambda_3 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) \frac{ds}{s}.
\end{aligned}$$

By (3.4), (3.8) and the same way that we used to search the upper solution, we conclude also that $\underline{x}$ is the lower solution of (1.1). Therefore, from Theorem 3.1, we conclude that (1.1) has at least one positive solution $x \in \mathcal{E}$ which verifies the inequalities (3.6) and (3.7). $\square$

**Corollary 3.2.** *Assume that* $(H1)$ *and* $(H2)$ *hold and there exists* $a_1, a_2 > 0$ *such that*

$$a_1 \leq g(t,x), \; a_2 \leq F(t,x), \quad (t,x) \in [1,e] \times [0,+\infty), \tag{3.9}$$

*and*

$$\lim_{x\to+\infty} g(t,x) < +\infty, \quad \lim_{x\to+\infty} F(t,x) < +\infty, \tag{3.10}$$

*then* (1.1) *has at least one positive solution.*

*Proof.* By (3.10), there exist positive constants $N_1$, $N_2$, $R_1$ and $R_2$ such that

$$g(t,x) \leq N_1, \quad \text{for any } x \geq R_1,\ t \in [1,e], \tag{3.11}$$

and

$$F(t,x) \leq N_2, \quad \text{for any } x \geq R_2,\ t \in [1,e]. \tag{3.12}$$

Let $C_1 = \max\limits_{1\leq t\leq e, 0\leq x\leq R_1} g(t,x)$ and $C_2 = \max\limits_{1\leq t\leq e, 0\leq x\leq R_2} F(t,x)$. Then, by (3.11) and (3.12), we have

$$a_1 \leq g(t,x) \leq N_1 + C_1, \quad \text{for any } x \geq 0,\ t \in [1,e],$$

and

$$a_2 \leq F(t,x) \leq N_2 + C_2, \quad \text{for any } x \geq 0,\ t \in [1,e].$$

Thus, from Corollary 3.1, (1.1) has at least one positive solution $x$ in $\mathcal{E}$ which satisfies the following inequalities

$$x(t) \geq (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + a_1 + a_2 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha)$$

and

$$\begin{aligned} x(t) \leq{}& (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + N_1 + C_1 \\ &+ (N_2 + C_2)(\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha). \end{aligned}$$ □

## 4. Uniqueness of Positive Solution

In this section, we shall prove the uniqueness of the positive solution using the contraction mapping principle.

**Theorem 4.1.** *Assume that* $(H1)$-$(H3)$ *hold and*

$$L_g + \frac{L_F}{\Gamma(\alpha+1)} < 1, \tag{4.1}$$

*then* (1.1) *has a unique positive solution* $x \in \mathcal{K}$.

*Proof.* From Theorem 3.1, it follows that (1.1) has at least one positive solution in $\mathcal{K}$. Hence, we need only to prove that the operator $\mathcal{H}$ defined in (3.1) is a contraction on $X$. In fact, since for any $x_1, x_2 \in \mathcal{K}$, $(H2)$ and $(H3)$ are verified, then we have

$$\begin{aligned}
&\left|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\right| \\
&\leq \left|g(t, x_1(t)) - g(t, x_2(t))\right| \\
&\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^{\alpha}\right) \left|F(s, x_1(s)) - F(s, x_2(s))\right| \frac{ds}{s} \\
&\leq L_g \|x_1 - x_2\| + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} L_F \|x_1 - x_2\| \\
&\leq \left(L_g + \frac{L_F}{\Gamma(\alpha+1)}\right) \|x_1 - x_2\|.
\end{aligned}$$

Thus,

$$\|\mathcal{H}x_1 - \mathcal{H}x_2\| \leq \left(L_g + \frac{L_F}{\Gamma(\alpha+1)}\right) \|x_1 - x_2\|.$$

Hence, the operator $\mathcal{H}$ is a contraction mapping by (4.1). Therefore, by the contraction mapping principle, we conclude that the problem (1.1) has a unique positive solution $x \in \mathcal{K}$. □

Finally, we give an example to illustrate our results.

*Example* 4.1. We consider the following nonlinear Caputo-Hadamard fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_1^{1/3}\left(x(t) - \dfrac{x(t)+2}{x(t)+3}\right) + x(t) \\ = \dfrac{(t+6)x^2(t) + (4t+21)x(t) + 5t + 18}{(t+3)(x^2(t) + 4x(t) + 3)}, \quad 1 < t \leq e, \\ x(1) = 1, \end{cases} \tag{4.2}$$

where

$$\alpha = \frac{1}{3}, \quad w = 1, \quad x_0 = 1, \quad g(t,x) = \frac{x+2}{x+3}, \quad g(1, x_0) = \frac{3}{4},$$
$$f(t,x) = \frac{(t+6)x^2 + (4t+21)x + 5t + 18}{(t+3)(x^2 + 4x + 3)}, \quad F(t,x) = \frac{1}{3+t}\left(\frac{t}{x+1} + 3\right).$$

Since $g$ is nondecreasing on $x$ and $F$ is decreasing on $x$

$$\frac{2}{3} \leq g(t,x) \leq 1, \quad \frac{3}{3+e} \leq F(t,x) \leq 1,$$

for $(t,x) \in [1,e] \times [0,\infty)$. Hence, by Corollary 3.1, (4.2) has a positive solution, which verifies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, where

$$\overline{x}(t) = \frac{1}{4} E_{1/3}\left(-(\log t)^{1/3}\right) + 1 + (\log t)^{1/3} E_{1/3,4/3}\left(-(\log t)^{1/3}\right)$$

and

$$\underline{x}(t) = \frac{1}{4}E_{1/3}\left(-(\log t)^{1/3}\right) + \frac{2}{3} + \frac{3}{3+e}(\log t)^{1/3} E_{1/3,4/3}\left(-(\log t)^{1/3}\right),$$

are respectively the upper and lower solutions of (4.2). Also, we have

$$L_g + \frac{L_F}{\Gamma(\alpha+1)} \simeq 0.64 < 1.$$

Then, by Theorem 4.1, (4.2) has a unique positive solution which is bounded by $\underline{x}$ and $\overline{x}$.

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[1]Department of Mathematics and Informatics,
University of Souk Ahras,
P.O. Box 1553, Souk Ahras, Algeria
*Email address*: abd_ardjouni@yahoo.fr

[2]Department of Mathematics,
University of Annaba,
P.O. Box 12, Annaba, Algeria
*Email address*: adjoudi@yahoo.com