

SPECTRAL INCLUSIONS OF BOUNDARY SETS OF FREDHOLM OPERATOR MATRIX PENCILS

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ABSTRACT. The problem of describing the eigenvalues of some perturbed operator pencils has made significant advances in many fields of mathematics, especially in the study of control systems with linear descriptors. Thus, the goal of this paper is to develop original stability results arising in the theory of perturbed classes of Fredholm operator pencils by using new arguments linked to a different approach to perturbation measures developed by S. Č. Živković-Zlatanović in [32]. Our approach allows us to define an abstract framework that yields real progress in analyzing the stability of certain boundary sets of the essential spectra of the closure of an unbounded block 2×2 operator matrix pencil defined on a maximal domain via two different perturbation approaches, expressed in terms of measures of non-strict cosingularity and strict singularity.

1. INTRODUCTION

One of the most important contributions of M. V. Keldysh's work [22] is the notion of the operator pencil. This notion of the operator pencil, which emerged in mathematical fields, has played an important role in the study of spectral properties in operator theory since 1951 and was later introduced by A. S. Markus in [23]. The introduction of this notion strongly motivated numerous authors to develop a new line of research and to enrich analytical problems in operator theory. The most significant contributions of these works, which embraced this notion, have had a considerable

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impact in many fields of study, particularly in the context of spectral analysis. In this regard, we refer readers to [17, 21] for a more comprehensive understanding.

Later, the above notion of an operator pencil has been extended and extensively studied in the matrix framework. In particular, in 1995, M. Faierman et al. in [18] extended the spectral description of linear operator pencil to the context of matrix pencil framework. Their main contribution relied on exploiting compactness criteria of perturbation to investigate spectral properties of unbounded operator matrix pencils block 2×2 and investigated an analytic approach to characterize the relative essential spectrum of their studied operator matrix.

In closely related areas of interest, in the study of characterization problems for the relative essential spectra of linear operator pencils, classes of Fredholm perturbations have proved to be a powerful mathematical tool. Since the early 2000s, the mathematical theory addressing analysis problems concerning the properties of Fredholm operators and their derived sets of perturbed operators, under additive arguments linked to sets of Fredholm perturbations, has developed rapidly; see [1–4, 7–11, 24, 25, 28, 34]. Meanwhile, considerable progress has been made using different approaches to investigate various essential spectra of the above-mentioned operator concepts; see [15, 26, 28, 34] for further details.

Even the invention by M. Faierman et al. in [18] opened up opportunities for researchers to make meaningful use of Fredholm perturbation criteria, allowing a detailed description of other types of relative essential spectra of mixed forms of operator matrix pencils (among the works in this direction we quote, for example, [5, 6, 15, 25]). However, we usually strive to find the greatest possible generality.

So, our main objective in this work is to reveal the connection between two different approaches of perturbations and the concept of matrix pencils block 2×2 . Particularly, our findings aim to draw an abstract setting and provide new perturbation criteria that allow us to establish spectral inclusions between some types of relative essential spectra of our considered matrix form and their operators pencil entries. By way of explanation, we discuss via two different approaches of perturbations linked to the notion of measures of non-strict cosingularity and strict singularity perturbations the stability problem of perturbed lower (resp. upper) semi Fredholm, right (resp. left) Fredholm and right (resp. left) Weyl operators pencil and explore some spectral links between their relative essential spectra. The results in this paper emerge as a natural extension in light of advancements in the field of matrix pencil framework.

The paper is organized as follows. In Section 2 we introduce some mathematical background around operator theory needed throughout the paper. The goal of Section 3 (resp. 4) is to introduce the notion of non-strict cosingularity (resp. singularity) measure and explore its properties. Such criteria of perturbations are used to develop some new stability results of perturbed lower (resp. upper) semi Fredholm, right (resp. left) Fredholm and right (resp. left) Weyl operators pencil and explore some spectral inclusions between their relative essential spectra. Finally, Section 5 is dedicated to illustrate the validity of our theoretical results to the case of operator matrix block

2×2 . Particularly, sufficient criteria based on the measures of non-strict cosingularity (resp. singularity) are established in order to provide some spectral inclusions on relative essential spectra of our considered matrix form and their operators pencils entries.

2. PRELIMINARIES ON LINEAR OPERATORS

In this section, we recall some basic notations, definitions and preliminary results.

Notations. Let $\mathcal{X}$ be a Banach space. We denote by:

- * $\mathcal{L}(\mathcal{X})$: the set of bounded operators on $\mathcal{X}$,
- * $\mathcal{C}(\mathcal{X})$: the set of closed densely defined linear operators,
- * $\mathcal{K}(\mathcal{X})$: the set of compact operators.

For $T \in \mathcal{C}(\mathcal{X})$, we write by:

- * $\mathcal{D}(T)$: for the domain of T ,
- * $\ker(T)$: for the null space of T ,
- * $R(T)$: for the range of T ,
- * $\alpha(T)$: for the dimension of $\ker(T)$,
- * $\beta(T)$: for the codimension of $R(T)$.

The purpose of this work is to develop some new spectral properties of linear operator pencil $\xi M - T$ for a couple of operators $(T, M) \in \mathcal{C}(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$ where $\xi \in \mathbb{C}$ and $M \neq 0$. To this end, we recall the definition of such pencils introduced by A. S. Markus [23] that we will need throughout this paper.

Definition 2.1 (Resolvent set and spectrum of operator pencils). Let $T \in \mathcal{C}(\mathcal{X})$ and M be a bounded linear operator on $\mathcal{X}$ such that $M \neq 0$. We denote by:

$$\begin{aligned} \rho(T, M) &:= \{\xi \in \mathbb{C}, \xi M - T \text{ has a bounded inverse}\} : \text{ as the } M\text{-resolvent set of } T, \\ \sigma(T, M) &:= \mathbb{C} \setminus \rho(T, M) : \text{ as the } M\text{-spectrum of } T. \end{aligned}$$

Remark 2.1. (i) It is known that for $T \in \mathcal{L}(\mathcal{X})$, its spectrum $\sigma(T)$ is a compact subset of $\mathbb{C}$, but this property fails to hold for the relative spectrum or M -spectrum of T , where $M \in \mathcal{L}(\mathcal{X})$. To explain this, we can recall an illustrative example developed by S. C. Živanović-Zlatanović et al. in [34].

Indeed, for $\mathcal{X} = L^2([0, 1])$, let consider the following operators:

$$Tf(x) = (x + 1)f(x) \quad \text{and} \quad Mf(x) = xf(x).$$

A short computation reveals that:

$$\sigma(T, M) = \{1\} \cup [2, +\infty) \quad \text{and} \quad \sigma(T) = [1, 2],$$

which asserts that $\sigma(T, M) \neq \sigma(T)$.

(ii) In particular, even if M is an invertible operator, we have not $\sigma(T, M) = \sigma(T)$.

To explain this, we consider for example the case where $\mathcal{X}$ is a finite-dimensional space and M is an invertible operator on $\mathcal{X}$ such that $\sigma(M) \setminus 1 \neq \emptyset$. Then, $\sigma(M, M) = \{1\}$, which implies that $\sigma(M, M) \neq \sigma(M)$ (see [34] for more details).

Definition 2.2 (Relatively bounded operator). Let $T \in \mathcal{C}(\mathcal{X})$. An operator K is said to be a T -bounded linear operator if it satisfies the following items:

- (i) $\mathcal{D}(T) \subset \mathcal{D}(K)$,
- (ii) the restriction of K on $\mathcal{D}(T)$, denoted by $\widehat{K}$, is a bounded operator from $\mathcal{X}_T$ into $\mathcal{X}$.

Remark 2.2. For $T \in \mathcal{C}(\mathcal{X})$ and K is an T -bounded linear operator, we have:

$$\alpha(\widehat{T}) = \alpha(T), \quad \beta(\widehat{T}) = \beta(T), \quad R(\widehat{T}) = R(T)$$

and

$$\alpha(\widehat{T} + \widehat{K}) = \alpha(T + K), \quad \beta(\widehat{T} + \widehat{K}) = \beta(T + K), \quad R(\widehat{T} + \widehat{K}) = R(T + K).$$

We start this section by recalling some classical classes of Fredholm operators needed in our formulation as follow.

Definition 2.3 (Sets of Fredholm operators). Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{C}(\mathcal{X})$.

- (i) The class of Fredholm operators on $\mathcal{X}$ is denoted by $\Phi(\mathcal{X})$ and defined if $\alpha(T) < +\infty$ and $\beta(T) < +\infty$.
- (ii) The class of upper (resp. lower) semi-Fredholm operators on $\mathcal{X}$ is denoted by $\Phi_+(\mathcal{X})$ (resp. $\Phi_-(\mathcal{X})$) and defined if $\alpha(T) < +\infty$ (resp. $\beta(T) < \infty$, $R(T)$ is closed).
- (iii) The class of left (resp. right) Fredholm operators on $\mathcal{X}$ is denoted by $\Phi_l(\mathcal{X})$ (resp. $\Phi_r(\mathcal{X})$) and defined if $\alpha(T) < +\infty$, $R(T)$ is closed and complemented subspace of $\mathcal{X}$ (resp. $\beta(T) < +\infty$, $R(T)$ is closed and $\ker(T)$ is complemented subspace of $\mathcal{X}$).
- (iv) The set of left (resp. right) Weyl operators on $\mathcal{X}$ is denoted by $\mathcal{W}_l(\mathcal{X})$ (resp. $\mathcal{W}_r(\mathcal{X})$) and defined if $T \in \Phi_l(\mathcal{X})$ and $\text{ind}(T) := \alpha(T) - \beta(T) \leq 0$ (resp. $T \in \Phi_r(\mathcal{X})$ and $\text{ind}(T) \geq 0$).
- (v) The class of Weyl operators on $\mathcal{X}$ is defined as: $\mathcal{W}(\mathcal{X}) := \{T \in \Phi(\mathcal{X}) : \text{ind}(T) = 0\}$.

The classes of Fredholm operators serve to define the corresponding perturbation classes.

Definition 2.4 (Sets of Fredholm perturbations). Let H be a subset of $\mathcal{L}(\mathcal{X})$. The perturbation class of H is denoted by $\mathcal{P}H$ and defined by

$$\mathcal{P}H := \{P \in \mathcal{L}(\mathcal{X}) : P + T \in H, \text{ for all } T \in H\}.$$

We list in the following some classical definitions of linear bounded operators used in our formulation.

Definition 2.5 (Sets of $\mathcal{WC}(\mathcal{X})$, $\mathcal{SC}(\mathcal{X})$, $\mathcal{SS}(\mathcal{X})$). Let $T \in \mathcal{L}(\mathcal{X})$.

- (i) T is said to be a weakly compact operator on $\mathcal{X}$, if $T(\mathcal{H})$ is relatively weakly compact, for every bounded subset $\mathcal{H} \in \mathcal{X}$.
- (ii) T is called a strictly cosingular operator if there exists no infinite-codimensional closed subspace V of $\mathcal{X}$ such that $Q_V T : \mathcal{X} \rightarrow \mathcal{X}|_V$ is surjective.
- (iii) T is called strictly singular operator if, for every infinite-dimensional subspace V , the restriction of T to V is not a homeomorphism.

Denote by:

- $\mathcal{WC}(\mathcal{X})$:the set of weakly compact operators on $\mathcal{X}$,
- $\mathcal{SC}(\mathcal{X})$:the set of strictly cosingular operators on $\mathcal{X}$,
- $\mathcal{SS}(\mathcal{X})$:the set of strictly singular operators on $\mathcal{X}$.

The relationships between these sets are given by

$$\mathcal{K}(\mathcal{X}) \subset \mathcal{SC}(\mathcal{X}) \subset \mathcal{P}\Phi_-(\mathcal{X}) \quad \text{and} \quad \mathcal{K}(\mathcal{X}) \subset \mathcal{WC}(\mathcal{X}) \subset \mathcal{SS}(\mathcal{X}) \subset \mathcal{P}\Phi_+(\mathcal{X}),$$

where $\mathcal{P}\Phi_-(\mathcal{X})$ and $\mathcal{P}\Phi_+(\mathcal{X})$ denote the sets of lower semi-Fredholm and upper semi-Fredholm perturbations, respectively.

Before tackling the stability problem of perturbed semi Fredholm operators pencils via the notions of measures of non-strict cosingularity perturbations and non-strict singularity as well as the concept of right strictly cosingular Fredholm inverse and left strictly singular Fredholm inverse, the following definition is required.

Definition 2.6 (Left-right strictly cosingular Fredholm inverse of bounded operator).

(i) An operator $T \in \mathcal{L}(\mathcal{X})$ is said to have a right Fredholm inverse modulo strictly cosingular operator if there exists $T_{sc}^r \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{SC}(\mathcal{X})$ such that $TT_{sc}^r = I - S$. The operator T_{sc}^r is called right strictly cosingular Fredholm inverse of T .

(ii) An operator $T \in \mathcal{L}(\mathcal{X})$ is said to have a left Fredholm inverse modulo strictly singular operator if there exists $T_{ss}^l \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{SS}(\mathcal{X})$ such that $T_{ss}^l T = I - S$. The operator T_{ss}^l is called left strictly singular Fredholm inverse of T .

Based on the Definition 2.6 and Remark 2.2, we will extend the previous definition for closed densely defined operator as follows.

Definition 2.7 (Strictly cosingular Fredholm inverse of closed densely defined linear operator). An operator $T \in \mathcal{C}(\mathcal{X})$ is said to have a right (resp. left) Fredholm inverse modulo strictly cosingular (resp. singular) operator if $\hat{T}$ has a right (resp. left) Fredholm inverse modulo strictly cosingular (resp. singular) operator from $\mathcal{X}_T$ into $\mathcal{X}$.

Several sets of Fredholm operators pencils generate their corresponding essential spectra defined as follows which we are interested throughout this study.

Definition 2.8 (M-essential spectra or essential spectra of operator pencils). Let $T \in \mathcal{C}(\mathcal{X})$ and M is a bounded linear operator on $\mathcal{X}$ such that $M \neq 0$.

We define some relative essential spectra or M -essential spectra of the operator T as:

$$\begin{aligned} \sigma_{e-}(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \Phi_-(\mathcal{X})\} : \text{M-Weidmann essential spectra of } T, \\ \sigma_{e+}(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \Phi_+(\mathcal{X})\} : \text{M-Gustafson essential spectra of } T, \\ \sigma_{er}(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \Phi_r(\mathcal{X})\} : \text{M-right Fredholm spectrum of } T, \\ \sigma_{el}(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \Phi_l(\mathcal{X})\} : \text{M-left Fredholm spectrum of } T, \\ \sigma_{ef}(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \Phi(\mathcal{X})\} : \text{M-Fredholm essential spectrum of } T, \end{aligned}$$

$$\begin{aligned}\sigma_{wr}(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \mathcal{W}_r(\mathcal{X})\} : \text{M-right Weyl spectrum of } T, \\ \sigma_{wl}(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \mathcal{W}_l(\mathcal{X})\} : \text{M-left Weyl spectrum of } T, \\ \sigma_w(T, M) &:= \{\xi \in \mathbb{C} : \xi M - T \notin \mathcal{W}(\mathcal{X})\} : \text{M-Weyl essential spectrum of } T.\end{aligned}$$

3. STABILITY RESULTS OF OPERATOR PENCILS VIA A MEASURE OF NON-STRICT COSINGULARITY PERTURBATION

Recently, the notion of measure of non-strict cosingularity appears as a generalization of the class of strict cosingularity operators, which was introduced in the pioneer paper of S. C. Živanović-Zlatanović in [32] to offer to the reader a frequency of spectral analysis via this kind of measure perturbation. Motivated by such concept of measure originated by L. Weis in [29] and the obtained results by S. C. Živanović-Zlatanović in [32], we are interested in this section we establish new conditions describing some sets of Fredholm perturbed operators pencil via the measure of non-strict cosingularity perturbation. Before doing so, we recall the definition of such measures.

Definition 3.1 (Measure of non-strict cosingularity: M.S.C. [29, 32]). Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. The non-strict co-singular measure of T , denoted by $\nabla(T)$, is defined as:

$$\nabla(T) := \sup_W \inf_{V \supset W} \|(Q_V T)\|,$$

where W and V are two closed infinite codimensional subspaces of $\mathcal{X}$ and Q_W denotes the quotient maps from $\mathcal{X}$ into $\mathcal{X}|_W$, satisfying

$$(3.1) \quad \nabla(T) = 0 \quad \text{if and only if} \quad T \in \mathcal{SC}(\mathcal{X}).$$

In what follows, we are in the position to explore some spectral inclusions of some perturbed semi Fredholm operators pencils.

Theorem 3.1 (Perturbation results by M.S.C.). *Let $T \in \mathcal{C}(\mathcal{X})$, $M \in \mathcal{L}(\mathcal{X})$ for which $M \neq 0$, and let A be a T -bounded linear operator on $\mathcal{X}$. Assume that, for $\xi \in \mathbb{C}$, there exists $T_{sc,M}^r$ a right strictly cosingular Fredholm inverse of the operator pencils $\xi M - T$ such that $\nabla((T_{sc,M}^r \hat{A})^n) < 1$, for some $n \geq 1$. Then, we have:*

(i) $\xi M - T \in \Phi_*(\mathcal{X})$ implies that $\xi M - T - A \in \Phi_*(\mathcal{X})$ with $\text{ind}(\xi M - T - A) = \text{ind}(\xi M - T)$,

(ii) $\xi M - T \in \mathcal{W}_r(\mathcal{X})$ implies that $\xi M - T - A \in \mathcal{W}_r(\mathcal{X})$.

Therefore, we get

$$\sigma_*(T + A, M) \subseteq \sigma_*(T, M),$$

with

$$(\Phi_*(\mathcal{X}), \sigma_*(\cdot, M)) \in \{(\Phi_-(\mathcal{X}), \sigma_{e-}(\cdot, M)), (\Phi_r(\mathcal{X}), \sigma_{er}(\cdot, M)), (\mathcal{W}_r(\mathcal{X}), \sigma_{wr}(\cdot, M))\}.$$

Proof. By Remark 2.2, since $T \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$, and A is T -bounded, the operator pencils $\xi M - T$ and $\xi M - T - A$ are closed with domains $\mathcal{D}(\xi M - T) = \mathcal{D}(T)$ and $\mathcal{D}(\xi M - T - A) = \mathcal{D}(T)$, respectively. Hence, their restrictions on the Banach

space $\mathcal{X}_T$ form bounded operators from $\mathcal{X}_T$ into $\mathcal{X}$. This implies that $\widehat{A} \in \mathcal{L}(\mathcal{X}_T, \mathcal{X})$, $\widehat{\xi M - T} \in \mathcal{L}(\mathcal{X}_T, \mathcal{X})$ and $\widehat{\xi M - T - \hat{A}} \in \mathcal{L}(\mathcal{X}_T, \mathcal{X})$.

Let $\xi \in \mathbb{C}$ and consider $\xi M - T \in \Phi_-(\mathcal{X})$ (resp. $\xi M - T \in \Phi_r(\mathcal{X})$). So, we deduce thanks to Definition 2.6 and Remark 2.2, that there exist $S \in \mathcal{SC}(\mathcal{X})$ and $T_{sc,M}^r \in \mathcal{L}(\mathcal{X}, \mathcal{X}_T)$ for which the operator $\widehat{\xi M - T - \hat{A}}$ may be written as follows:]

$$(3.2) \quad \widehat{\xi M - T - \hat{A}} = (\widehat{\xi M - T})(I - T_{sc,M}^r \hat{A}) - S \hat{A}.$$

Based on the assumption that $\nabla((T_{sc,M}^r \hat{A})^n) < 1$, for some $n \geq 1$, and in view of Theorem 3.1 in [33], we conclude that $I - T_{sc,M}^r \hat{A} \in \Phi_-^b(\mathcal{X}_T)$, with $\text{ind}(I - T_{sc,M}^r \hat{A}) = 0$. Therefore, the use of Corollary 1.33 in [14] (resp. Theorem 2.5 in [20]), asserts that $(\widehat{\xi M - T})(I - T_{sc,M}^r \hat{A}) \in \Phi_-^b(\mathcal{X}_T, \mathcal{X})$ with $\text{ind}((\widehat{\xi M - T})(I - T_{sc,M}^r \hat{A})) = \text{ind}(\widehat{\xi M - T})$ (resp. $(\widehat{\xi M - T})(I - T_{sc,M}^r \hat{A}) \in \Phi_r^b(\mathcal{X}_T, \mathcal{X})$). Due to the criterion of perturbation made on the operator $S \hat{A}$, as $S \hat{A} \in \mathcal{SC}(\mathcal{X}_T, \mathcal{X})$, we deduce that $\widehat{\xi M - T - \hat{A}} \in \Phi_-^b(\mathcal{X}_T, \mathcal{X})$ with $\text{ind}(\widehat{\xi M - T - \hat{A}}) = \text{ind}((\widehat{\xi M - T})(I - T_{sc,M}^r \hat{A})) = \text{ind}(\widehat{\xi M - T})$ (resp. $\widehat{\xi M - T - \hat{A}} \in \Phi_r^b(\mathcal{X}_T, \mathcal{X})$).

Now, to reach the required result, we consider two cases to prove the stability of the index between the right Fredholm operators $\widehat{\xi M - T - \hat{A}}$ and $\widehat{\xi M - T}$. In fact, the following hold.

Case 1. Let us assume that $\widehat{\xi M - T - \hat{A}} \in \Phi^b(\mathcal{X}_T, \mathcal{X})$. Then, by (3.2), we have $(\widehat{\xi M - T})(I - T_{sc,M}^r \hat{A}) \in \Phi^b(\mathcal{X}_T, \mathcal{X})$. Therefore, one has $I - T_{sc,M}^r \hat{A} \in \Phi^b(\mathcal{X})$ with $\text{ind}(I - T_{sc,M}^r \hat{A}) = 0$, it follows that $(\widehat{\xi M - T}) \in \Phi^b(\mathcal{X}_T, \mathcal{X})$. That is, we obtain

$$\begin{aligned} \text{ind}(\widehat{\xi M - T - \hat{A}}) &= \text{ind}((\widehat{\xi M - T})(I - T_{sc,M}^r \hat{A})) \\ &= \text{ind}(I - T_{sc,M}^r \hat{A}) + \text{ind}(\widehat{\xi M - T}) \\ &= \text{ind}(\widehat{\xi M - T}). \end{aligned}$$

Case 2. If not, we have $\widehat{\xi M - T - \hat{A}} \in \Phi_r^b(\mathcal{X}_T, \mathcal{X}) \setminus \Phi^b(\mathcal{X}_T, \mathcal{X})$, then its index $\text{ind}(\widehat{\xi M - T - \hat{A}}) = +\infty$. Therefore, we infer that:

$$\text{ind}(\widehat{\xi M - T - \hat{A}}) = \text{ind}(\widehat{\xi M - T}) = +\infty,$$

as $\widehat{\xi M - T} \notin \Phi^b(\mathcal{X}_T, \mathcal{X})$. Consequently, from Remark 2.2, we have:

$$\xi M - T - A \in \Phi_-(\mathcal{X}) \quad (\text{resp. } \xi M - T - A \in \Phi_r(\mathcal{X})),$$

with $\text{ind}(\xi M - T - A) = \text{ind}(\xi M - T)$.

In what follows, we can deduce the stability results of the sets of right Weyl operators which asserts the stability result in term of essential spectra. $\square$

4. SPECTRAL DESCRIPTION OF SEMI FREDHOLM OPERATORS VIA MEASURE OF NON-STRICT SINGULARITY PERTURBATION

Definition 4.1 (Measure of non-strict singularity: M.S.S. [27]). Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. The strict singular measure of T , denoted by $\Delta(T)$ is defined as: $\Delta(T) := \sup_{N \subset M} \inf_{M \subset \mathcal{X}} \|T|_N\|$, where N, M denote infinite-dimensional subspaces of $\mathcal{X}$ and $T|_N$ denote the restriction of T with the subspace N , satisfying:

$$(4.1) \quad \Delta(T) = 0 \quad \text{if and only if} \quad T \in \mathcal{SS}(\mathcal{X}).$$

Theorem 4.1 (Stability of relative essential spectra via M.S.S). Let $T \in \mathcal{C}(\mathcal{X})$, $M \in \mathcal{L}(\mathcal{X})$ for which $M \neq 0$, and A be an T -bounded linear operator on $\mathcal{X}$.

Assume that, for $\xi \in \mathbb{C}$, there exists T_{ss}^l a left strictly singular Fredholm inverse of the operator pencils $\xi M - T$, such that:

$$\Delta(\widehat{A}(T_{ss}^l)^n) < 1, \quad \text{for some } n \geq 1.$$

Then, we have

(i) $\xi M - T \in \Phi_*(\mathcal{X})$ implies that $\xi M - T - A \in \Phi_*(\mathcal{X})$, with $\text{ind}(\xi M - T - A) = \text{ind}(\xi M - T)$,

(ii) $\xi M - T \in \mathcal{W}_l(\mathcal{X})$ implies that $\xi M - T - A \in \mathcal{W}_l(\mathcal{X})$.

Therefore, we get

$$\sigma_*(T + A, M) \subseteq \sigma_*(T, M),$$

for

$$(\Phi_*(\mathcal{X}), \sigma_*(\cdot, M)) \in \{(\Phi_+(\mathcal{X}), \sigma_{e+}(\cdot, M)), (\Phi_l(\mathcal{X}), \sigma_{el}(\cdot, M)), (\mathcal{W}_l(\mathcal{X}), \sigma_{wl}(\cdot, M))\}.$$

Proof. Obviously, while $T \in \mathcal{C}(\mathcal{X})$, $M \in \mathcal{L}(\mathcal{X})$ and A be an T -bounded linear operator on $\mathcal{X}$, we infer that the operators pencils $\xi M - T$ and $\xi M - T - A$ are closed with $\mathcal{D}(\xi M - T) = \mathcal{D}(T)$ and $\mathcal{D}(\xi M - T - A) = \mathcal{D}(T)$, respectively. Thus, the operators $\widehat{\xi M - T}$ and $\widehat{\xi M - T - A}$ are bounded from $\mathcal{X}_T$ into $\mathcal{X}$.

Consider $\xi \notin \sigma_*(T, M)$, then $\xi M - T \in \Phi_*(\mathcal{X})$. Thus, by Remark 2.2, we have $\widehat{\xi M - T} \in \Phi_*^b(\mathcal{X}_T, \mathcal{X})$. Thus, there exists an operator $S \in \mathcal{SS}(\mathcal{X}_T)$ such that

$$T_{ss}^l(\widehat{\xi M - T}) = I - S, \quad \text{on } \mathcal{X}_T.$$

Since A is an T -bounded linear operator on $\mathcal{X}$, then we conclude that $\widehat{A} \in \mathcal{L}(\mathcal{X}_T, \mathcal{X})$. Thus, we express the operator $\widehat{\xi M - T - A}$ as follows:

$$(4.2) \quad \widehat{\xi M - T - A} = (I - \widehat{A}T_{ss}^l)(\widehat{\xi M - T}) - \widehat{A}S.$$

One has $\Delta((\widehat{A}T_{ss}^l)^n) < 1$, for some $n \geq 1$, asserts that $I - \widehat{A}T_{ss}^l \in \Phi^b(\mathcal{X})$ with $i(I - \widehat{A}T_{ss}^l) = 0$, in view of the results given in [33]. According with Corollary 1.33 in [14] (resp. Theorem 2.5 in [20]), we obtain:

$$(I - \widehat{A}T_{ss}^l)(\widehat{\xi M - T}) \in \Phi_+^b(\mathcal{X}_T) \quad (\text{resp. } (I - \widehat{A}T_{ss}^l)(\widehat{\xi M - T}) \in \Phi_l^b(\mathcal{X}_T)).$$

As $\widehat{AS} \in \mathcal{SS}(\mathcal{X}_T, \mathcal{X}) \subset \mathcal{PH}(\mathcal{X}_T, \mathcal{X})$, we show in view of Corollary 1.37 in [14] (resp. Theorem 2.7 in [20]) that:

$$(I - \widehat{AT}_{ss}^l)(\widehat{\xi M - T}) - \widehat{AS} \in \Phi_+^b(\mathcal{X}_T, \mathcal{X}) \quad (\text{resp. } (I - \widehat{AT}_{ss}^l)(\widehat{\xi M - T}) - \widehat{AS} \in \Phi_l^b(\mathcal{X}_T, \mathcal{X})),$$

with $\text{ind}((I - \widehat{AT}_{ss}^l)(\widehat{\xi M - T}) - \widehat{AS}) = \text{ind}((I - \widehat{AT}_{ss}^l)(\widehat{\xi M - T}))$. Thus, we conclude that

$$\widehat{\xi M - T} - \widehat{A} \in \Phi_+^b(\mathcal{X}_T, \mathcal{X}) \quad (\text{resp. } \widehat{\xi M - T} - \widehat{A} \in \Phi_l^b(\mathcal{X}_T, \mathcal{X})),$$

with $\text{ind}(\widehat{\xi M - T} - \widehat{A}) = \text{ind}((I - \widehat{AT}_{ss}^l)(\widehat{\xi M - T}))$. Next, we will discuss two cases to prove that $\text{ind}(\widehat{\xi M - T} - \widehat{A}) = \text{ind}(\widehat{\xi M - T})$. Indeed, assume that $\widehat{\xi M - T} - \widehat{A} \in \Phi^b(\mathcal{X}_T, \mathcal{X})$. Then by (4.2), we have $(I - \widehat{AT}_{ss}^l)(\widehat{\xi M - T}) \in \Phi^b(\mathcal{X}_T, \mathcal{X})$. Consequently, one has $I - \widehat{AT}_{ss}^l \in \Phi^b(\mathcal{X})$ with $\text{ind}(I - \widehat{AT}_{ss}^l) = 0$, that is $(\widehat{\xi M - T}) \in \Phi^b(\mathcal{X}_T, \mathcal{X})$. Hence, we obtain

$$\begin{aligned} \text{ind}(\widehat{\xi M - T} - \widehat{A}) &= \text{ind}((I - \widehat{AT}_{ss}^l)(\widehat{\xi M - T})) \\ &= \text{ind}(I - \widehat{AT}_{ss}^l) + \text{ind}(\widehat{\xi M - T}) \\ &= \text{ind}(\widehat{\xi M - T}). \end{aligned}$$

If not, that is, if we have $\widehat{\xi M - T} - \widehat{A} \in \Phi_l^b(\mathcal{X}_T, \mathcal{X}) \setminus \Phi^b(\mathcal{X}_T, \mathcal{X})$, then its index $\text{ind}(\widehat{\xi M - T} - \widehat{A}) = -\infty$. Therefore,

$$\text{ind}(\widehat{\xi M - T} - \widehat{A}) = \text{ind}(\widehat{\xi M - T}) = -\infty,$$

since $\widehat{\xi M - T} \notin \Phi^b(\mathcal{X}_T, \mathcal{X})$. Consequently, by Remark 2.2, we have

$$\xi M - T - A \in \Phi_+(\mathcal{X}) \quad (\text{resp. } \xi M - T - A \in \Phi^l(\mathcal{X})),$$

with $\text{ind}(\xi M - T - A) = \text{ind}(\xi M - T)$. Thus, we deduce that $\xi M - T \in \mathcal{W}_l(\mathcal{X})$ implies $\xi M - T - A \in \mathcal{W}_l(\mathcal{X})$, which asserts that $\sigma_*(T + A, M) \subseteq \sigma_*(T, M)$, with

$$(\Phi_*(\mathcal{X}), \sigma_*(\cdot, M)) \in \{(\Phi_+(\mathcal{X}), \sigma_{e+}(\cdot, M)), (\Phi_l(\mathcal{X}), \sigma_{el}(\cdot, M)), (\mathcal{W}_l(\mathcal{X}), \sigma_{wl}(\cdot, M))\}.$$

□

5. RELATIVE ESSENTIAL SPECTRA OF MATRIX PENCIL BLOCK 2×2 VIA M.S.C AND M.S.S

The purpose of this section is to provide a new technique allowing to investigate some relative essential spectra of unbounded block 2×2 operator matrix pencil defined on maximal domains. This approach involves the concepts of measures of non-strict cosingularity (resp. non-strict singularity) perturbations within the theory of left strictly cosingular (resp. right strictly singular) Fredholm inverses. To this end, we consider unbounded linear operators A, B, C and D such that: A (resp. D) acts on the Banach space $\mathcal{X}$ and has domain $\mathcal{D}(A)$ (resp. $\mathcal{D}(D)$) and B (resp. C) is defined on the domain $\mathcal{D}(B)$ and acts on $\mathcal{X}$ and M_1, M_2, M_3 and M_4 are bounded linear

operators on $\mathcal{X}$. For these operators, we introduce in the product of Banach space $\mathcal{X} \times \mathcal{X}$, the unbounded 2×2 block operator matrix pencil $\xi\mathbb{M} - \mathbb{A}$ as follows:

$$\xi\mathbb{M} - \mathbb{A} := \xi \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \xi M_1 - A & \xi M_2 - B \\ \xi M_3 - C & \xi M_4 - D \end{pmatrix}, \quad \text{for } \xi \in \mathbb{C},$$

defined on its maximal domain

$$\begin{aligned} \mathcal{D}(\xi\mathbb{M} - \mathbb{A}) &= (\mathcal{D}(\xi M_1 - A) \cap \mathcal{D}(\xi M_3 - C)) \times (\mathcal{D}(\xi M_2 - B) \cap \mathcal{D}(\xi M_4 - D)) \\ &:= (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D)). \end{aligned}$$

The entries of the operator pencil $\xi\mathbb{M} - \mathbb{A}$ satisfy the following conditions.

(**H₁**) The linear operator D is densely defined and closed in X , with a non-empty M_4 resolvent set.

(**H₂**) The operator B satisfies $\mathcal{D}(D) \subset \mathcal{D}(B)$, for which $B(\xi M_1 - D)^{-1}$ is bounded on $\mathcal{X}$, for some (hence for all) $\xi \in \rho(D, M_4)$.

(**H₃**) The operator C is assumed to be densely defined on $\mathcal{X}$ satisfy that $(\xi M_4 - D)^{-1}C$ is a bounded operator on its domain, for some (hence for all) $\xi \in \rho(D, M_4)$.

(**H₄**) The Schur complement of the operator pencil $\xi\mathbb{M} - \mathbb{A}$ is supposed closable and defined as

$$S(\xi) := A - (\xi M_2 - B)(\xi M_4 - D)^{-1}(\xi M_3 - C), \quad \text{for some } \xi \in \rho(D, M_4),$$

on the dense set $\mathcal{D}(A) \cap \mathcal{D}(C)$ of $\mathcal{X}$.

We will denote its closure by $\overline{S}(\xi)$ as:

$$\overline{S}(\xi) := \overline{A - (\xi M_2 - B)(\xi M_4 - D)^{-1}(\xi M_3 - C)}, \quad \text{for some } \xi \in \rho(D, M_4).$$

For further proceed, put for $\xi \in \rho(D, M_4)$, the following operators:

$$\mathbf{U}(\xi) := B(\xi M_1 - D)^{-1} \in \mathcal{L}(\mathcal{X}),$$

$$\mathbf{L}(\xi) := \overline{(\xi M_1 - D)^{-1}C} \in \mathcal{L}(\mathcal{X}),$$

in view of the above assumptions and under the closed graph theorem.

Theorem 5.1 (Closure of the operator matrix pencil $\xi\mathbb{M} - \mathbb{A}$). *Assume that the hypotheses (**H_i**), $1 \leq i \leq 4$, are satisfied. Then, the operator pencil $\xi\mathbb{M} - \mathbb{A}$ is closable in $\mathcal{X} \times \mathcal{X}$, for some $\xi \in \rho(D, M_4)$.*

Moreover, for some (hence for all) $\xi \in \rho(D, M_4)$, its closure is denoted by $\xi\mathbb{M} - \overline{\mathbb{A}}$ and described by the following relation:

$$(5.1) \quad \xi\mathbb{M} - \overline{\mathbb{A}} := \begin{pmatrix} I & \mathbf{U}(\xi) \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi M_1 - \overline{S}(\xi) & 0 \\ 0 & \xi M_4 - D \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathbf{L}(\xi) & I \end{pmatrix} = \mathbb{P}(\xi)\mathbb{V}_\xi(\xi)\mathbb{Q}(\xi).$$

Proof. For $\xi \in \rho(D, M_4)$, we compute our matrix model $\xi\mathbb{M} - \mathbb{A}$ as follows:

$$(5.2) \quad \xi\mathbb{M} - \mathbb{A} := \begin{pmatrix} I & \mathbf{U}(\xi) \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi M_1 - S(\xi) & 0 \\ 0 & \xi M_4 - D \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathbf{L}(\xi) & I \end{pmatrix},$$

inspired by the Frobenius-Schur form invested by Issai Schur (see [19]).

Therefore, as

$$\mathbb{P}(\xi)^{-1} := \begin{pmatrix} I & -\mathbf{U}(\xi) \\ 0 & I \end{pmatrix} \in \mathcal{L}(\mathcal{X} \times \mathcal{X}) \quad \text{and} \quad \mathbb{Q}(\xi)^{-1} := \begin{pmatrix} I & 0 \\ -\mathbf{L}(\xi) & I \end{pmatrix} \in \mathcal{L}(\mathcal{X} \times \mathcal{X}),$$

we conclude that $\widetilde{\mathbb{V}}_\xi(\xi) := \mathbb{P}(\xi)^{-1}(\xi\mathbb{M} - \mathbb{A})\mathbb{Q}(\xi)^{-1}$. Hence, $\widetilde{\mathbb{V}}_\xi(\xi)$ is closable if and only if $\xi M_1 - S(\xi)$ is closable if and only if $\xi\mathbb{M} - \mathbb{A}$ is closable if and only if $\mathbb{A}$ is closable too. This yields our required result while $\mathbb{M}$ is a bounded operator matrix. $\square$

5.1. Insights into some sets of semi-Fredholm operators and relative essential spectra by M.S.C. In this subsection, we investigate how perturbation criteria.. linked to the notion of measure of non-strict cosingularity is intended to provide new insights into the analysis properties of lower (right) semi-Fredholm and relative essential spectra of operator matrix pencil block 2×2 . To reach such behavior, we start firstly by proving our main outcome linked to measure of non-strict cosingularity of operator matrix framework.

Lemma 5.1 (On M.S.C of operator matrix block 2×2). *Consider the bounded matrix operator*

$$\mathbb{G} := \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}, \quad \text{for given } (G_i)_{1 \leq i \leq 4} \in \mathcal{L}(\mathcal{X}).$$

We define a measure of non-strict cosingularity associated with the operator matrix $\mathbb{G}$, as well

$$\mathcal{M}_{sc}(\mathbb{G}) := \max \{ \nabla(G_1) + \nabla(G_2), \nabla(G_3) + \nabla(G_4) \}.$$

Proof. The proof may be checked in the following steps.

Step 1. Proof of the subadditivity: $\mathcal{M}_{sc}(\mathbb{G} + \widetilde{\mathbb{G}}) \leq \mathcal{M}_{sc}(\mathbb{G}) + \mathcal{M}_{sc}(\widetilde{\mathbb{G}})$, for two bounded operator matrices $\mathbb{G}$ and $\widetilde{\mathbb{G}}$.

Consider two bounded operator matrices $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ as the following form:

$$\mathbb{G} := \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \quad \text{and} \quad \widetilde{\mathbb{G}} := \begin{pmatrix} \widetilde{G}_1 & \widetilde{G}_2 \\ \widetilde{G}_3 & \widetilde{G}_4 \end{pmatrix}.$$

A short computation reveals that:

$$\begin{aligned} \mathcal{M}_{sc}(\mathbb{G} + \widetilde{\mathbb{G}}) &= \max \{ \nabla(G_1 + \widetilde{G}_1) + \nabla(G_2 + \widetilde{G}_2), \nabla(G_3 + \widetilde{G}_3) + \nabla(G_4 + \widetilde{G}_4) \} \\ &\leq \max \{ \nabla(G_1) + \nabla(\widetilde{G}_1) + \nabla(G_2) + \nabla(\widetilde{G}_2), \\ &\quad \nabla(G_3) + \nabla(\widetilde{G}_3) + \nabla(G_4) + \nabla(\widetilde{G}_4) \} \\ &\leq \max \{ \nabla(G_1) + \nabla(G_2), \nabla(G_3) + \nabla(G_4) \} \\ &\quad + \max \{ \nabla(\widetilde{G}_1) + \nabla(\widetilde{G}_2), \nabla(\widetilde{G}_3) + \nabla(\widetilde{G}_4) \}. \end{aligned}$$

Hence, we conclude that

$$\mathcal{M}_{sc}(\mathbb{G} + \widetilde{\mathbb{G}}) \leq \mathcal{M}_{sc}(\mathbb{G}) + \mathcal{M}_{sc}(\widetilde{\mathbb{G}}), \quad \text{for all } (\mathbb{G}, \widetilde{\mathbb{G}}) \in \mathcal{L}^2(\mathcal{X} \times \mathcal{X}).$$

Step 2. Homogeneity: $\mathcal{M}_{sc}(\nu\mathbb{G}) = |\nu|\mathcal{M}_{sc}(\mathbb{G})$, for a scalar ν and a bounded operator matrix $\mathbb{G}$.

Let $\nu \in \mathbb{C}$ and $\mathbb{G} := \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \in \mathcal{L}(\mathcal{X} \times \mathcal{X})$. Then, we get

$$\begin{aligned} \mathcal{M}_{sc}(\nu\mathbb{G}) &:= \max \{ \nabla(\nu G_1) + \nabla(\nu G_2), \nabla(\nu G_3) + \nabla(\nu G_4) \} \\ &= \max \{ |\nu| \nabla(G_1) + |\nu| \nabla(G_2), |\nu| \nabla(G_3) + |\nu| \nabla(G_4) \} \\ &= |\nu| \mathcal{M}_{sc}(\mathbb{G}). \end{aligned}$$

Step 3. We prove that $\mathcal{M}_{sc}(\mathbb{G}) = 0$ if and only if $\nabla(G_i) = 0$, for $1 \leq i \leq 4$,

$$\begin{aligned} \mathcal{M}_{sc}(\mathbb{G}) = 0 &\Leftrightarrow \max \{ \nabla(G_1) + \nabla(G_2), \nabla(G_3) + \nabla(G_4) \} = 0 \\ &\Leftrightarrow \nabla(G_i) = 0, \quad \text{for all } 1 \leq i \leq 4. \end{aligned}$$

Keeping into account from the properties of the measure of non-strict cosingularity, we infer that

$$\mathcal{M}_{sc}(\mathbb{G}) = 0 \Leftrightarrow \nabla(G_i) = 0 \Leftrightarrow G_i \in \mathcal{SC}(\mathcal{X}), \quad \text{for all } 1 \leq i \leq 4. \quad \square$$

In what follows, we derive from the above steps that $\mathcal{M}_{sc}(\cdot)$ is a measure of non-strict cosingularity on $\mathcal{X} \times \mathcal{X}$ associated to the bounded operator matrix.

Before further proceed, the following proposition will be required as a follow-up.

Proposition 5.1 (Left-right Fredholm inverse of operator matrix). *Let $D_1 \in \mathcal{C}(\mathcal{X})$, $D_2 \in \mathcal{C}(\mathcal{Y})$ and considered the operator matrix*

$$\mathbb{M}_d := \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Assume that:

- (i) D_1^l (resp. D_1^r) is a left (resp. right) Fredholm inverse of D_1 ,
- (ii) D_2^l (resp. D_2^r) is a left (resp. right) Fredholm inverse of D_2 .

Then, we have a left (resp. right) Fredholm inverse of $\mathbb{M}_d$ given by

$$\mathbb{M}_d^* := \begin{pmatrix} D_1^* & 0 \\ 0 & D_2^* \end{pmatrix},$$

for $(\mathbb{M}_d^, D_i^*) \in (\mathbb{M}_d^l, D_i^l)$ (resp. $(\mathbb{M}_d^*, D_i^*) \in (\mathbb{M}_d^r, D_i^r)$), $i = 1, 2$. Moreover, if $\mathbb{M}_l$ and $\mathbb{M}_r$ are bounded operators with bounded inverses, we get*

$$(5.3) \quad \mathbb{M}_l^{-1} \mathbb{M}_d^l \mathbb{M}_r^{-1} \quad (\text{resp. } \mathbb{M}_l^{-1} \mathbb{M}_d^r \mathbb{M}_r^{-1})$$

is a left (resp. right) Fredholm inverse of $\mathbb{M}_r \mathbb{M}_d^l \mathbb{M}_l$ (resp. $\mathbb{M}_r \mathbb{M}_d^r \mathbb{M}_l$).

Proof. Obviously, a short computation reveals that our required statements follows from the use of Definition 2.7. $\square$

Theorem 5.2 (Fredholmness properties of matrix pencil via M.S.C). *Assume that the conditions $(\mathbf{H}_1)$ – $(\mathbf{H}_4)$ are fulfilled, for some (hence for all) $\xi \in \rho(D, M_4)$. If, for $\tau \in \mathbf{D}(\xi, 1)$, the following assertion holds:*

- (i) $M_i \in \mathcal{K}(\mathcal{X})$, $i = 2, 3$,

- (ii) $\|\mathbf{L}(\xi)\| < 1$,
 (iii) there exist $S_{sc}^r(\tau)$ and $D_{sc}^r(\tau)$ a right strictly cosingular Fredholm inverses of $\tau M_1 - \bar{S}(\xi)$ and $\tau M_4 - D$, respectively, satisfying $\nabla(S_{sc}^r(\tau)\mathbf{U}(\xi)M_4) < \frac{1}{2}$ and $\nabla(D_{sc}^r(\tau)M_4\mathbf{L}(\xi)) < \frac{1}{2}$.

Then, we get

$$\tau\mathbb{M} - \bar{\mathbb{A}} \in \Phi_*(\mathcal{X} \times \mathcal{X}), \quad \text{for } \Phi_*(\mathcal{X} \times \mathcal{X}) := \{\Phi_-(\mathcal{X} \times \mathcal{X}), \Phi_r(\mathcal{X} \times \mathcal{X})\},$$

with $\text{ind}(\tau\mathbb{M} - \bar{\mathbb{A}}) = \text{ind}(\tau M_1 - \bar{S}(\xi)) + \text{ind}(\tau M_4 - D)$.

Proof. Let $\xi \in \rho(D, M_4)$ and $\tau \in \mathbf{D}(\xi, 1)$. Based on the Frobenius-Schur decomposition of our considered operator matrix pencil given by (5.1), we use the following factorization of the operator matrix $\tau\mathbb{M} - \bar{\mathbb{A}}$ which is rewritten as follows:

$$(5.4) \quad \tau\mathbb{M} - \bar{\mathbb{A}} := \mathbb{P}(\xi)\mathbb{V}_\xi(\tau)\mathbb{Q}(\xi) + (\tau - \xi)\mathbb{R}(\xi)$$

$$(5.5) \quad := \mathbb{T}(\tau) + (\tau - \xi)(\mathbb{R}_1(\xi) + \mathbb{R}_2(\xi)),$$

where

$$\mathbb{T}(\tau) := \mathbb{P}(\xi)\mathbb{V}_\xi(\tau)\mathbb{Q}(\xi),$$

$$\mathbb{R}_1(\xi) := \begin{pmatrix} \mathbf{U}(\xi)M_4\mathbf{L}(\xi) & \mathbf{U}(\xi)M_4 \\ M_4\mathbf{L}(\xi) & 0 \end{pmatrix},$$

$$\mathbb{R}_2(\xi) := \begin{pmatrix} 0 & -M_2 \\ -M_3 & 0 \end{pmatrix}.$$

To complete the proof of our results, we shall proceed relying on the following steps.

Step 1. Our purpose is to construct a right strictly cosingular Fredholm inverse of the matrix operator $\mathbb{T}(\tau)$.

In fact to this, imply that $S_{sc}^r(\tau)$ and $D_{sc}^r(\tau)$ are right strictly cosingular Fredholm inverses of $\tau M_1 - \bar{S}(\xi)$ and $\tau M_4 - D$, respectively. Thus, Proposition 5.1 yields that the matrix operator

$$\mathbb{V}_{sc}^r(\tau) := \begin{pmatrix} S_{sc}^r(\tau) & 0 \\ 0 & D_{sc}^r(\tau) \end{pmatrix}$$

is a right strictly cosingular Fredholm inverse of $\mathbb{V}_\xi(\tau)$. Therefore, as $\mathbb{P}(\xi)$ and $\mathbb{Q}(\xi)$ are two bounded invertible with bounded inverses operators, we conclude that

$$\begin{aligned} \mathbb{T}_{sc}^r(\tau) &:= \mathbb{Q}^{-1}(\xi)\mathbb{V}_{sc}^r(\tau)\mathbb{P}^{-1}(\xi) \\ &= \begin{pmatrix} S_{sc}^r(\tau) & -S_{sc}^r(\tau)\mathbf{U}(\xi) \\ -\mathbf{L}(\xi)S_{sc}^r(\tau) & \mathbf{L}(\xi)S_{sc}^r(\tau)\mathbf{U}(\xi) + D_{sc}^r(\tau) \end{pmatrix} \end{aligned}$$

is a right strictly cosingular Fredholm inverse of $\mathbb{T}(\tau)$, from Proposition 5.1.

Step 2. We will proof that the inequality $\mathcal{M}_{sc}((\tau - \xi)\mathbb{T}_{sc}^r(\tau)\mathbb{R}_1(\xi)) < 1$, holds for all $\xi \in \rho(D, M_4)$ and $\tau \in \mathbf{D}(\xi, 1)$.

Indeed, let $\xi \in \rho(D, M_4)$ and $\tau \in \mathbf{D}(\xi, 1)$. Using the above expression of the operator $\mathbb{T}_{sc}^r(\tau)\mathbb{R}_1(\xi)$

$$\mathbb{T}_{sc}^r(\tau)\mathbb{R}_1(\xi) := \begin{pmatrix} 0 & S_{sc}^r(\tau)\mathbf{U}(\xi)M_4 \\ D_{sc}^r(\tau)M_4\mathbf{L}(\xi) & \mathbf{L}(\xi)S_{sc}^r(\tau)\mathbf{U}(\xi)M_4 \end{pmatrix},$$

and applying Lemma 5.1, we show that

$$\begin{aligned} \mathcal{M}_{sc}(\mathbb{T}_{sc}^r(\tau)\mathbb{R}_1(\xi)) &= \max \{ \nabla(S_{sc}^r(\tau)\mathbf{U}(\xi)M_4), \\ &\quad \nabla(D_{sc}^r(\tau)M_4\mathbf{L}(\xi)) + \nabla(\mathbf{L}(\xi)S_{sc}^r(\tau)\mathbf{U}(\xi)M_4) \} \\ &\leq \max \{ \nabla(S_{sc}^r(\tau)\mathbf{U}(\xi)M_4), \\ &\quad \nabla(D_{sc}^r(\tau)M_4\mathbf{L}(\xi)) + \|\mathbf{L}(\xi)\| \nabla(S_{sc}^r(\tau)\mathbf{U}(\xi)M_4) \}. \end{aligned}$$

Hence, according with our assumptions we obtain that

$$\mathcal{M}_{sc}((\tau - \xi)\mathbb{T}_{sc}^r(\tau)\mathbb{R}_1(\xi)) < 1.$$

By leveraging our required result within the context of semi Fredholm operator framework, regarding Theorem 3.1, we arrive at the conclusion in view of the above-mentioned steps that $\mathbb{T}(\tau) + (\tau - \xi)\mathbb{R}_1(\xi) \in \Phi_*(\mathcal{X} \times \mathcal{X})$ for $\Phi_*(\mathcal{X} \times \mathcal{X}) := \{\Phi_-(\mathcal{X} \times \mathcal{X}), \Phi_r(\mathcal{X} \times \mathcal{X})\}$ with $\text{ind}(\mathbb{T}(\tau) + (\tau - \xi)\mathbb{R}_1(\xi)) = \text{ind}(\mathbb{T}(\tau))$. Consequently, item (i) proves that $\mathbb{R}_2(\xi) \in \mathcal{K}(\mathcal{X} \times \mathcal{X})$. Thus, we conclude that

$$\mathbb{T}(\tau) + (\tau - \xi)\mathbb{R}_1(\xi) + (\tau - \xi)\mathbb{R}_2(\xi) := \tau\mathbb{M} - \overline{\mathbb{A}} \in \Phi_*(\mathcal{X} \times \mathcal{X}),$$

with $\text{ind}(\tau\mathbb{M} - \overline{\mathbb{A}}) = \text{ind}(\mathbb{A}(\tau))$, for $\Phi_*(\mathcal{X} \times \mathcal{X}) := \{\Phi_-(\mathcal{X} \times \mathcal{X}), \Phi_r(\mathcal{X} \times \mathcal{X})\}$.

Moreover, as $\mathbb{P}(\xi)$ and $\mathbb{Q}(\xi)$ are two bounded operators with bounded inverses, we deduce that the index of $\mathbb{T}(\tau)$, $\text{ind}(\mathbb{T}(\tau))$ is equal to the index of the diagonal operators entries of the matrix pencil $\mathbb{V}_\xi(\tau)$, that is

$$\text{ind}(\mathbb{T}(\tau)) = \text{ind}(\xi M_1 - \overline{S}(\tau)) + \text{ind}(\xi M_4 - D).$$

This completes the proof. $\square$

As an immediate consequence of Theorem 5.2, we obtain the following corollary concerning the relative essential spectra of operator matrix pencil block 2×2 linked to more complex perturbation criteria than the classical perturbation criteria for Fredholm operators.

Corollary 5.1 (On relative essential spectra of matrix pencil by M.S.C). *Assume that the conditions $(\mathbf{H}_1)$ - $(\mathbf{H}_4)$ are fulfilled, for some (hence for all) $\xi \in \rho(D, M_4)$. If, for $\tau \in \mathbf{D}(\xi, 1)$, the following assertion holds:*

- (i) $M_i \in \mathcal{K}(\mathcal{X})$, $i = 2, 3$,
- (ii) $\|\mathbf{L}(\xi)\| < 1$,
- (iii) *there exist $S_{sc}^r(\tau)$ and $D_{sc}^r(\tau)$ a right strictly cosingular Fredholm inverses of $\tau M_1 - \overline{S}(\xi)$ and $\tau M_4 - D$, respectively, satisfying $\nabla(S_{sc}^r(\tau)\mathbf{U}(\xi)M_4) < \frac{1}{2}$ and $\nabla(D_{sc}^r(\tau)M_4\mathbf{L}(\xi)) < \frac{1}{2}$.*

Then, we have

$$\sigma_*(\overline{\mathbb{A}}, \mathbb{M}) \cup {}^C\{\tau : \tau \in \mathbf{D}(\xi, 1)\} \subset \sigma_*(\overline{S}(\xi), M_1) \cup \sigma_*(D, M_4) \cup {}^C\{\tau : \tau \in \mathbf{D}(\xi, 1)\},$$

where $\sigma_*(\cdot, M_k) := \{\sigma_{e-}(\cdot, M_k), \sigma_{er}(\cdot, M_k), \sigma_{wr}(\cdot, M_k)\}$, $k \in \{1, 4\}$.

Proof. Our required claims may be inferred from the use of Theorem 5.2 with Definition 2.8. $\square$

In what follows, we ask the following question to reach the applicability of our investigations.

Question. Can you find some other arguments of perturbations which cover the value of our analytic properties done in Theorem 5.2 and Corollary 5.1?

Namely, it is worthy to point out that our spectral properties shown in Theorem 5.2 and Corollary 5.1 still preserved in the presence of strictly cosingular arguments of perturbations, particularly, under the following criteria $(\mathbf{L}(\xi), \mathbf{U}(\xi)) \in \mathcal{SC}^2(\mathcal{X})$, for $\xi \in \rho(D, M_4)$ with $\tau \in \mathbf{D}(\xi, 1)$, which provides an affirmative answer to this question. Additionally, further criteria of perturbations in term of strictly singular operators linked to particular space of superprojective will be derived subsequently to answer the last posed question. That is, if we change the above assumptions

$\curvearrowright$ the Banach space $\mathcal{X}$,

$\curvearrowright \nabla(\mathbf{L}(\xi)) < 1$,

$\curvearrowright \nabla(S_{sc}^r(\tau)\mathbf{U}(\xi)M_4) < \frac{1}{2}$,

$\curvearrowright \nabla(D_{sc}^r(\tau)M_4\mathbf{L}(\xi)) < \frac{1}{2}$,

by

$\curvearrowright$ the superprojective weakly compact generating Banach space $\mathcal{X}$,

$\curvearrowright \mathbf{L}(\xi) \in \mathcal{SS}(\mathcal{X})$,

$\curvearrowright S_{sc}^r(\tau)\mathbf{U}(\xi)M_4 \in \mathcal{SS}(\mathcal{X})$, for $\xi \in \rho(D, M_4)$ such that $\tau \in \mathbf{D}(\xi, 1)$.

5.2. Insights into sets of semi-Fredholm operators and relative essential spectra by a M.S.S. In this subsequent part of this work, we are interested to provide new insights into the description of upper (left) semi-Fredholm and relative essential spectra of operator matrix pencil block 2×2 . To show such behavior, we first introduce the following result.

Lemma 5.2 (On M.S.S of operator matrix). *Consider the bounded matrix operator*

$$\mathbb{G} := \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}, \quad \text{for given } (G_i)_{1 \leq i \leq 4} \in \mathcal{L}(\mathcal{X}).$$

We define a measure of non-strict singularity associated with the operator matrix $\mathbb{G}$, as well

$$\mathcal{M}_{ss}(\mathbb{G}) := \max \{ \Delta(G_1) + \Delta(G_2), \Delta(G_3) + \Delta(G_4) \}.$$

Proof. The proof is analogous to that of Lemma 5.1, for further details, we refer the reader to [11]. $\square$

Our first result of this subsection will be presented as follows.

Theorem 5.3 (Upper and Left semi-Fredholm operator matrix pencils block 2×2). *Let the conditions $(\mathbf{H}_1)$ – $(\mathbf{H}_4)$ are fulfilled, for some (hence for all) $\xi \in \rho(D, M_4)$. Assume that, for $\tau \in \mathbf{D}(\xi, 1)$,*

- (i) $M_i \in \mathcal{K}(\mathcal{X})$, $i = 2, 3$,
 - (ii) $\Delta(\mathbf{U}(\xi)) < 1$,
 - (iii) $S_{ss}^l(\tau)$ and $D_{ss}^l(\tau)$ a left strictly singular Fredholm inverse of $\tau M_1 - \bar{S}(\xi)$ and $\tau M_4 - D$, respectively, such that $\Delta(M_4 \mathbf{L}(\xi) S_{ss}^l(\tau)) < \frac{1}{2}$ and $\Delta(\mathbf{U}(\xi) M_4 D_{ss}^l(\tau)) < 1$.
- Then, we get*

$$\tau \mathbb{M} - \bar{\mathbb{A}} \in \Phi_*(\mathcal{X} \times \mathcal{X}), \quad \text{for } \Phi_*(\mathcal{X} \times \mathcal{X}) := \{\Phi_+(\mathcal{X} \times \mathcal{X}), \Phi_l(\mathcal{X} \times \mathcal{X})\},$$

with $\text{ind}(\tau \mathbb{M} - \bar{\mathbb{A}}) = \text{ind}(\tau M_1 - \bar{S}(\xi)) + \text{ind}(\tau M_4 - D)$.

Proof. Let $\xi \in \rho(D, M_4)$ and $\tau \in \mathbf{D}(\xi, 1)$. In order to achieve our goal, we will start to show that the operator matrix $\mathbb{T}(\tau)$ given by (5.5) has a left strictly singular Fredholm inverse.

Indeed, taking into account from (5.4)–(5.5) and Proposition 5.1, one has

$$\begin{aligned} \mathbb{T}_{ss}^l(\tau) &:= \mathbb{Q}^{-1}(\xi) \mathbb{V}_{ss}^l(\tau) \mathbb{P}^{-1}(\xi) \\ &= \begin{pmatrix} S_{ss}^l(\tau) & -S_{ss}^l(\tau) \mathbf{U}(\xi) \\ -\mathbf{L}(\xi) S_{ss}^l(\tau) & \mathbf{L}(\xi) S_{ss}^l(\tau) \mathbf{U}(\xi) + D_{ss}^l(\tau) \end{pmatrix} \end{aligned}$$

is a left strictly singular Fredholm inverse of $\mathbb{T}(\tau)$, while $\mathbb{P}(\xi)$ and $\mathbb{Q}(\xi)$ are two bounded with bounded inverses operators and $S_{ss}^l(\tau)$ (resp. $D_{ss}^l(\tau)$) is a left strictly singular Fredholm inverse of $\xi M_1 - \bar{S}(\tau)$ (resp. $\xi M_4 - D$) and $\mathbb{V}_{ss}^l(\tau)$ is a left strictly singular Fredholm inverse of $\mathbb{V}_\xi(\tau)$. Therefore, the operator $\mathbb{R}_1(\xi) \mathbb{T}_{ss}^l(\tau)$ is expressed as follows:

$$\mathbb{R}_1(\xi) \mathbb{T}_{ss}^l(\tau) := \begin{pmatrix} 0 & \mathbf{U}(\xi) M_4 D_{ss}^l(\tau) \\ M_4 \mathbf{L}(\xi) S_{ss}^l(\tau) & -M_4 \mathbf{L}(\xi) S_{ss}^l(\tau) \mathbf{U}(\xi) \end{pmatrix}.$$

According with the properties of the measure of non-strict singularity provided in [11] and Lemma 5.2 (i), we conclude that

$$\begin{aligned} \mathcal{M}_{ss}(\mathbb{R}_1(\xi) \mathbb{T}_{ss}^l(\tau)) &= \max \left\{ \Delta(\mathbf{U}(\xi) M_4 D_{ss}^l(\tau)), \right. \\ &\quad \left. \Delta(M_4 \mathbf{L}(\xi) S_{ss}^l(\tau)) + \Delta(-M_4 \mathbf{L}(\xi) S_{ss}^l(\tau) \mathbf{U}(\xi)) \right\} \\ &\leq \max \left\{ \Delta(\mathbf{U}(\xi) M_4 D_{ss}^l(\tau)), \Delta(M_4 \mathbf{L}(\xi) S_{ss}^l(\tau)) [1 + \Delta(\mathbf{U}(\xi))] \right\}. \end{aligned}$$

Therefore, under our assumptions, we conclude that

$$\mathcal{M}_{ss} \left\{ (\tau - \xi) \mathbb{R}_1(\xi) \mathbb{T}_{ss}^l(\tau) \right\} < 1.$$

Consequently, Theorem 4.1 asserts that

$$\mathbb{T}(\tau) + (\tau - \xi) \mathbb{R}_1(\xi) \in \Phi_*(\mathcal{X} \times \mathcal{X}), \quad \text{for } \Phi_*(\mathcal{X} \times \mathcal{X}) := \{\Phi_+(\mathcal{X} \times \mathcal{X}), \Phi_l(\mathcal{X} \times \mathcal{X})\},$$

with

$$\begin{aligned}\operatorname{ind}(\mathbb{T}(\nu) + (\tau - \xi)\mathbb{R}_1(\xi)) &= \operatorname{ind}(\mathbb{T}(\tau)) = \operatorname{ind}(\mathbb{V}_\xi(\tau)) \\ &= \operatorname{ind}(\xi M_1 - \overline{S}(\tau)) + \operatorname{ind}(\xi M_4 - D),\end{aligned}$$

while $\mathbb{V}_\xi(\tau)$ is a diagonal operator matrix, $\mathbb{P}(\xi)$ and $\mathbb{Q}(\xi)$ are two bounded operators with bounded inverses.

As $\mathbb{R}_2(\xi) \in \mathcal{K}(\mathcal{X} \times \mathcal{X})$, from item **(i)**, it follows that

$$\mathbb{T}(\tau) + (\tau - \xi)\mathbb{R}_1(\xi) + (\tau - \xi)\mathbb{R}_2(\xi) = \tau\mathbb{M} - \overline{\mathbb{A}} \in \Phi_*(\mathcal{X} \times \mathcal{X}),$$

with $\operatorname{ind}(\tau\mathbb{M} - \overline{\mathbb{A}}) = \operatorname{ind}(\mathbb{T}(\tau)) = \operatorname{ind}(\xi M_1 - \overline{S}(\tau)) + \operatorname{ind}(\xi M_4 - D)$, for $\Phi_*(\mathcal{X} \times \mathcal{X}) := \{\Phi_+(\mathcal{X} \times \mathcal{X}), \Phi_l(\mathcal{X} \times \mathcal{X})\}$, this proves our required claim. $\square$

In view of the preceding results, Theorem 5.3 allows us to establish some relative essential spectral inclusions of the operator matrix pencil in terms of its Schur complement linked to the notion of a measure of non-strict singularity.

Corollary 5.2 (On relative essential spectra of matrix pencil block 2×2 via M.S.S). *Let the conditions $(\mathbf{H}_1)$ – $(\mathbf{H}_4)$ be fulfilled, for some (hence for all) $\xi \in \rho(D, M_4)$. Assume that, for $\tau \in \mathbf{D}(\xi, 1)$,*

- (i)** $M_i \in \mathcal{K}(\mathcal{X})$, $i = 2, 3$,
- (ii)** $\Delta(\mathbf{U}(\xi)) < 1$,
- (iii)** $S_{ss}^l(\tau)$ and $D_{ss}^l(\tau)$ a left strictly singular Fredholm inverse of $\tau M_1 - \overline{S}(\xi)$ and $\tau M_4 - D$, respectively, such that $\Delta(M_4 \mathbf{L}(\xi) S_{ss}^l(\tau)) < \frac{1}{2}$ and $\Delta(\mathbf{U}(\xi) M_4 D_{ss}^l(\tau)) < 1$.

Then, we get

$$\sigma_*(\overline{\mathbb{A}}, \mathbb{M}) \cup {}^C \{\tau : \tau \in \mathbf{D}(\xi, 1)\} \subset \sigma_*(\overline{S}(\xi), M_1) \cup \sigma_*(D, M_4) \cup {}^C \{\tau : \tau \in \mathbf{D}(\xi, 1)\},$$

where $\sigma_*(\cdot, M_k) := \{\sigma_{e+}(\cdot, M_k), \sigma_{el}(\cdot, M_k), \sigma_{wl}(\cdot, M_k)\}$, $k \in \{1, 4\}$.

Proof. The desired results follow from Theorem 5.3 and Definition 2.8. $\square$

To illustrate the scope of our theoretical results, we consider the following problem.

Question. Are the analytic properties established in Theorem 5.3 and Corollary 5.2 preserved under other, simpler perturbation criteria?

Subsequently, a positive answer to the last proposed question will be derived in the presence of some particular spaces likewise: subprojective space linked to other criteria of perturbations such as: strictly singular and strictly cosingular operators. More precisely, our descriptive results shown in Theorem 5.3 and Corollary 5.2 are obviously trivial for particular example of perturbations linked to strictly singular operators (that is if $(\mathbf{U}(\xi), \mathbf{L}(\xi)) \in \mathcal{SS}^2(\mathcal{X})$, for $\xi \in \rho(D, M_4)$ with $\tau \in \mathbf{D}(\xi, 1)$) as well as are also true in the following considerations. That is, if the above assumptions:

- $\curvearrowright$ the Banach space $\mathcal{X}$,
- $\curvearrowright$ $\Delta(\mathbf{U}(\xi)) < 1$,
- $\curvearrowright$ $\Delta(M_4 \mathbf{L}(\xi) S_{ss}^l(\tau)) < \frac{1}{2}$,
- $\curvearrowright$ $\Delta(\mathbf{U}(\xi) M_4 D_{ss}^l(\tau)) < 1$,

will be changed by

↪ a superprojective, weakly compactly generated Banach space $\mathcal{X}$,

↪ $\mathbf{U}(\xi) \in \mathcal{SC}(\mathcal{X})$,

↪ $M_4 \mathbf{L}(\xi) S_{ss}^l(\tau) \in \mathcal{SC}(\mathcal{X})$,

for $\xi \in \rho(D, M_4)$ such that $\tau \in \mathbf{D}(\xi, 1)$.

At the end of this paper, we summarize our main contribution in the following conclusion.

6. CONCLUSION

This paper provides a comprehensive analysis linked to new concept of Fredholm perturbations, which is used to develop some spectral inclusions in terms of some types of M-essential spectra of perturbed linear operator pencils. Instead, our analysis allows us to derive several interesting results in operator theory, particularly within the framework of 2×2 block operator matrix pencils. The measures of non-strict cosingularity and non-strict singularity have been used to establish sufficient criteria, develop new arguments and propose a novel methodology for deriving some relative essential spectral inclusions between 2×2 block operator matrix pencils and these entries, enriching the theoretical landscape of operator matrices pencils.

A natural question arising from our main results is whether it is possible to identify new and more general perturbation criteria.

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