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CHEN-LIKE INEQUALITIES ON LIGHTLIKE HYPERSURFACE OF A LORENTZIAN PRODUCT MANIFOLD WITH QUARTER-SYMMETRIC NONMETRIC CONNECTION

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Abstract. In this paper, we introduce k-Ricci curvature and k-scalar curvature on lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection. Using these curvatures, we establish some Chen-type inequalities for lighlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection. Considering the equality case, we obtain some results.

1. Introduction

In [16], Golab introduced the idea of a quarter-symmetric linear connections in a differential manifold. Later, the properties of Riemannian manifolds with quarter-symmetric metric (nonmetric) connection have been studied by some authours [19,24].

Warped products were first defined by Bishop and O'Neill in [6]. In [2], Atçeken and Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [20], Kılıç and Oğuzhan considered lightlike hypersurfaces with respect to a quarter-symmetric nonmetric connection which is determined by the product structure. They also gave some equivalent conditions for integrability of disributions with respect to the Levi-Civita connection of semi-Riemannian manifold and the quarter-symmetric nonmetric connection, and obtained some results.

In 1993, B. Y. Chen [9] introduced a new Riemannian invariant for a Riemannian manifold M as follows:

$$\delta_M(p) = \tau(p) - \inf(K)(p),$$

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where $\tau(p)$ is scalar curvature of M and

$$\inf(K)(p) = \inf\{K(\Pi) : K(\Pi) \text{ is a plane section of } T_pM\}.$$

In [9], B. Chen established a sharp inequality for submanifolds in a real space form involving δ_M and the main extrinsic invariant, namely the squared mean curvature.

Afterwards, B. Y. Chen and some geometers studied similar problems for non-degenerate submanifolds of different spaces such as in [8, 9, 17, 28]. Later, Mihai and Özgür in [22] proved Chen inequalities for submanifolds of real space forms endowed with a semi-symmetric metric connection.

In degenerate submanifolds, M. Gülbahar, E. Kılıç and S. Keleş introduced k-Ricci curvature, k-scalar curvature, k-degenerate Ricci curvature, k-degenerate scalar curvature and they established some inequalities that characterize lightlike hypersurface of a Lorentzian manifold in [17]. After, they established some inequalities involving k-Ricci curvature, k-scalar curvature, the screen scalar curvature on a screen homothetic lightlike hypersurface of a Lorentzian manifold and they computed Chen-Ricci inequality and Chen inequality on a screen homothetic lightlike hypersurface of a Lorentzian manifold in [18].

In this paper, we study Chen-type inequalities for screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection. Considering these inequalities, we obtain the relation between Ricci curvature and scalar curvature endowed with the quarter-symmetric nonmetric connection.

2. Preliminaries

Let M be a hypersurface of an $(n+1)-$dimensional, $n > 1$, semi-Riemannian manifold $\widetilde{M}$ with semi-Riemannian metric $\widetilde{g}$ of index $1 \leq \nu \leq n$. We consider

$$T_xM^{\perp} = \left\{ Y_x \in T_x\widetilde{M} \mid \widetilde{g}_x(Y_x, X_x) = 0, \quad \text{for all } X_x \in T_xM \right\},$$

for any $x \in M$. Then we say that M is a *lightlike (null, degenerate) hypersurface* of $\widetilde{M}$ or equivalently, the immersion

$$i : M \to \widetilde{M}$$

of M in $\widetilde{M}$ is *lightlike (null, degenerate)* if $T_xM \cap T_xM^{\perp} \neq \{0\}$ at any $x \in M$. Henceforth we identify $i(M)$ with M and we denote the differential di, immersing a vector field X in M to a vector field ϕX in $\widetilde{M}$, by ϕ. Thus the induced metric tensor $g = \widetilde{g}_{|_M}$ is defined by

$$g(X, Y) = \widetilde{g}(\phi X, \phi Y), \quad \text{for all } X, Y \in \Gamma(TM).$$

An orthogonal complementary vector bundle of $TM^{\perp}$ in TM is non-degenerate subbundle of TM called the *screen distribution* on M and denoted by $S(TM)$. We have the following splitting into orthogonal direct sum:

$$TM = S(TM) \perp TM^{\perp}. \tag{2.1}$$

The subbundle $S\,(TM)$ is non-degenerate, so is $S\,(TM)^{\perp}$, and the following holds:

$$T\widetilde{M} = S\,(TM)\perp S\,(TM)^{\perp}\,, \tag{2.2}$$

where $S\,(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S\,(TM)$ in $T\widetilde{M}\Big|_{M}$.

Let $\mathrm{tr}(TM)$ denote the complementary vector bundle of $TM^{\perp}$ in $S\,(TM)^{\perp}$. Then we have

$$S\,(TM)^{\perp} = TM^{\perp} \oplus \mathrm{tr}(TM). \tag{2.3}$$

Let $\mathcal{U}$ be a coordinate neighbourhood in M and ξ be a basis of $\Gamma(TM^{\perp}|_{\mathcal{U}})$. Then there exists a basis N of $\mathrm{tr}\,(TM)|_{\mathcal{U}}$ satisfying the following conditions:

$$\widetilde{g}(N,\xi) = 1,$$

and

$$\widetilde{g}\,(N,N) = \widetilde{g}(W,N) = 0, \qquad \text{for all } W \in \Gamma(\,S\,(TM)|_{\mathcal{U}}).$$

The subbundle $\mathrm{tr}\,(TM)$ is called a *lightlike transversal vector bundle* of M. We note that $\mathrm{tr}\,(TM)$ is never orthogonal to TM. From (2.1), (2.2) and (2.3) we have

$$T\widetilde{M}\Big|_{M} = S\,(TM)\perp \big(TM^{\perp}\oplus \mathrm{tr}\,(TM)\big) = TM \oplus \mathrm{tr}\,(TM)\,.$$

Let $\overset{\circ}{\widetilde{\nabla}}$ be the Levi-Civita connection of $\widetilde{M}$ and P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. The Gauss and Weingarten formulas are given

$$\begin{aligned}
\overset{\circ}{\widetilde{\nabla}}_X Y &= \overset{\circ}{\nabla}_X Y + B(X,Y)N,\\
\overset{\circ}{\widetilde{\nabla}}_X Y &= -\overset{\circ}{A}_N X + \omega(X)N,\\
\overset{\circ}{\nabla}_X PY &= \overset{*}{\overset{\circ}{\nabla}}_X PY + C(X,PY)\xi,\\
\overset{\circ}{\nabla}_X \xi &= -\overset{*}{\overset{\circ}{A}}_\xi X - \omega(X)\xi,
\end{aligned} \tag{2.4}$$

for any $X,Y \in \Gamma(TM)$, where $\overset{\circ}{\nabla}$ and $\overset{*}{\overset{\circ}{\nabla}}$ are the induced linear connection on TM and $S(TM)$, respectively; B and C are the local second fundemental forms on TM and $S(TM)$, respectively; $\overset{\circ}{A}_N$ and $\overset{*}{\overset{\circ}{A}}_\xi$ are the shape operators on TM and $S(TM)$, respectively; and ω is a 1-form on TM [14, 15]. Also, the local second fundamental forms B and C of TM and $S(TM)$, respectively; are related to their shape operators $\overset{\circ}{A}_N$ and $\overset{*}{\overset{\circ}{A}}_\xi$ by

$$\begin{aligned}
B(X,Y) &= g(\overset{*}{\overset{\circ}{A}}_\xi X, Y),\\
C(X,PY) &= g(\overset{\circ}{A}_N X, PY).
\end{aligned}$$

If $B = 0$, then the lightlike hypsersurface M is called totally geodesic in $\widetilde{M}$. A point $p \in M$ is said to be umbilical if

$$B(X,Y)_p = Hg_p(X,Y), \quad X,Y \in \Gamma(T_pM),$$

where $H \in R$. The lightlike hypsersurface M is called totally umbilical in $\widetilde{M}$ if every points of M is umbilical [14].

The mean curvature μ of M with respect to an orthonormal basis $\{e_1, \dots, e_n\}$ of $\Gamma(S(TM))$ is defined in [5] as follows:

$$\mu = \frac{1}{n}\operatorname{tr}(B) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i B(e_i,e_i), \quad g(e_i,e_i) = \varepsilon_i.$$

A Lightlike hypersurface (M,g) of a semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$ is called *screen locally conformal* if the shape operators $\overset{\circ}{A}_N$ and $\overset{\overset{*}{\circ}}{A}_\xi$ of M and $S(TM)$, respectively, are related by

$$\overset{\circ}{A}_N = \varphi \overset{\overset{*}{\circ}}{A}_\xi,$$

where φ is a non-vanishing smooth function on a neighbourhood $\mathcal{U}$ on M. In particular, M is called *screen homothetic* if φ is non-zero constant [3].

We denote by $\overset{\circ}{\widetilde{R}}$ the curvature tensor of $\widetilde{M}$ with respect to Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ and by $\overset{\circ}{R}$ that of M with respect to induced connection $\overset{\circ}{\nabla}$. Then the *Gauss equations of* M is given by

$$\begin{aligned}\overset{\circ}{\widetilde{R}}(X,Y)Z = \overset{\circ}{R}(X,Y)Z &+ A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &+ (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),\end{aligned}$$

for X, Y, Z, $W \in \Gamma(TM)$.

Let M be a two-dimensional non-degenerate plane. The number

$$K_{ij} = \frac{g(R(e_j,e_i)e_i,e_j)}{g(e_i,e_i)g(e_j,e_j) - g(e_i,e_j)^2}$$

is called the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$ [15].

Let $p \in M$ and ξ be null vector of T_pM. A plane Π of T_pM is said to be null plane if it contains ξ and e_i such that $g(\xi,e_i) = 0$ and $g(e_i,e_i) = \varepsilon_i = \pm 1$. The null sectional curvature of Π is given in [4] as follows

$$K_i^{null} = \frac{g(R_p(e_i,\xi)\xi,e_i)}{g_p(e_i,e_i)}.$$

The Ricci tensor $\widetilde{\mathrm{Ric}}$ of $\widetilde{M}$ and the induced Ricci type tensor $R^{(0,2)}$ of M are defined by

$$\widetilde{\mathrm{Ric}}(X,Y) = \mathrm{trace}\{Z \to \widetilde{R}(Z,X)Y\}, \quad \text{for all } X,Y \in \Gamma(T\widetilde{M}),$$
$$R^{(0,2)}(X,Y) = \mathrm{trace}\{Z \to R(Z,X)Y\}, \quad \text{for all } X,Y \in \Gamma(TM),$$

where

$$R^{(0,2)}(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(R(e_i,X)Y,e_i) + g(R(\xi,X)Y,N),$$

for the quasi-orthonormal frame $\{e_1,\dots,e_n,\xi\}$ of T_pM.

If M admits that an induced symmetric Ricci tensor Ric and Ricci tensor satisfy

$$\mathrm{Ric}(X,Y) = kg(X,Y),$$

where k is a constant, then M is called an *Einstein hypersurface* [15].

3. Lorentzian Product Manifolds

In this section, we use the same notations and terminologies as in [20].

Let (M_1,g_1) and (M_2,g_2) be two (m_1+1) and (m_2+1) dimensional Lorentzian manifolds with constant indexes $q_1>0$, $q_2>0$, respectively, and $\widetilde{M}=(M_1\times M_2,\widetilde{g})$ be (m_1+m_2+2)-dimensional differentiable manifold with a tensor field F of type $(1,1)$ on $\widetilde{M}$ such that

$$F^2 = I. \tag{3.1}$$

Let $\pi : M_1\times M_2 \to M_1$ and $\sigma : M_1\times M_2 \to M_2$ be the projections which are given by $\pi(x,y)=x$ and $\sigma(x,y)=y$ for any $(x,y)\in M_1\times M_2$. Then $\widetilde{M}=M_1\times M_2$ is called an almost product manifold with almost product structure F. If we put

$$\pi = \frac{1}{2}(I+F), \quad \sigma = \frac{1}{2}(I-F),$$

then we have

$$\pi^2=\pi, \quad \sigma^2=\sigma, \quad \pi\sigma=\sigma\pi=0, \quad \pi+\sigma=I, \quad F=\pi-\sigma,$$

where π and σ define two complementary distributions [20].

If an almost product manifold $\widetilde{M}$ admits a Lorentzian metric $\widetilde{g}$ such that

$$\widetilde{g}(FX,FY)=\widetilde{g}(X,Y), \tag{3.2}$$

for any vector fields $X,Y\in\Gamma(T\widetilde{M})$, then $\widetilde{M}=M_1\times M_2$ is called Lorentzian almost product manifold. From (3.1) and (3.2), we can easily see that

$$\widetilde{g}(FX,Y)=\widetilde{g}(X,FY).$$

If, for any vector fields X,Y on $\widetilde{M}$,

$$(\overset{\circ}{\widetilde{\nabla}}_X F)Y = 0, \ \text{ that is } \overset{\circ}{\widetilde{\nabla}}_X FY = F(\overset{\circ}{\widetilde{\nabla}}_X Y),$$

then $\widetilde{M}$ is called a Lorentzian product manifold, where $\overset{\circ}{\widetilde{\nabla}}$ is the Levi-Civita connection on $\widetilde{M}$ (see, [20]).

Now, let M_1 and M_2 be real space forms with constant sectional curvatures c_1 and c_2 respectively. Then the Riemannian curvature tensor $\overset{\circ}{\widetilde{R}}$ of $\widetilde{M} = M_1(c_1) \times M_2(c_2)$ is given by

$$
\begin{aligned}
\overset{\circ}{\widetilde{R}}(X,Y)Z = {} & \frac{1}{16}(c_1 + c_2)\Big\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y \\
& + \widetilde{g}(FY,Z)FX - \widetilde{g}(FX,Z)FY\Big\} \\
& + \frac{1}{16}(c_1 - c_2)\Big\{\widetilde{g}(FY,Z)X - \widetilde{g}(FX,Z)Y \\
& + \widetilde{g}(Y,Z)FX - \widetilde{g}(X,Z)FY\Big\},
\end{aligned}
\tag{3.3}
$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$ [29].

Let $(\widetilde{M}, \widetilde{g}, F)$ be Lorentzian product manifold and $\overset{\circ}{\widetilde{\nabla}}$ a Levi-Civita connection on $\widetilde{M}$. A linear connection $\widetilde{\nabla}$ is said to be *quarter-symmetric nonmetric connection* if the torsion tensor $\widetilde{T}$ is of the form

$$\widetilde{T}(X,Y) = \widetilde{\pi}(Y)FX - \widetilde{\pi}(X)FY,$$

where $\widetilde{\pi}$ is a 1-form on $\widetilde{M}$ with $\widetilde{Q}$ as associated vector field, that is

$$\widetilde{g}(\widetilde{Q}, X) = \widetilde{\pi}(X).$$

A linear connection $\widetilde{\nabla}$ is called a nonmetric connection if

$$(\widetilde{\nabla}_X \widetilde{g})(Y,Z) \neq 0.$$

Let M be a lightlike hypersurface of a Lorentzian product manifold $(\widetilde{M}, \widetilde{g})$. For any $X \in \Gamma(TM)$ we can write

$$FX = fX + w(X)N, \tag{3.4}$$

where f is a $(1,1)$ tensor field and w is a 1-form on M given by $w(X) = \widetilde{g}(FX, \xi) = \widetilde{g}(X, F\xi)$.

Following [16], a quarter-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ is given by

$$\widetilde{\nabla}_X Y = \overset{\circ}{\widetilde{\nabla}}_X Y + \widetilde{\pi}(Y)FX, \tag{3.5}$$

for any vector fields X and Y of M.

From (3.5) the curvature tensor $\widetilde{R}$ of the quarter-symmetric nonmetric connection $\widetilde{\nabla}$ is given by

$$\widetilde{R}(X,Y)Z = \overset{\circ}{\widetilde{R}}(X,Y)Z + \widetilde{\lambda}(X,Z)FY - \widetilde{\lambda}(Y,Z)FX, \tag{3.6}$$

for any vector fields $X, Y \in \Gamma(TM)$, where $\widetilde{\lambda}$ is a $(0,2)$ tensor given by $\widetilde{\lambda}(X,Z) = (\widetilde{\nabla}_X \pi)((Z) - \pi(Z)\pi(FX)$.

Let M be a lightlike hypersurface of a Lorentzian product manifold $(\widetilde{M}, \widetilde{g})$ with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then the Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ are given by, respectively,

$$\widetilde{\nabla}_X Y = \nabla_X Y + \bar{B}(X,Y)N, \tag{3.7}$$

$$\widetilde{\nabla}_X N = -\bar{A}_N X + \bar{\tau}(X)N, \tag{3.8}$$

for any $X, Y \in \Gamma(TM)$.

From (2.4), (3.4), (3.5), (3.7) and (3.8) we obtain

$$\begin{aligned} \nabla_X Y &= \overset{\circ}{\nabla}_X Y + \widetilde{\pi}(Y) fX, \\ \bar{B}(X,Y) &= B(X,Y) + \widetilde{\pi}(Y) w(X), \\ \bar{A}_N X &= A_N X - \widetilde{\pi}(N) fX, \\ \bar{\tau}(X) &= \tau(X) + \widetilde{\pi}(N) w(X), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

Using (3.7) we have

$$R(X,Y,Z,PW) = \widetilde{R}(X,Y,Z,PW) + \bar{B}(Y,Z)\bar{C}(X,PW) - \bar{B}(X,Z)\bar{C}(Y,PW), \tag{3.9}$$

for any any $X, Y, Z, W \in \Gamma(TM)$.

From (3.6) and (3.9)

$$\begin{aligned} \widetilde{g}(R(X,Y)Z,PW) = {} & \widetilde{g}(\overset{\circ}{\widetilde{R}}(X,Y)Z,PW) + \bar{B}(Y,Z)\bar{C}(X,PW) - \bar{B}(X,Z)\bar{C}(Y,PW) \\ & + \widetilde{\lambda}(X,Z) g(FY,PW) - \widetilde{\lambda}(Y,Z) g(FX,PW), \end{aligned} \tag{3.10}$$

for any any $X, Y, Z, W \in \Gamma(TM)$.

From now on, we will consider a Lorentzian product manifold $\widetilde{M}$ endowed with a quarter-symmetric nonmetric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

4. Chen-Ricci Inequality

In this section, we use the same notations and terminologies as in [17].

Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian product manifold $\widetilde{M} = M_1 \times M_2$ with a quarter-symmetric nonmetric connection and $\{e_1, \ldots, e_n, \xi\}$ be a basis of $\Gamma(TM)$ where $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$ and $n = m_1 + m_2$. For $k \leq n$, we set $\pi_{k,\xi} = \mathrm{Span}\{e_1, \ldots, e_k, \xi\}$ is a $(k+1)$ dimensional degenerate plane section and $\pi_k = \mathrm{Span}\{e_1, \ldots, e_k\}$ is k-dimensional non degenerate plane section. Define $k-$degenerate Ricci curvature and $k-$Ricci curvature at a unit vector $X \in \Gamma(TM)$ as follows:

$$\mathrm{Ric}_{\pi_{k,\xi}}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j) + \widetilde{g}(R(\xi,X)X,N),$$

$$\mathrm{Ric}_{\pi_k}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j),$$

respectively [17]. Furthermore, k-degenerate scalar curvature and k-scalar curvature at $p \in M$ are given by

$$\tau_{\pi_{k,\xi}}(p) = \sum_{i,j=1}^{k} K_{ij} + \sum_{i=1}^{k} K_i^{\mathrm{null}} + K_{iN},$$

$$\tau_{\pi_k}(p) = \sum_{i,j=1}^{k} K_{ij},$$

respectively [17]. For $k = n$, $\pi_n = \mathrm{Span}\{e_1,\dots,e_n\} = \Gamma(S(TM))$, we have the screen Ricci curvature and the screen scalar curvature given by

$$\mathrm{Ric}_{S(TM)}(e_1) = \mathrm{Ric}_{\pi_n}(e_1) = \sum_{j=1}^{n} K_{1j} = K_{12} + \cdots + K_{1n},$$

and

$$\tau_{S(TM)} = \sum_{i,j=1}^{n} K_{ij},$$

respectively [17].

From (3.3) and (3.10) we can write

$$\tau_{S(TM)}(p) = \frac{1}{16}(c_1+c_2)\big((izF)^2 + n(n-1)\big) + \frac{1}{8}(c_1-c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}$$
$$+ \sum_{i,j=1}^{n} \bar{B}_{ii}\bar{C}_{jj} - \bar{B}_{ij}\bar{C}_{ji}, \tag{4.1}$$

where $\bar{B}_{ij} = \bar{B}(e_i,e_j)$, $\bar{C}_{ij} = \bar{C}(e_i,e_j)$ and $m(e_i,e_j) = m_{ij} = \widetilde{\lambda}(e_i,e_j)g(Fe_j,e_i) - \widetilde{\lambda}(e_j,e_j)g(Fe_i,e_i)$, for $i,j \in \{1,\dots,n\}$.

Let M be a screen homothetic lightlike hypersurface of an $(n+2)$-dimensional Lorentzian space form $\widetilde{M}(c)$. Then, from (4.1) we get

$$\tau_{S(TM)}(p) = \frac{1}{16}(c_1+c_2)\big((izF)^2 + n(n-1)\big) + \frac{1}{8}(c_1-c_2)(izF)$$
$$+ \sum_{i,j=1}^{n} m_{ij} + \varphi n^2\mu^2 - \varphi \sum_{i,j=1}^{n} (\bar{B}_{ij})^2. \tag{4.2}$$

Since the sectional curvature of screen homothetic lightlike hypersurface is symmetric, we can denote the screen scalar curvature by $r_{S(TM)}$ as follows:

$$r_{S(TM)}(p) = \sum_{1\leq i<j\leq n} K_{ij} = \frac{1}{2}\sum_{i,j=1}^{n} K_{ij} = \frac{1}{2}\tau_{S(TM)}(p). \tag{4.3}$$

By (4.3), (4.2) equality become

$$\begin{aligned} 2r_{S(TM)}(p) &= \frac{1}{16}(c_1+c_2)\left((izF)^2+n(n-1)\right)+\frac{1}{8}(c_1-c_2)(izF) \\ &\quad+\sum_{i,j=1}^{n} m_{ij}+\varphi n^2\mu^2-\varphi\sum_{i,j=1}^{n}\left(\bar{B}_{ij}\right)^2. \end{aligned} \tag{4.4}$$

Theorem 4.1. *Let M be a screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1)\times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\bar{\nabla}}$. Then, the following statements are true.*
(i) *For $X \in S^1(TM) = \{X \in S(TM) : \langle X, X\rangle = 1\}$*

$$\begin{aligned} \mathrm{Ric}_{S(TM)}(X) &\leq \frac{1}{4}\varphi n^2\mu^2+\frac{1}{32}(c_1+c_2)\left(2(izF)\bar{g}(FX,X)+3n-4\right) \\ &\quad+\frac{1}{16}(c_1-c_2)(n-1)\bar{g}(FX,X)-\frac{1}{2}\sum_{2\leq i<j\leq n} m_{ij} \\ &\quad+\frac{1}{2}\left(\sum_{i=1}^{n} m_{ii}+\sum_{1\leq j<i\leq n} m_{ij}+\sum_{j=2}^{n} m(X,e_j)\right). \end{aligned} \tag{4.5}$$

(ii) *The equality case of (4.5) is satisfied by $X \in T_p^1(M)$ if and only if*

$$\begin{aligned} &\bar{B}(X,Y) = 0, \quad \textit{for all } Y \in T_p(M) \textit{ orthogonal to } X, \\ &\bar{B}(X,X) = \frac{n}{2}\mu. \end{aligned} \tag{4.6}$$

(iii) *The equality case of (4.5) holds for all $X \in T_p^1(M)$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. From (4.4) we get

$$\begin{aligned} \frac{1}{4}\varphi n^2\mu^2 &= r_{S(TM)}(p)-\frac{1}{32}(c_1+c_2)\left((izF)^2+n(n-1)\right)-\frac{1}{16}(c_1-c_2)(izF) \\ &\quad-\frac{1}{2}\sum_{i,j=1}^{n} m_{ij}+\frac{1}{4}\varphi\left(\bar{B}_{11}-\bar{B}_{22}-\cdots-\bar{B}_{nn}\right)^2+\varphi\sum_{j=2}^{n}\left(\bar{B}_{1j}\right)^2 \\ &\quad-\varphi\sum_{2\leq i<j\leq n}^{m}\left(\bar{B}_{ii}\bar{B}_{jj}-\left(\bar{B}_{ij}\right)^2\right). \end{aligned} \tag{4.7}$$

Using (3.3) and (3.10) we also have

$$\begin{aligned}
\varphi \sum_{2\le i<j\le n}^{m} \left(\bar{B}_{ii}\bar{B}_{jj} - \left(\bar{B}_{ij}\right)^2\right) &= \sum_{2\le i<j\le n} K_{ij} - \sum_{2\le i<j\le n} \widetilde{K}_{ij} \\
&= \sum_{2\le i<j\le n} K_{ij} - \frac{1}{32}(c_1+c_2)\big((izF)^2 - 2\,(izF)\,\bar{g}\,(Fe_1,e_1)\big) \\
&\quad - \frac{1}{16}(c_1-c_2)\,((izF)-(n-1)\bar{g}\,(Fe_1,e_1)) \\
&\quad - \frac{1}{32}(c_1+c_2)(n-2)^2 - \sum_{2\le i<j\le n} m_{ij}. \qquad (4.8)
\end{aligned}$$

From (4.7) and (4.8) we obtain

$$\begin{aligned}
\mathrm{Ric}_{S(TM)}(e_1) = {}& \frac{1}{4}\varphi n^2\mu^2\varphi - \frac{1}{4}\varphi\left(\bar{B}_{11}-\bar{B}_{22}-\cdots-\bar{B}_{nn}\right)^2 - \varphi\sum_{j=2}^{n}\left(\bar{B}_{1j}\right)^2 \\
&+ \frac{1}{32}(c_1+c_2)\left(2(izF)\bar{g}(Fe_1,e_1)+3n-4\right) - \sum_{2\le i<j\le n} m_{ij} \\
&+ \frac{1}{16}(c_1-c_2)(n-1)\bar{g}(Fe_1,e_1) \\
&+ \frac{1}{2}\left(\sum_{i=1}^{n} m_{ii} + \sum_{1\le j<i\le n} m_{ij} + \sum_{j=2}^{n} m_{1j}\right). \qquad (4.9)
\end{aligned}$$

If we put $e_1 = X$ as any vector of $T_p^1(M)$ in (4.9) we obtain (4.5).

The equality case of (4.5) holds for $X \in T_p^1(M)$ if and only if

$$\bar{B}_{12} = \bar{B}_{13} = \cdots = \bar{B}_{1n} = 0 \text{ and } \bar{B}_{11} = \bar{B}_{22} + \cdots + B_{nn}, \qquad (4.10)$$

equivalent to (4.6).

Now we prove the statement (iii). Assuming the equality case of (4.5) for all $X \in T_p^1(M)$, in view of (4.10), we have

$$\bar{B}_{ij} = 0, \quad i \ne j, \qquad (4.11)$$

and

$$2\bar{B}_{ii} = \bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn}, \quad i \in \{1,\ldots,n\}. \qquad (4.12)$$

From (4.12) we have $2\bar{B}_{11} = 2\bar{B}_{22} = \cdots = 2\bar{B}_{nn} = \bar{B}_{11}+\bar{B}_{22}+\cdots+\bar{B}_{nn}$ which implies that

$$(n-2)\left(\bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn}\right) = 0.$$

Thus, either $\bar{B}_{11}+\bar{B}_{22}+\cdots+\bar{B}_{nn} = 0$ or $n = 2$. If $\bar{B}_{11}+\bar{B}_{22}+\cdots+\bar{B}_{nn} = 0$, then in view of (4.12), we get $\bar{B}_{ii} = 0$ for all $i \in \{1,\ldots,n\}$. This together with (4.11) gives $\bar{B}_{ij} = 0$ for all $i,j \in \{1,\ldots,n\}$, that is, p is a totally geodesic point. If $n = 2$, then from (4.12), $2\bar{B}_{11} = 2\bar{B}_{22} = \bar{B}_{11} + \bar{B}_{22}$, which shows that p is a totally umbilical point. The proof of the converse part is straightforward. □

We recall the following algebraic Lemma from [27].

Lemma 4.1. *Let $a_1, a_2, \ldots, a_n$, be n-real number $(n > 1)$, then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 4.2. *Let M be a screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$*

$$\tau_{S(TM)}(p) \leq \varphi n(n-1)\mu^2 + \frac{1}{16}(c_1 + c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}, \tag{4.13}$$

with equality if and only if p is a totally umbilical point.

Proof. From (4.2) we have

$$\varphi n^2\mu^2 = \tau_{S(TM)}(p) + \varphi\sum_{i=1}^{n}(B_{ii})^2 + \varphi\sum_{i\neq j}(B_{ij})^2 - \sum_{i,j=1}^{n} m_{ij} - \frac{1}{16}(c_1 + c_2)\left((izF)^2 + n(n-1)\right) - \frac{1}{8}(c_1 - c_2)(izF). \tag{4.14}$$

Using Lemma 4.1 we get

$$n\mu^2 \leq \sum_{i=1}^{n}(B_{ii})^2. \tag{4.15}$$

Considering (4.14) and (4.15) we obtain (4.13). Equality case of (4.13) holds if and only if

$$\bar{B}_{11} = \bar{B}_{22} = \cdots = \bar{B}_{nn},$$

the shape operator A^*_ξ take the form:

$$A^*_\xi = \begin{bmatrix} \bar{B}_{11} & 0 & \cdots & 0 & 0 \\ 0 & \bar{B}_{11} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{B}_{11} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{4.16}$$

which shows that M is totally umbilical. This completes the proof of the theorem. □

Also, the components of the second fundamental form $\bar{B}$ and the screen second fundamental form $\bar{C}$ satisfy

$$\sum_{i,j=1}^{n} \bar{B}_{ij}\bar{C}_{ji} = \frac{1}{2}\left\{\sum_{i,j=1}^{n} \left(\bar{B}_{ij}+\bar{C}_{ji}\right)^2 - \sum_{i,j=1}^{n} \left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2\right\}, \tag{4.17}$$

and

$$\sum_{i,j} \bar{B}_{ii}\bar{C}_{jj} = \frac{1}{2}\left\{\left(\sum_{i,j} \bar{B}_{ii}+\bar{C}_{jj}\right)^2 - \left(\sum_{i} \bar{B}_{ii}\right)^2 - \left(\sum_{j} C_{jj}\right)^2\right\}. \tag{4.18}$$

Theorem 4.3. *Let M be lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then*
(i)

$$\begin{aligned}\tau_{S(TM)}(p) \leq{}& n\mu \operatorname{trace} A_N + \frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2\right) + \sum_{i,j=1}^{n} m_{ij}\\ &+ \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF).\end{aligned} \tag{4.19}$$

The equality case of (4.19) *holds for all $p \in M$ if and only if either M is a screen homothetic lightlike hypersurface with $\varphi = -1$ or M is a totally geodesic lightlike hypersurface.*
(ii)

$$\begin{aligned}\tau_{S(TM)}(p) \geq{}& n\mu \operatorname{trace} A_N - \frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2\right) + \sum_{i,j=1}^{n} m_{ij}\\ &+ \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF).\end{aligned} \tag{4.20}$$

The equality case of (4.20) *holds for all $p \in M$ if and only if either M is a screen homothetic lightlike hypersurface with $\varphi = 1$ or M is a totally geodesic lightlike hypersurface.*
(iii) *The equalities case of* (4.19) *and* (4.20) *hold at $p \in M$ if and only if p is a totally geodesic point.*

Proof. Using (4.1) and (4.17), we get

$$\begin{aligned}\tau_{S(TM)}(p) ={}& \sum_{i,j=1}^{n} \bar{B}_{ii}\bar{C}_{jj} - \frac{1}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}+\bar{C}_{ji}\right) + \frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2\right)\\ &+ \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF) + \sum_{i,j=1}^{n} m_{ij},\end{aligned} \tag{4.21}$$

which yields (4.19).

Since

$$\frac{1}{2}\left(\left(\bar{B}_{ij}\right)^2+\left(\bar{C}_{ji}\right)^2\right)=\frac{1}{4}\left(\bar{B}_{ij}+\bar{C}_{ji}\right)^2+\frac{1}{4}\left(\bar{B}_{ij}-\bar{C}_{ji}\right)^2, \tag{4.22}$$

we obtain

$$\begin{aligned}\tau_{S(TM)}(p) &= \sum_{i,j=1}^{n}\bar{B}_{ii}C_{jj}-\frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\bar{B}_{ij}\right)^2+\left(\bar{C}_{ji}\right)^2\right)+\frac{1}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}-\bar{C}_{ji}\right)^2\\ &+\frac{1}{16}(c_1+c_2)\left((izF)^2+n(n-1)\right)+\frac{1}{8}(c_1-c_2)(izF)+\sum_{i,j=1}^{n}m_{ij},\end{aligned} \tag{4.23}$$

which yields (4.20). From (4.19), (4.20), (4.21) and (4.23) it is easy to get (i), (ii) and (iii) statements.

By Theorem 4.3 we have the following corollary. □

Corollary 4.1. *Let M be a screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c)=M_1(c_1)\times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then, we have*

$$\begin{aligned}\tau_{S(TM)}(p) &\leq \varphi n^2\mu^2+\frac{(1+\varphi^2)}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}\right)^2+\frac{1}{16}(c_1+c_2)\left((izF)^2+n(n-1)\right)\\ &+\frac{1}{8}(c_1-c_2)(izF)+\sum_{i,j=1}^{n}m_{ij}.\end{aligned}$$

and

$$\begin{aligned}\tau_{S(TM)}(p) &\geq \varphi n^2\mu^2-\frac{(1+\varphi^2)}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}\right)^2+\frac{1}{16}(c_1+c_2)\left((izF)^2+n(n-1)\right)\\ &+\frac{1}{8}(c_1-c_2)(izF)+\sum_{i,j=1}^{n}m_{ij}.\end{aligned}$$

Theorem 4.4. *Let M be lightlike hypersurface of a real product space form $\widetilde{M}(c)=M_1(c_1)\times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then, we have*

$$\begin{aligned}\tau_{S(TM)}(p) &\leq \frac{1}{2}\left(\operatorname{trace}\bar{A}\right)^2-\frac{1}{2}\left(\operatorname{trace}A_N\right)^2-\frac{1}{4}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}+\bar{C}_{ji}\right)^2\\ &+\frac{1}{4}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}-\bar{C}_{ji}\right)^2+\frac{1}{16}(c_1+c_2)\left((izF)^2+n(n-1)\right)\\ &+\frac{1}{8}(c_1-c_2)(izF)+\sum_{i,j=1}^{n}m_{ij},\end{aligned} \tag{4.24}$$

where

$$\bar{A} = \begin{bmatrix} \bar{B}_{11} + \bar{C}_{11} & \bar{B}_{12} + \bar{C}_{21} & \cdots & \bar{B}_{1n} + \bar{C}_{n1} \\ \bar{B}_{21} + \bar{C}_{12} & \bar{B}_{22} + \bar{C}_{22} & \cdots & \bar{B}_{2n} + \bar{C}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{B}_{n1} + C_{1n} & \bar{B}_{n2} + \bar{C}_{2n} & \cdots & \bar{B}_{nn} + \bar{C}_{nn} \end{bmatrix}. \tag{4.25}$$

The equality case of (4.24) *holds for all* $p \in M$ *if and only if* M *is minimal.*

Proof. From (4.1), (4.17) and (4.18) we get

$$\begin{aligned} \tau_{S(TM)}(p) = {} & \frac{1}{2}\left(\sum_{i,j} \bar{B}_{ii} + C_{jj}\right)^2 - \frac{1}{2}\left(\sum_{i} \bar{B}_{ii}\right)^2 - \frac{1}{2}\left(\sum_{j} C_{jj}\right)^2 \\ & - \frac{1}{2}\sum_{i,j=1}^{n} \left(\bar{B}_{ij} + \bar{C}_{ji}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{n} \left((\bar{B}_{ij})^2 + (\bar{C}_{ji})^2\right) + \sum_{i,j=1}^{n} m_{ij} \\ & + \frac{1}{16}(c_1 + c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1 - c_2)(izF). \end{aligned} \tag{4.26}$$

From (4.22) we have

$$\begin{aligned} & - \frac{1}{2}\sum_{i,j=1}^{n} \left(\bar{B}_{ij} + \bar{C}_{ji}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{n} \left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2 \\ & = - \frac{1}{4}\sum_{i,j=1}^{n} \left(\bar{B}_{ij} + \bar{C}_{ji}\right)^2 + \frac{1}{4}\sum_{i,j=1}^{n} \left(\bar{B}_{ij} - \bar{C}_{ji}\right)^2. \end{aligned} \tag{4.27}$$

If we put (4.27) in (4.26), we obtain

$$\begin{aligned} \tau_{S(TM)}(p) = {} & \frac{1}{2}\left(\sum_{i,j} \bar{B}_{ii} + C_{jj}\right)^2 - \frac{1}{2}\left(\sum_{i} \bar{B}_{ii}\right)^2 - \frac{1}{2}\left(\sum_{j} C_{jj}\right)^2 \\ & - \frac{1}{4}\sum_{i,j=1}^{n} \left(\bar{B}_{ij} + \bar{C}_{ji}\right)^2 + \frac{1}{4}\sum_{i,j=1}^{n} \left(\bar{B}_{ij} - \bar{C}_{ji}\right)^2 + \sum_{i,j=1}^{n} m_{ij} \\ & + \frac{1}{16}(c_1 + c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1 - c_2)(izF). \end{aligned}$$

The equality case of (4.24) satisfies then

$$\sum_{i} \bar{B}_{ii} = 0.$$

This shows that M is minimal. $\square$

By Theorem 4.4 we have the following corollary.

Corollary 4.2. *Let M be a screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$*

$$\tau_{S(TM)}(p) \leq \frac{(2\varphi+1)}{2} n^2\mu^2 - \varphi \sum_{i,j=1}^{n} \left(\bar{B}_{ij}\right)^2 + \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right)$$
$$+ \frac{1}{8}(c_1-c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}. \tag{4.28}$$

The equality case of (4.28) *holds for all $p \in M$ if and only if M is minimal.*

Theorem 4.5. *Let M be lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then, we have*

$$\tau_{S(TM)}(p) \leq \frac{n-1}{2n}\left(\operatorname{trace}\bar{A}\right)^2 - \frac{1}{2}(\operatorname{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 - \frac{1}{2}\sum_{i\neq j}\left(\bar{B}_{ij} + \bar{C}_{ji}\right)^2$$
$$+ \frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2\right) + \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right)$$
$$+ \frac{1}{8}(c_1-c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}, \tag{4.29}$$

where $\bar{A}$ is equal to (4.25).

The equality case of (4.29) *holds for all $p \in M$ if and only if $n\mu = -\operatorname{trace}A_N$.*

Proof. From (4.26)

$$\tau_{S(TM)}(p) = \frac{1}{2}\left(\operatorname{trace}\bar{A}\right)^2 - \frac{1}{2}(\operatorname{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 - \frac{1}{2}\sum_{i}(\bar{B}_{ii} + \bar{C}_{ii})^2$$
$$- \frac{1}{2}\sum_{i\neq j}\left(\bar{B}_{ij} + \bar{C}_{ji}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2\right) + \sum_{i,j=1}^{n} m_{ij}$$
$$+ \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF). \tag{4.30}$$

Using Lemma 4.1 and equality case of (4.30), we have

$$\tau_{S(TM)}(p) \leq \frac{1}{2}(\operatorname{trace}\bar{A})^2 - \frac{1}{2}(\operatorname{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 - \frac{1}{2n}\sum_{i}(\bar{B}_{ii} + \bar{C}_{ii})^2$$
$$- \frac{1}{2}\sum_{i\neq j}(\bar{B}_{ij} + \bar{C}_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}\right)^2 + \left(\bar{C}_{ji}\right)^2 + \sum_{i,j=1}^{n} m_{ij}$$
$$+ \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF),$$

which implies (4.29). The equality case of (4.29) holds, then

$$\bar{B}_{11} + \bar{C}_{11} = \cdots = \bar{B}_{nn} + \bar{C}_{nn}. \tag{4.31}$$

From (4.31) we get

$$\begin{aligned}
&(1-n)\bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn} + (1-n)\bar{C}_{11} + \bar{C}_{22} + \cdots + \bar{C}_{nn} = 0,\\
&\bar{B}_{11} + (1-n)\bar{B}_{22} + \cdots + \bar{B}_{nn} + \bar{C}_{11} + (1-n)\bar{C}_{22} + \cdots + \bar{C}_{nn} = 0,\\
&\qquad\vdots\\
&\bar{B}_{11} + \bar{B}_{22} + \cdots + (1-n)\bar{B}_{nn} + \bar{C}_{11} + \bar{C}_{22} + \cdots + (1-n)\bar{C}_{nn} = 0.
\end{aligned}$$

By the above equations, we have

$$(n-1)^2(\operatorname{trace} A_N + n\mu) = 0.$$

Since $n \neq 1$, we obtain $n\mu = -\operatorname{trace} A_N$. □

By Theorem 4.5 we have the following corollary.

Corollary 4.3. *Let M be screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then*

$$\begin{aligned}
\tau_{S(TM)}(p) \leq{}& \varphi n(n-1)\mu^2 - \frac{(1+\varphi^2)}{2} n\mu^2 - \frac{(1+\varphi)^2}{2} \sum_{i\neq j} \left(\bar{B}_{ij}\right)^2 + \frac{(1+\varphi^2)}{2} \sum_{i,j=1}^{n} \left(\bar{B}_{ij}\right)^2\\
&+ \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}.
\end{aligned} \tag{4.32}$$

The equality case of (4.32) holds for all $p \in M$ if and only if either $\varphi = -1$ or M is minimal.

Theorem 4.6. *Let M be lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then*

$$\begin{aligned}
\tau_{S(TM)}(p) \geq{}& \frac{1}{2}\left(\operatorname{trace} \bar{A}\right)^2 - \frac{1}{2}(\operatorname{trace} A_N)^2 - \frac{1}{2} n(n-1)\mu^2 - \frac{1}{2} \sum_{i,j=1}^{n} \left(\bar{B}_{ij} + \bar{C}_{ji}\right)^2\\
&+ \frac{1}{2} \sum_{i\neq j} \left(\bar{B}_{ij}\right)^2 + \frac{1}{2} \sum_{i,j=1}^{n} \left(\bar{C}_{ji}\right)^2 + \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right)\\
&+ \frac{1}{8}(c_1-c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}.
\end{aligned} \tag{4.33}$$

The equality case of (4.33) holds for all $p \in M$ if and only if p is a totally umbilical point.

Proof. From (4.26)

$$
\begin{aligned}
\tau_{S(TM)}(p) &= \frac{1}{2}\left(\operatorname{trace}\bar{A}\right)^2 - \frac{1}{2}(\operatorname{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 + \frac{1}{2}\sum_i \left(\bar{B}_{ii}\right)^2 + \frac{1}{2}\sum_{i\neq j}\left(\bar{B}_{ij}\right)^2 \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\left(\bar{C}_{ji}\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}+\bar{C}_{ji}\right)^2 + \sum_{i,j=1}^{n} m_{ij} \\
&\quad + \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF). \qquad (4.34)
\end{aligned}
$$

Using Lemma 4.1 and equality case of (4.34) we have

$$
\begin{aligned}
\tau_{S(TM)}(p) &\geq \frac{1}{2}\left(\operatorname{trace}\bar{A}\right)^2 - \frac{1}{2}(\operatorname{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 + \frac{1}{2n}\left(\sum_i \bar{B}_{ii}\right)^2 \\
&\quad + \frac{1}{2}\sum_{i\neq j}\left(\bar{B}_{ij}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{n}\left(\bar{C}_{ji}\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}+\bar{C}_{ji}\right)^2 + \sum_{i,j=1}^{n} m_{ij} \\
&\quad + \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF),
\end{aligned}
$$

which implies (4.33). Equality case of (4.33) holds if and only if $\bar{B}_{11} = \cdots = \bar{B}_{nn}$ the shape operator A^*_ξ take the form as (4.16), which shows that M is totally umbilical. This completes the proof of the theorem. □

By Theorem 4.6 we have the following corollary.

Corollary 4.4. *Let M be screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature c, endowed with quarter-symmetric nonmetric connection $\widetilde{\nabla}$. Then*

$$
\begin{aligned}
\tau_{S(TM)}(p) &\geq \frac{(2\varphi+1)}{2}n^2\mu^2 - \frac{1}{2}n(n-1)\mu^2 - \frac{(2\varphi+1)}{2}\sum_{i,j=1}^{n}\left(\bar{B}_{ij}\right)^2 \\
&\quad + \frac{1}{16}(c_1+c_2)\left((izF)^2 + n(n-1)\right) + \frac{1}{8}(c_1-c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}. \qquad (4.35)
\end{aligned}
$$

The equality case of (4.35) holds for all $p \in M$ if and only if p is a totally umbilical point.

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