Kragujevac Journal of Mathematics
Volume 51(3) (2027), Pages 459–477.

FEKETE-SZEGÖ TYPE COEFFICIENT INEQUALITIES FOR CLASSES OF ANALYTIC FUNCTION ASSOCIATED WITH (p,q)-DERIVATIVE OF SĂLĂGEAN OPERATOR AND HORADAM POLYNOMIALS

RABHA M. EL-ASHWAH[1] AND WAFAA Y. KOTA[1]

Abstract. Using the concept of subordination principle, we propose to make use of the Horadam polynomials and (p,q)-analogue of Sălăgean differential operator. We introduce and study new subclasses of analytic functions. The objective of this paper is to obtain the Fekete-Szegö inequalities associated with k-th root transformations $[f(\varsigma^k)]^{\frac{1}{k}}$ for functions in the subclasses of analytic univalent functions $\mathfrak{M}^n_{p,q}(\zeta,\sigma,r)$, $\mathfrak{N}^n_{p,q}(\zeta,\sigma,r)$ and $\mathfrak{L}^n_{p,q}(\rho,r)$ defined by the generalized fractional derivative operator $\mathfrak{D}^n_{p,q}$. Similar problems are investigated for $\frac{\varsigma}{f(\varsigma)}$, when $f(\varsigma)$ belongs to these classes.

1. Introduction

Quantum calculus is widely used in mathematical sciences because of its numerous applications in combinatorics [15], number theory [34], fundamental hypergeometric functions [14] and orthogonal polynomials [38]. The ideal basis for applying the q-calculus within geometric function theory was provided by Srivastava's 1989 book chapter [30]. Jackson [22, 23] was the first to successfully build the q-integral and q-derivative. The geometrical indicating of the q-analysis was subsequently discovered as a result of quantum group investigations. In 1990, the q-calculus techniques were implemented to define the notion of a q-starlike function in geometric function theory [21].

Key words and phrases. Analytic and univalent functions, Fekete-Szegö inequalities, (p,q)-derivative operator, Salagean operator, Horadam polynomials.

2020 *Mathematics Subject Classification.* Primary 30C45. Secondary 30C80.

DOI

Received: September 06, 2025.

Accepted: September 29, 2025.

The development of this field of study was subsequently facilitated by the appearance of numerous q-calculus operators that were employed in different investigations, including the definition of new classes of analytic functions and the discovery of various features for them, the most typical of these being geometric properties and coefficient estimates, through techniques specific to geometric function theory. In an attempt to modify the classical operators to the q-calculus features incorporated into geometric function theory, Kanas and Răducanu [24] began the field of exploration by building the q-analogue of the Ruscheweyh differential operator utilizing the concept of convolution. Srivastava [32] established operators for basic (or q-)calculus and fractional q-calculus together with their applications in their study of geometric function theory of complex analysis. In [37], new subclasses of bi-univalent functions are obtained by applying a q-integral operator.

The general background information for the current research is created by the collection of analytic functions in the unit disc $\mathbb{D} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, which are observed with the geometric position presented by the star-likeness qualities that some analytic function classes display.

The normalized analytic family of functions f, designated by $\mathcal{A}$, can be written as follows:

$$f(\varsigma) = \varsigma + \sum_{\kappa \geq 2} a_\kappa \varsigma^\kappa \quad (\varsigma \in \mathbb{D}), \tag{1.1}$$

with $f(0) = f'(0) - 1 = 0$.

Let

$$\Psi(\varsigma) = 1 + \mathrm{C}_1\varsigma + \mathrm{C}_2\varsigma^2 + \mathrm{C}_3\varsigma^3 + \cdots \tag{1.2}$$

be an analytic function on $\mathbb{D}$, with $\Psi(0) = 1$, $\Psi'(0) > 0$ and $\mathrm{Re}(\Psi(\varsigma)) > 0$, mapping the unit disc $\mathbb{D}$ onto a starlike region with regards to 1 and being symmetric with regards to the real axis.

Definition 1.1 ([9])**.** The formula $f(\varsigma) \prec h(\varsigma)$ indicates that two analytic functions are subordinate to one another if there is a function ω that is analytic in $\mathbb{D}$ and satisfies $\omega(0) = 0$, $|\omega(\varsigma)| < 1$ for all $\varsigma \in \mathbb{D}$, such that $f(\varsigma) = h(\omega(\varsigma))$, $\varsigma \in \mathbb{D}$.

More precisely, if h in $\mathbb{D}$ is univalent, we have:

$$f(\varsigma) \prec h(\varsigma) \quad \text{if and only if} \quad f(0) = h(0) \text{ and } f(\mathbb{D}) \subset h(\mathbb{D}).$$

It is possible to extend the q-calculus to quantum calculus, represented by the (p, q)-calculus. The q-calculus can be created when $p = 1$ in the (p, q)-calculus (see [18, 28]). First, we define (p, q)-calculus, which is helpful in understanding the subject of the present paper.

Definition 1.2 ([4, 33])**.** The (p, q)-derivative operator of a function f is defined by

$$D_{p,q} f(\varsigma) = \frac{f(p\varsigma) - f(q\varsigma)}{(p - q)\varsigma} = 1 + \sum_{\kappa \geq 2} [\kappa]_{p,q} a_\kappa \varsigma^{\kappa - 1} \quad (\varsigma \in \mathbb{D}, \ \varsigma \neq 0), \tag{1.3}$$

where $[\kappa]_{p,q}$ is given by

$$[\kappa]_{p,q} = \frac{p^\kappa - q^\kappa}{p-q} \quad (p \neq q,\, 0 < q < p \leq 1).$$

If $p = 1$ and $q \to 1^-$, then $D_{p,q}f(\varsigma) = f'(\varsigma)$ (see also [28]).

Note that, for $p = 1$, the (p,q)-derivative reduces to (see [22])

$$D_q f(\varsigma) = \frac{f(\varsigma) - f(q\varsigma)}{(1-q)\varsigma} = 1 + \sum_{\kappa \geq 0} [\kappa]_q a_\kappa \varsigma^{\kappa-1} \quad (\varsigma \neq 0). \tag{1.4}$$

For $f(z) \in \mathcal{A}$, the (p,q)-analogue of Sălăgean differential operator $\mathfrak{D}^n_{p,q} : \mathcal{A} \to \mathcal{A}$ ($n \in \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$) defined by Panigrahi and El-Ashwah [28] as:

$$\begin{aligned} \mathfrak{D}^0_{p,q} f(\varsigma) &= f(\varsigma), \\ \mathfrak{D}^1_{p,q} f(\varsigma) &= \varsigma(D_{p,q} f(\varsigma)), \\ &\vdots \\ \mathfrak{D}^n_{p,q} f(\varsigma) &= \mathfrak{D}^1_{p,q}\left(\mathfrak{D}^{n-1}_{p,q} f(\varsigma)\right) \quad (0 < q < p \leq 1). \end{aligned} \tag{1.5}$$

It follows from (1.1) and (1.5) that

$$\mathfrak{D}^n_{p,q} f(\varsigma) = \varsigma + \sum_{\kappa \geq 2} [\kappa]^n_{p,q} a_\kappa \varsigma^\kappa \quad (\varsigma \in \mathbb{D}). \tag{1.6}$$

Note that the following hold.

- For $p = 1$, $\mathfrak{D}^n_{p,q} f(\varsigma) = D^n_q f(\varsigma) = \varsigma + \sum_{\kappa \geq 2} [\kappa]^n_q a_\kappa \varsigma^\kappa$, where D^n_q is defined in [17].
- For $p = 1$, $\lim_{q \to 1^-} \mathfrak{D}^n_{p,q} f(\varsigma) = D^n f(\varsigma) = \varsigma + \sum_{\kappa \geq 2} \kappa^n a_\kappa \varsigma^\kappa$, where D^n defined in [29].

Using the (p,q)-analogue of Sălăgean differential operator $\mathfrak{D}^n_{p,q}$, we introduce a new classes of analytic functions as follows.

Definition 1.3. A function $f \in \widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ ($0 < q < p \leq 1$, $n \in \mathbb{N} \cup \{0\}$, $\sigma \geq 0$, $0 \leq \zeta < 1$) if

$$(1-\zeta)\left(\frac{\mathfrak{D}^n_{p,q} f(\varsigma)}{\varsigma}\right)^\sigma + \zeta D_{p,q}\left(\mathfrak{D}^n_{p,q} f(\varsigma)\right)\left(\frac{\varsigma}{\mathfrak{D}^n_{p,q} f(\varsigma)}\right)^{\sigma-1} \prec \Psi(\varsigma) \quad (f \in \mathcal{A},\, \varsigma \in \mathbb{D}),$$

where $\Psi(\varsigma)$ is defined in (1.2).

Remark 1.1. The following subclasses can be obtained by specializing the parameters p, q, ζ, σ, n and Ψ in (1.2).

(i) For $p = 1$, the class $\widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $\widetilde{\mathfrak{M}}^n_q(\zeta, \sigma, \Psi)$, which satisfied the following condition:

$$(1-\zeta)\left(\frac{\mathfrak{D}^n_q f(\varsigma)}{\varsigma}\right)^\sigma + \zeta D_q\left(\mathfrak{D}^n_q f(\varsigma)\right)\left(\frac{\varsigma}{\mathfrak{D}^n_q f(\varsigma)}\right)^{\sigma-1} \prec \Psi(\varsigma).$$

(ii) For $p = 1$ and $n = 0$, the class $\widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $\widetilde{\mathfrak{M}}_q(\zeta, \sigma, \Psi)$, which satisfies the following condition:

$$(1-\zeta)\left(\frac{f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta D_q\left(f(\varsigma)\right)\left(\frac{\varsigma}{f(\varsigma)}\right)^{\sigma-1} \prec \Psi(\varsigma).$$

(iii) For $p = 1$ and $q \to 1^-$, the class $\widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $\widetilde{\mathfrak{M}}^n(\zeta, \sigma, \Psi)$, which satisfies the following condition:

$$(1-\zeta)\left(\frac{D^n f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta\left(D^n f(\varsigma)\right)'\left(\frac{\varsigma}{D^n f(\varsigma)}\right)^{\sigma-1} \prec \Psi(\varsigma).$$

(iv) For $p = 1$, $q \to 1^-$ and $n = 0$, the class $\widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $M_{\zeta,\sigma}(\Psi)$ [7].

(v) For $p = 1$, $q \to 1^-$, $\sigma = 1$, $n = 0$ and $\Psi = \frac{1+(1-2\beta)\varsigma}{1-\varsigma}$ $(0 \leq \beta < 1)$, the class $\widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $Q_\zeta(\beta)$ [8] (see also [10, with $h(z) = \frac{z}{1-z}$]).

(vi) For $p = 1$, $q \to 1^-$, $\sigma = 1$, $n = 0$ and $\Psi = \frac{1+\varsigma}{1-\varsigma}$, the class $\widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to Q_ζ [5].

(vii) For $p = 1$, $q \to 1^-$, $\sigma = 1$, $n = 0$, $\zeta = 1$ and $\Psi = \frac{1+\varsigma}{1-\varsigma}$, the class $\widetilde{\mathfrak{M}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to Q [27].

Definition 1.4. A function $f \in \widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ $(0 < q < p \leq 1,\ n \in \mathbb{N},\ \sigma \geq 0,\ 0 \leq \zeta < 1)$ if

$$(1-\zeta)\left(\frac{\mathfrak{D}^n_{p,q} f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta D_{p,q}\left(\mathfrak{D}^n_{p,q} f(\varsigma)\right)\left(\frac{\mathfrak{D}^n_{p,q} f(\varsigma)}{\varsigma}\right)^{\sigma-1} \prec \Psi(\varsigma) \quad (f \in \mathcal{A},\ \varsigma \in \mathbb{D}),$$

where $\Psi(\varsigma)$ is defined in (1.2).

Remark 1.2. By specializing the parameters p, q, ζ, σ, n and Ψ in (1.2), we obtain the following subclasses as follows.

(i) For $p = 1$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $\widetilde{\mathfrak{N}}^n_q(\zeta, \sigma, \Psi)$, which satisfies the following condition:

$$(1-\zeta)\left(\frac{\mathfrak{D}^n_q f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta D_q\left(\mathfrak{D}^n_q f(\varsigma)\right)\left(\frac{\mathfrak{D}^n_q f(\varsigma)}{\varsigma}\right)^{\sigma-1} \prec \Psi(\varsigma).$$

(ii) For $p = 1$ and $n = 0$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $\widetilde{\mathfrak{N}}_q(\zeta, \sigma, \Psi)$, which satisfies the following condition:

$$(1-\zeta)\left(\frac{f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta D_q\left(f(\varsigma)\right)\left(\frac{f(\varsigma)}{\varsigma}\right)^{\sigma-1} \prec \Psi(\varsigma).$$

(iii) For $p = 1$ and $q \to 1^-$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $\widetilde{\mathfrak{N}}^n(\zeta, \sigma, \Psi)$, which satisfies the following condition:

$$(1-\zeta)\left(\frac{D^n f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta\left(D^n f(\varsigma)\right)'\left(\frac{D^n f(\varsigma)}{\varsigma}\right)^{\sigma-1} \prec \Psi(\varsigma).$$

(iv) For $p = 1$, $q \to 1^-$ and $n = 0$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $N_{\zeta,\sigma}(\Psi)$ [7].
(v) For $p = 1$, $q \to 1^-$, $\sigma = n = 0$ and $\zeta = 1$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to $S^*(\Psi)$ [26].
(vi) For $p = 1$, $q \to 1^-$, $n = 0$, $\zeta = 1$ and $\Psi = \frac{1+\varsigma}{1-\varsigma}$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to N_σ [11].
(vii) For $p = 1$, $q \to 1^-$, $n = 0$, $\zeta = 1$, $\sigma = \frac{1}{2}$ and $\Psi = \frac{1+\varsigma}{1-\varsigma}$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to N [12].
(viii) For $p = 1$, $q \to 1^-$, $n = 0$, $\zeta = 1$, $\sigma = 0$ and $\Psi = \frac{1+\varsigma}{1-\varsigma}$, the class $\widetilde{\mathfrak{N}}^n_{p,q}(\zeta, \sigma, \Psi)$ reduces to S^* [16].

Horadam polynomials sequence $\hbar_\imath(r, s, t; c, d)$ or $\hbar_\imath(s)$ is defined by (see [19, 20])

$$\hbar_\imath(r) = sr\hbar_{\imath-1}(r) + t\hbar_{\imath-2}(r) \quad (c, d, r, s, t \in \mathbb{R},\ \imath \in \mathbb{N}\backslash\{1, 2\}), \tag{1.7}$$

with

$$\hbar_1(r) = c, \quad \hbar_2(r) = dr \quad \text{and} \quad \hbar_3(r) = sdr^2 + cd. \tag{1.8}$$

The generating function of the Horadam polynomial sequence can be established by

$$\aleph(\varsigma, r) = \sum_{\imath\geq 1} \hbar_\imath(r)\varsigma^{\imath-1} = -\frac{c + (d - cs)r\varsigma}{t\varsigma^2 + sr\varsigma - 1}. \tag{1.9}$$

Remark 1.3. The following are specific instances of the polynomials $\hbar_m(s)$ that we present (see [19, 20]).

- The Fibonacci polynomials $F_\imath(r)$, if we take $c = d = 1 = t = s$.
- The Lucas polynomials $L_\imath(r)$, if we take $d = s = t = 1$ and $c = 2$.
- The Pell polynomials $P_\imath(r)$, if we take $d = s = 2$ and $c = t = 1$.
- The Pell-Lucas polynomials $Q_\imath(r)$, if we take $t = 1$ and $c = s = d = 2$.
- The Chebyshev polynomials $U_\imath(r)$ of the second kind, if we take $t = -1$, $c = 1$ and $s = d = 2$.

Definition 1.5. A function $f \in \mathfrak{M}^n_{p,q}(\zeta, \sigma, r)$ $(0 < q < p \leq 1,\ n \in \mathbb{N}\cup\{0\},\ \sigma \geq 0,\ 0 \leq \zeta < 1,\ c, d, r, s, t \in \mathbb{R},\ \imath \in \mathbb{N}\backslash\{1, 2\})$ if

$$(1 - \zeta)\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^\sigma + \zeta D_{p,q}\left(\mathfrak{D}^n_{p,q}f(\varsigma)\right)\left(\frac{\varsigma}{\mathfrak{D}^n_{p,q}f(\varsigma)}\right)^{\sigma-1} \prec \aleph(\varsigma, r) + 1 - c, \tag{1.10}$$

where $\aleph(\varsigma, r)$ is defined in (1.9).

Definition 1.6. A function $f \in \mathfrak{N}^n_{p,q}(\zeta, \sigma, r)$ $(0 < q < p \leq 1,\ n \in \mathbb{N},\ \sigma \geq 0,\ 0 \leq \zeta < 1,\ c, d, r, s, t \in \mathbb{R},\ \imath \in \mathbb{N}\backslash\{1, 2\})$ if

$$(1 - \zeta)\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^\sigma + \zeta D_{p,q}\left(\mathfrak{D}^n_{p,q}f(\varsigma)\right)\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma-1} \prec \aleph(\varsigma, r) + 1 - c, \tag{1.11}$$

where $\aleph(\varsigma, r)$ is defined in (1.9).

Definition 1.7. A function $f \in \mathfrak{L}^n_{p,q}(\rho, r)$ $(0 < q < p \leq 1,\ n \in \mathbb{N},\ \rho \geq 0)$ if

$$(1 + e^{i\rho}) \left(\frac{\varsigma D_{p,q} \left(\mathfrak{D}^n_{p,q} f(\varsigma) \right)}{\mathfrak{D}^n_{p,q} f(\varsigma)} \right) - e^{i\rho} \prec \aleph(\varsigma, r) + 1 - c, \tag{1.12}$$

where $\aleph(\varsigma, r)$ is defined in (1.9).

The following publications offer an interesting summary of the development and applications of Fekete-Szegö inequalities and bi-univalent functions. El-Ashwah and Hassan [13] derived Fekete-Szegö inequalities for functions within a specific subclass of analytic functions by applying the Sălăgean operator. Srivastava established and examined the first Maclaurin coefficient restrictions for new subclasses of analytic and m-fold symmetric bi-univalent functions produced by a linear combination [31]. In their study of fractional q-calculus, Srivastava et al. [36] also connected p-valent q-convex functions of complex order and p-valent q-starlike functions to the Fekete-Szegö problem. For reference, one can see [1, 3, 6, 35]. Using Horadam polynomials and (p, q)-analogue of the Sălăgean differential operator, we introduce and study new subclasses of analytic functions. The objective of this paper is to obtain the Fekete-Szegö inequalities associated with k-th root transformations $[f(\varsigma^k)]^{\frac{1}{k}}$ for functions in the subclasses of analytic univalent functions $\mathfrak{M}^n_{p,q}(\zeta, \sigma, r)$, $\mathfrak{N}^n_{p,q}(\zeta, \sigma, r)$ and $\mathfrak{L}^n_{p,q}(\rho, r)$ defined by the generalized fractional derivative operator $\mathfrak{D}^n_{p,q}$. Similar problems are investigated for $\frac{\varsigma}{f(\varsigma)}$ when $f(\varsigma)$ belonging to these classes. Our main conclusions will be presented using the following lemmas.

Lemma 1.1. ([2, Lemma 1, p. 36]). *Let*

$$\varpi(\varsigma) = \omega_1 \varsigma + \omega_2 \varsigma^2 + \cdots \tag{1.13}$$

be analytic function in $\mathbb{D}$ *satisfying the condition* $|\varpi(\varsigma)| < 1$. *Then,*

$$\left| \omega_2 - \nu \omega_1^2 \right| \leq \begin{cases} -\nu, & \text{if } \nu \leq -1, \\ 1, & \text{if } -1 \leq \nu \leq 1, \\ \nu, & \text{if } \nu \geq 1. \end{cases}$$

For $\nu > 1$ *or* $\nu < -1$, *equality is satisfied if and only if* $\varpi(\varsigma) = \varsigma$ *or one of its rotations. For* $-1 < \nu < 1$, *the equality is satisfied if and only if* $\varpi(\varsigma) = \varsigma^2$ *or one of its rotations. For* $\nu = -1$, *equality satisfies if and only if* $\varpi(\varsigma) = \frac{(\gamma+\varsigma)\varsigma}{(1+\gamma\varsigma)}$ $(0 \leq \gamma \leq 1)$ *or one of its rotations. For* $\nu = 1$, *equality is satisfied if and only if* $\varpi(\varsigma) = \frac{-(\gamma+\varsigma)\varsigma}{(1+\gamma\varsigma)}$ $(0 \leq \gamma \leq 1)$ *or one of its rotations.*

Lemma 1.2. ([25, inequality 7, p. 10]). *If* $\varpi(\varsigma)$ *is defined by* (1.13), *then*

$$\left| \omega_2 - \nu \omega_1^2 \right| \leq \max\{1, |\nu|\} \quad (\nu \in \mathbb{C}).$$

The result is sharp for $\varpi(\varsigma) = \varsigma$ *or* $\varpi(\varsigma) = \varsigma^2$.

2. Main Results

Unless otherwise mentioned, we shall assume throughout the paper that $\varsigma \in \mathbb{D}$, $0 < q < p \leq 1$, $n \in \mathbb{N}$, $\sigma \geq 0$, $0 \leq \zeta < 1$, $c, d, r, s, t \in \mathbb{R}$, $\imath \in \mathbb{N}\backslash\{1, 2\}$ and $\rho \geq 0$.

Definition 2.1. For $f(\varsigma) \in \mathcal{A}$ given by (1.1), the k-th root transform is defined by:

$$\begin{aligned} G(\varsigma) &= \left(f(\varsigma^k)\right)^{\frac{1}{k}} = \varsigma + \sum_{i \geq 1} b_{ik+1}\varsigma^{ik+1} \\ &= \varsigma + \frac{a_2}{k}\varsigma^{k+1} + \left(\frac{a_3}{k} - \frac{k-1}{2k^2}a_2^2\right)\varsigma^{2k+1} + \cdots . \end{aligned} \tag{2.1}$$

For $k = 1$ in (2.1), the function $G(\varsigma)$ reduces to $f(\varsigma)$.

Theorem 2.1. *If $f(\varsigma) \in \mathfrak{M}^n_{p,q}(\zeta, \sigma, r)$ and $G(\varsigma)$ is defined by (2.1), then*

$$|b_{2k+1} - \nu b_{k+1}^2| \leq \begin{cases} -\frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]^{2n}_{p,q}+(k-1+2\nu)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}{2k^2(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[2]^{2n}_{p,q}[3]^n_{p,q}}\hbar_2^2(r) \\ +\frac{\hbar_3(r)}{k(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}, & \nu \leq \tau_1, \\ \frac{\hbar_3(r)}{k(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}, & \tau_1 \leq \nu \leq \tau_2, \\ \frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]^{2n}_{p,q}+(k-1+2\nu)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}{2k^2(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[2]^{2n}_{p,q}[3]^n_{p,q}}\hbar_2^2(r) \\ -\frac{\hbar_3(r)}{k(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}, & \nu \geq \tau_2, \end{cases}$$

where

$$\begin{aligned} \tau_1 =& \frac{1}{2(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))[3]^n_{p,q}\hbar_2(r)}\bigg\{ \left(\frac{\hbar_3(r)}{\hbar_2(r)} - 1\right)\left(2k(\sigma - \zeta(2\sigma - [2]_{p,q} - 1))^2[2]^{2n}_{p,q}\right) \\ &- (\sigma - 1)(\sigma - 2[2]_{p,q}\zeta)k[2]^{2n}_{p,q}\hbar_2(r) - (k-1)(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))[3]^n_{p,q}\hbar_2(r)\bigg\}, \\ \tau_2 =& \frac{1}{2(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))[3]^n_{p,q}\hbar_2(r)}\bigg\{ \left(\frac{\hbar_3(r)}{\hbar_2(r)} + 1\right)\left(2k(\sigma - \zeta(2\sigma - [2]_{p,q} - 1))^2[2]^{2n}_{p,q}\right) \\ &- (\sigma - 1)(\sigma - 2[2]_{p,q}\zeta)k[2]^{2n}_{p,q}\hbar_2(r) - (k-1)(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))[3]^n_{p,q}\hbar_2(r)\bigg\} \end{aligned}$$

and

$$|b_{2k+1} - \nu b_{k+1}^2| \leq \frac{\hbar_2(r)}{k\left(\sigma - \zeta(2\sigma - [3]_{p,q} - 1)\right)[3]^n_{p,q}}\max\{1, |\xi|\},$$

where ν is any complex number and

$$\xi = \frac{(\sigma - 1)(\sigma - 2[2]_{p,q}\zeta)k[2]^{2n}_{p,q} + (k - 1 + 2\nu)(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))[3]^n_{p,q}}{2k(\sigma - \zeta(2\sigma - [2]_{p,q} - 1))^2[2]^{2n}_{p,q}}\hbar_2(r) - \frac{\hbar_3(r)}{\hbar_2(r)}, \tag{2.2}$$

where $\hbar_2(r)$ and $\hbar_3(r)$ are denoted by (1.8).

Proof. Let $f(\varsigma) \in \mathfrak{M}^n_{p,q}(\zeta, \sigma, r)$. Then,

$$(1-\zeta)\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta D_{p,q}\left(\mathfrak{D}^n_{p,q}f(\varsigma)\right)\left(\frac{\varsigma}{\mathfrak{D}^n_{p,q}f(\varsigma)}\right)^{\sigma-1} = \aleph(\varpi(\varsigma), r) + 1 - c,$$

where $\varpi(\varsigma)$ is given by (1.13). Since

$$\begin{aligned}\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma} =&1 + \sigma[2]^n_{p,q}a_2\varsigma + \left\{\sigma[3]^n_{p,q}a_3 + \frac{\sigma(\sigma-1)}{2}[2]^{2n}_{p,q}a_2^2\right\}\varsigma^2 + \cdots,\\ \left(\frac{\varsigma}{\mathfrak{D}^n_{p,q}f(\varsigma)}\right)^{\sigma-1} =&1 - (\sigma-1)[2]^n_{p,q}a_2\varsigma + \left\{\frac{\sigma(\sigma-1)}{2}[2]^{2n}_{p,q}a_2^2 - (\sigma-1)[3]^n_{p,q}a_3\right\}\varsigma^2 + \cdots,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&(1-\zeta)\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma} + \zeta D_{p,q}\left(\mathfrak{D}^n_{p,q}f(\varsigma)\right)\left(\frac{\varsigma}{\mathfrak{D}^n_{p,q}f(\varsigma)}\right)^{\sigma-1}\\ =&1 + (\sigma - \zeta(2\sigma - 1 - [2]_{p,q}))[2]^n_{p,q}a_2\varsigma\\ &+ \left\{\frac{(\sigma - 2[2]_{p,q}\zeta)(\sigma-1)}{2}[2]^{2n}_{p,q}a_2^2 + (\sigma - \zeta(2\sigma - [3]_{p,q} - 1))[3]^n_{p,q}a_3\right\}\varsigma^2 + \cdots\end{aligned} \tag{2.4}$$

and

$$\aleph(\varpi(\varsigma), r) + 1 - c = 1 + \hbar_2(r)\omega_1\varsigma + \left(\hbar_2(r)\omega_2 + \hbar_3(r)\omega_1^2\right)\varsigma^2 + \cdots, \tag{2.5}$$

we have, from (2.4) and (2.5),

$$a_2 = \frac{\hbar_2(r)\omega_1}{(\sigma - \zeta(2\sigma - 1 - [2]_{p,q}))\,[2]^n_{p,q}} \tag{2.6}$$

and

$$a_3 = \frac{1}{(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))\,[3]^n_{p,q}}\left\{\hbar_2(r)\omega_2 + \left(\hbar_3(r) - \frac{(\sigma-1)(\sigma - 2[2]_{p,q}\zeta)}{2(\sigma - \zeta(2\sigma - 1 - [2]_{p,q}))^2}\hbar_2^2(r)\right)\omega_1^2\right\}. \tag{2.7}$$

Using (2.1), (2.6) and (2.7), we have

$$b_{k+1} = \frac{\hbar_2(r)\omega_1}{k\,(\sigma - \zeta(2\sigma - 1 - [2]_{p,q}))\,[2]^n_{p,q}}$$

and

$$\begin{aligned}b_{2k+1} =&\frac{1}{k\,(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))\,[3]^n_{p,q}}\left\{\hbar_2(r)\omega_2 + \hbar_3(r)\omega_1^2\right.\\ &\left.- \frac{(\sigma-1)(\sigma - 2[2]_{p,q}\zeta)k[2]^{2n}_{p,q} + (k-1)(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))[3]^n_{p,q}}{2k(\sigma - \zeta(2\sigma - 1 - [2]_{p,q}))^2[2]^{2n}_{p,q}}\hbar_2^2(r)\omega_1^2\right\}.\end{aligned}$$

Therefore,

$$b_{2k+1} - \nu b_{k+1}^2 = \frac{\hbar_2(r)}{k\,(\sigma - \zeta(2\sigma - [3]_{p,q} - 1))\,[3]^n_{p,q}}\left\{\omega_2 - \xi\omega_1^2\right\},$$

where ξ is defined by (2.2). Applying Lemma 1.1, we obtain the following.

(i) If $\xi \leq -1$, then

$$\begin{aligned}\nu \leq &\frac{1}{2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)}\\ &\times\bigg\{\left(\frac{\hbar_3(r)}{\hbar_2(r)}-1\right)\left(2k(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2[2]_{p,q}^{2n}\right)\\ &-(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]_{p,q}^{2n}\hbar_2(r)-(k-1)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)\bigg\}\end{aligned}$$

and

$$\begin{aligned}&|b_{2k+1}-\nu b_{k+1}^2|\\ \leq&-\left(\frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]_{p,q}^{2n}+(k-1+2\nu)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}}{2k^2(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2\left(\sigma-\zeta(2\sigma-[3]_{p,q}-1)\right)[2]_{p,q}^{2n}[3]_{p,q}^{n}}\right)\hbar_2^2(r)\\ &+\frac{\hbar_3(r)}{k\left(\sigma-\zeta(2\sigma-[3]_{p,q}-1)\right)[3]_{p,q}^{n}}.\end{aligned}$$

(ii) If $-1 \leq \xi \leq 1$, then

$$\begin{aligned}&\frac{1}{2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)}\bigg\{\left(\frac{\hbar_3(r)}{\hbar_2(r)}-1\right)\left(2k(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2[2]_{p,q}^{2n}\right)\\ &-(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]_{p,q}^{2n}\hbar_2(r)-(k-1)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)\bigg\}\\ \leq&\nu\\ \leq&\frac{1}{2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)}\bigg\{\left(\frac{\hbar_3(r)}{\hbar_2(r)}+1\right)\left(2k(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2[2]_{p,q}^{2n}\right)\\ &-(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]_{p,q}^{2n}\hbar_2(r)-(k-1)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)\bigg\}\end{aligned}$$

and

$$|b_{2k+1}-\nu b_{k+1}^2| \leq \frac{\hbar_2(r)}{k\left(\sigma-\zeta(2\sigma-[3]_{p,q}-1)\right)[3]_{p,q}^{n}}.$$

(iii) If $\xi \geq 1$, then

$$\begin{aligned}\nu \geq&\frac{1}{2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)}\bigg\{\left(\frac{\hbar_3(r)}{\hbar_2(r)}+1\right)\left(2k(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2[2]_{p,q}^{2n}\right)\\ &-(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]_{p,q}^{2n}\hbar_2(r)-(k-1)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}\hbar_2(r)\bigg\}\end{aligned}$$

and

$$\begin{aligned}&|b_{2k+1}-\nu b_{k+1}^2|\\ \leq&\frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)k[2]_{p,q}^{2n}+(k-1+2\nu)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^{n}}{2k^2(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2\left(\sigma-\zeta(2\sigma-[3]_{p,q}-1)\right)[2]_{p,q}^{2n}[3]_{p,q}^{n}}\hbar_2^2(r)\\ &-\frac{\hbar_3(r)}{k\left(\sigma-\zeta(2\sigma-[3]_{p,q}-1)\right)[3]_{p,q}^{n}}.\end{aligned}$$

Applying Lemma 1.2, we obtain the second part of the theorem. □

Theorem 2.2. *If $f(\varsigma) \in \mathfrak{N}^n_{p,q}(\zeta,\sigma,r)$ and $G(\varsigma)$ is defined by (2.1), then*

(2.8)
$$|b_{2k+1}-\nu_1 b_{k+1}^2| \leq \begin{cases} \frac{-\{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))k[2]^{2n}_{p,q}+(k-1+2\nu)(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}\}\hbar_2^2(r)}{2k^2(\sigma-\zeta+[2]_{p,q}\zeta)^2(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}[2]^{2n}_{p,q}} \\ \quad +\frac{\hbar_3(r)}{k(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}, & \nu_1 \leq \tau_3, \\ \frac{\hbar_2(r)}{k(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}, & \tau_3 \leq \nu_1 \leq \tau_4, \\ \frac{\{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))k[2]^{2n}_{p,q}+(k-1+2\nu)(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}\}\hbar_2^2(r)}{2k^2(\sigma-\zeta+[2]_{p,q}\zeta)^2(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}[2]^{2n}_{p,q}} \\ \quad -\frac{\hbar_3(r)}{k(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}, & \nu_1 \geq \tau_4, \end{cases}$$

where
$$\begin{aligned}\tau_3 =& \left(\frac{\hbar_3(r)}{\hbar_2(r)}-1\right)\left(\frac{k(\sigma-\zeta+[2]_{p,q}\zeta)^2[2]^{2n}_{p,q}}{(\sigma-\zeta+[3]_{p,q}\zeta)[3]^n_{p,q}\hbar_2(r)}\right)\\ &-\frac{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))k[2]^{2n}_{p,q}+(k-1)(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}{2(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}},\\ \tau_4 =& \left(\frac{\hbar_3(r)}{\hbar_2(r)}+1\right)\left(\frac{k(\sigma-\zeta+[2]_{p,q}\zeta)^2[2]^{2n}_{p,q}}{(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}\hbar_2(r)}\right)\\ &-\frac{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))k[2]^{2n}_{p,q}+(k-1)(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}{2(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}\end{aligned}$$

and

$$|b_{2k+1}-\nu b_{k+1}^2| \leq \frac{\hbar_2(r)}{k(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}\max\{1,|\xi_2|\}, \tag{2.9}$$

where ν_1 is any complex number and

(2.10)
$$\xi_2 = -\frac{\hbar_3(r)}{\hbar_2(r)}+\frac{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))k[2]^{2n}_{p,q}+(k-1+2\nu)(\sigma-\zeta(1-[3]_{p,q}))[3]^n_{p,q}}{2k(\sigma-\zeta+[2]_{p,q}\zeta)^2[2]^{2n}_{p,q}}\hbar_2(r),$$

where $\hbar_2(r)$ and $\hbar_3(r)$ are given by (1.8).

Proof. Let $f(\varsigma) \in \mathfrak{N}^n_{p,q}(\zeta,\sigma,r)$. Then,

$$(1-\zeta)\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma}+\zeta D_{p,q}\left(\mathfrak{D}^n_{p,q}f(\varsigma)\right)\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma-1} = \aleph(\varpi(\varsigma),r)+1-c,$$

where $\varpi(\varsigma)$ is given by (1.13). Since

$$\begin{aligned}\left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma} =& 1+\sigma[2]^n_{p,q}a_2\varsigma+\left\{\sigma[3]^n_{p,q}a_3+\frac{\sigma(\sigma-1)[2]^{2n}_{p,q}}{2}a_2^2\right\}\varsigma^2+\cdots,\\ \left(\frac{\mathfrak{D}^n_{p,q}f(\varsigma)}{\varsigma}\right)^{\sigma-1} =& 1+(\sigma-1)[2]^n_{p,q}a_2\varsigma+\left\{(\sigma-1)[3]^n_{p,q}a_3+\frac{(\sigma-1)(\sigma-2)[2]^{2n}_{p,q}}{2}a_2^2\right\}\varsigma^2\\ &+\cdots\end{aligned}$$

and

$$
\begin{aligned}
&(1-\zeta)\left(\frac{\mathfrak{D}_{p,q}^{n} f(\varsigma)}{\varsigma}\right)^{\sigma}+\zeta D_{p,q}\left(\mathfrak{D}_{p,q}^{n} f(\varsigma)\right)\left(\frac{\mathfrak{D}_{p,q}^{n} f(\varsigma)}{\varsigma}\right)^{\sigma-1} \\
=&1+(\sigma-\zeta+[2]_{p,q}\zeta)[2]_{p,q}^{n} a_2\varsigma+\Big\{(\sigma-\zeta+[3]_{p,q}\zeta)[3]_{p,q}^{n} a_3 \\
&+\frac{(\sigma-1)(\sigma-2\zeta+2\zeta[2]_{p,q})}{2}[2]_{p,q}^{2n} a_2^2\Big\}\varsigma^2+\cdots .
\end{aligned} \tag{2.11}
$$

We have from (2.11) and (2.5)

$$
a_2=\frac{\hbar_2(r)\omega_1}{\left(\sigma-\zeta+[2]_{p,q}\zeta\right)[2]_{p,q}^{n}} \tag{2.12}
$$

and

$$
a_3=\frac{\hbar_2(r)\omega_2+\left(\hbar_3(r)-\frac{(\sigma-1)(\sigma-2\zeta+2[2]_{p,q}\zeta)}{2(\sigma-\zeta+[2]_{p,q}\zeta)^2}\hbar_2^2(r)\right)\omega_1^2}{\left(\sigma-\zeta+[3]_{p,q}\zeta\right)[3]_{p,q}^{n}}. \tag{2.13}
$$

Using (2.12), (2.13) and (2.1), we have

$$
b_{k+1}=\frac{\hbar_2(r)\omega_1}{k\left(\sigma-\zeta+[2]_{p,q}\zeta\right)[2]_{p,q}^{n}}
$$

and

$$
\begin{aligned}
b_{2k+1}=&\frac{1}{k\left(\sigma-\zeta+[3]_{p,q}\zeta\right)[3]_{p,q}^{n}}\Big\{\hbar_2(r)\omega_2+\hbar_3(r)\omega_1^2 \\
&-\frac{(\sigma-1)(\sigma-2\zeta+2[2]_{p,q}\zeta)[2]_{p,q}^{2n}k+(k-1)(\sigma-\zeta+[3]_{p,q}\zeta)[3]_{p,q}^{n}}{2k(\sigma-\zeta+[2]_{p,q}\zeta)^2[2]_{p,q}^{2n}}\mathrm{C}_1^2\omega_1^2\Big\}.
\end{aligned}
$$

Therefore,

$$
b_{2k+1}-\nu_1 b_{k+1}^2=\frac{\hbar_2(r)}{k(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^{n}}\Big\{\omega_2-\xi_2\omega_1^2\Big\},
$$

where ξ_2 is given in (2.10). The results (2.8) are established by Lemma 1.1 and inequality (2.9) by Lemma 1.2. $\square$

Theorem 2.3. *If $f(\varsigma) \in \mathfrak{L}^n_{p,q}(\rho, r)$ and $G(\varsigma)$ is defined by (2.1), then*

(2.14)
$$|b_{2k+1} - \nu_2 b_{k+1}^2| \leq \begin{cases} \frac{\hbar_3(r)}{([3]_{p,q}-1)[3]^n_{p,q}k(1+e^{i\rho})} \\ \quad -\frac{\left\{(k-1+2\nu)([3]_{p,q}-1)[3]^n_{p,q}-2([2]_{p,q}-1)k[2]^{2n}_{p,q}\right\}\hbar_2^2(r)}{2([2]_{p,q}-1)^2([3]_{p,q}-1)k^2(1+e^{i\rho})^2[3]^n_{p,q}[2]^{2n}_{p,q}}, & \nu_2 \leq \tau_5, \\ \frac{\hbar_2(r)}{([3]_{p,q}-1)[3]^n_{p,q}k(1+e^{i\rho})}, & \tau_5 \leq \nu_2 \leq \tau_6, \\ \frac{-\hbar_3(r)}{([3]_{p,q}-1)[3]^n_{p,q}k(1+e^{i\rho})} \\ \quad +\frac{\left\{(k-1+2\nu)([3]_{p,q}-1)[3]^n_{p,q}-2([2]_{p,q}-1)k[2]^{2n}_{p,q}\right\}\hbar_2^2(r)}{2([2]_{p,q}-1)^2([3]_{p,q}-1)k^2(1+e^{i\rho})^2[3]^n_{p,q}[2]^{2n}_{p,q}}, & \nu_2 \geq \tau_6, \end{cases}$$

where

$$\begin{aligned}\tau_5 =& \left(\frac{\hbar_3(r)}{\hbar_2(r)} - 1\right)\left(\frac{([2]_{p,q}-1)^2[2]^{2n}_{p,q}k(1+e^{i\rho})}{([3]_{p,q}-1)[3]^n_{p,q}\hbar_2(r)}\right) \\ &+ \frac{2([2]_{p,q}-1)k[2]^{2n}_{p,q} - (k-1)([3]_{p,q}-1)[3]^n_{p,q}}{2([3]_{p,q}-1)[3]^n_{p,q}}, \\ \tau_6 =& \left(\frac{\hbar_3(r)}{\hbar_2(r)} + 1\right)\left(\frac{([2]_{p,q}-1)^2[2]^{2n}_{p,q}k(1+e^{i\rho})}{([3]_{p,q}-1)[3]^n_{p,q}\hbar_2(r)}\right) \\ &+ \frac{2([2]_{p,q}-1)k[2]^{2n}_{p,q} - (k-1)([3]_{p,q}-1)[3]^n_{p,q}}{2([3]_{p,q}-1)[3]^n_{p,q}},\end{aligned}$$

and

$$|b_{2k+1} - \nu_2 b_{k+1}^2| \leq \frac{\hbar_2(r)}{([3]_{p,q}-1)[3]^n_{p,q}k(1+e^{i\rho})}\max\{1, |\xi_3|\}, \tag{2.15}$$

where ν_2 is any complex number and

$$\xi_3 = \frac{(k-1+2\nu)([3]_{p,q}-1)[3]^n_{p,q} - 2([2]_{p,q}-1)[2]^{2n}_{p,q}k}{2([2]_{p,q}-1)^2[2]^{2n}_{p,q}k(1+e^{i\rho})}\hbar_2(r) - \frac{\hbar_3(r)}{\hbar_2(r)}, \tag{2.16}$$

where $\hbar_2(r)$ and $\hbar_3(r)$ are given by (1.8).

Proof. Let $f(\varsigma) \in \mathfrak{L}^n_{p,q}(\rho, r)$. Then,

$$(1+e^{i\rho})\left(\frac{\varsigma D_{p,q}\left(\mathfrak{D}^n_{p,q}f(\varsigma)\right)}{\mathfrak{D}^n_{p,q}f(\varsigma)}\right) - e^{i\rho} = \aleph(\varpi(\varsigma), r) + 1 - c \quad (\varsigma \in \mathbb{D}),$$

where $\varpi(\varsigma)$ is given by (1.13). Since

$$\begin{aligned}&(1+e^{i\rho})\left(\frac{\varsigma D_{p,q}\left(\mathfrak{D}^n_{p,q}f(\varsigma)\right)}{\mathfrak{D}^n_{p,q}f(\varsigma)}\right) - e^{i\rho} \\ =&1 + ([2]_{p,q}-1)[2]^n_{p,q}(1+e^{i\rho})a_2\varsigma\end{aligned} \tag{2.17}$$

$$+\left\{([3]_{p,q}-1)[3]_{p,q}^{n}a_3+(1-[2]_{p,q})\,[2]_{p,q}^{2n}a_2^2\right\}(1+e^{i\rho})\varsigma^2+\cdots,$$

we have from (2.17) and (2.5)

$$a_2=\frac{\hbar_2(r)\omega_1}{([2]_{p,q}-1)[2]_{p,q}^{n}(1+e^{i\rho})} \tag{2.18}$$

and

$$a_3=\frac{\hbar_2(r)\omega_2+\left(\hbar_3(r)+\frac{\hbar_2^2(r)}{([2]_{p,q}-1)(1+e^{i\rho})}\right)\omega_1^2}{([3]_{p,q}-1)[3]_{p,q}^{n}(1+e^{i\rho})}. \tag{2.19}$$

Using (2.18), (2.19) and (2.1), we have

$$b_{k+1}=\frac{\hbar_2(r)\omega_1}{k([2]_{p,q}-1)[2]_{p,q}^{n}(1+e^{i\rho})}$$

and

$$\begin{aligned}b_{2k+1}=&\frac{1}{k([3]_{p,q}-1)[3]_{p,q}^{n}(1+e^{i\rho})}\Bigg(\hbar_2(r)\omega_2+\hbar_3(r)\omega_1^2\\&+\left(\frac{2([2]_{p,q}-1)[2]_{p,q}^{2n}k-(k-1)([3]_{p,q}-1)[3]_{p,q}^{n}}{2([2]_{p,q}-1)^2[2]_{p,q}^{2n}k(1+e^{i\rho})}\right)\hbar_2^2(r)\omega_1^2\Bigg).\end{aligned}$$

Therefore,

$$b_{2k+1}-\nu_2 b_{k+1}^2=\frac{\hbar_2(r)}{([3]_{p,q}-1)[3]_{p,q}^{n}k(1+e^{i\rho})}\left\{\omega_2-\xi_3\omega_1^2\right\},$$

where ξ_3 is defined by (2.16). The results (2.14) are established by Lemma 1.1 and the inequality (2.15) is established by Lemma 1.2. □

3. Fekete-Szegö Result

In this section, the Fekete-Szegö type coefficient inequalities associated with the rational function $\Omega(\varsigma)$ of the form

$$\Omega(\varsigma)=\frac{\varsigma}{f(\varsigma)}=1+\sum_{\kappa\geq 1}d_\kappa\varsigma^\kappa=1-a_2\varsigma+\left(a_2^2-a_3\right)\varsigma^2+\cdots, \tag{3.1}$$

where $d_1=-a_2$, $d_2=a_2^2-a_3$.

Theorem 3.1. *If $f(\varsigma) \in \mathfrak{M}^n_{p,q}(\zeta,\sigma,r)$ and $\Omega(\varsigma)$ is defined by* (3.1) *then*
(3.2)
$$|d_2-\nu_3 d_1^2| \leq \begin{cases} \frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)[2]^{2n}_{p,q}+2(1-\nu)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}{2(\sigma-\zeta(2\sigma-1-[2]_{p,q}))^2[2]^{2n}_{p,q}(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}\hbar_2^2(r) \\ -\frac{\hbar_3(r)}{(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}, & \nu_3 \leq \beta_1, \\ \frac{\hbar_2(r)}{(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}, & \beta_1 \leq \nu_3 \leq \beta_2, \\ -\frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)[2]^{2n}_{p,q}+2(1-\nu)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}{2(\sigma-\zeta(2\sigma-1-[2]_{p,q}))^2[2]^{2n}_{p,q}(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}\hbar_2^2(r) \\ +\frac{\hbar_3(r)}{(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}, & \nu_3 \geq \beta_2, \end{cases}$$

where
$$\begin{aligned}\beta_1 =& \frac{1}{2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}\hbar_2(r)}\Bigg\{\left(\frac{\hbar_3(r)}{\hbar_2(r)}-1\right)\left(2(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2[2]^{2n}_{p,q}\right) \\ &-(\sigma-1)(\sigma-2[2]_{p,q}\zeta)[2]^{2n}_{p,q}\hbar_2(r)-2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}\hbar_2(r)\Bigg\}, \\ \beta_2 =& \frac{1}{2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}\hbar_2(r)}\Bigg\{\left(\frac{\hbar_3(r)}{\hbar_2(r)}+1\right)\left(2(\sigma-\zeta(2\sigma-[2]_{p,q}-1))^2[2]^{2n}_{p,q}\right) \\ &-(\sigma-1)(\sigma-2[2]_{p,q}\zeta)[2]^{2n}_{p,q}\hbar_2(r)-2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}\hbar_2(r)\Bigg\}\end{aligned}$$

and

$$|d_2-\nu_3 d_1^2| \leq \frac{\hbar_2(r)}{(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}\max\{1,|\eta_1|\}, \tag{3.3}$$

where ν_3 is any complex number and
(3.4)
$$\eta_1 = -\frac{\hbar_3(r)}{\hbar_2(r)}+\frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)[2]^{2n}_{p,q}+2(1-\nu)(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}{2(\sigma-\zeta(2\sigma-1-[2]_{p,q}))^2[2]^{2n}_{p,q}}\hbar_2(r),$$

where $\hbar_2(r)$ and $\hbar_3(r)$ are given by (1.8).

Proof. Let $f(\varsigma) \in \mathfrak{M}^n_{p,q}(\zeta,\sigma,r)$. Using (2.6), (2.7) and (3.1), we have
$$d_1 = -\frac{\hbar_2(r)\omega_1}{(\sigma-\zeta(2\sigma-1-[2]_{p,q}))\,[2]^n_{p,q}}$$
and
$$\begin{aligned}d_2 =& -\frac{\hbar_2(r)\omega_2}{(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}} \\ &+\frac{1}{(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}\Bigg(-\hbar_3(r)\omega_1^2 \\ &+\frac{(\sigma-1)(\sigma-2[2]_{p,q}\zeta)[2]^{2n}_{p,q}+2(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]^n_{p,q}}{2(\sigma-\zeta(2\sigma-1-[2]_{p,q}))^2[2]^{2n}_{p,q}}\hbar_2^2(r)\omega_1^2\Bigg).\end{aligned}$$

Therefore,

$$d_2-\nu_3 d_1^2=\frac{-\hbar_2(r)}{(\sigma-\zeta(2\sigma-[3]_{p,q}-1))[3]_{p,q}^n}\left\{\omega_2-\eta_1\omega_1^2\right\},$$

where η_1 is defined by (3.4). The results (3.2) are established by Lemma 1.1 and the inequality (3.3) is established by Lemma 1.2. □

Theorem 3.2. *If $f(\varsigma)\in\mathfrak{N}_{p,q}^n(\zeta,\sigma,r)$ and $\Omega(\varsigma)$ is defined by* (3.1)*, then*

(3.5)

$$|d_2-\nu_4 d_1^2|\le\begin{cases}\frac{\{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))[2]_{p,q}^{2n}+2(1-\nu)(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n\}\hbar_2^2(r)}{2(\sigma-\zeta+[2]_{p,q}\zeta)^2(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n[2]_{p,q}^{2n}}\\ -\frac{\hbar_3(r)}{(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}, & \nu_4\le\beta_3,\\ \frac{\hbar_2(r)}{(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}, & \beta_3\le\nu_4\le\beta_4,\\ -\frac{\{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))[2]_{p,q}^{2n}+2(1-\nu)(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n\}\hbar_2^2(r)}{2(\sigma-\zeta+[2]_{p,q}\zeta)^2(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n[2]_{p,q}^{2n}}\\ +\frac{\hbar_3(r)}{(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}, & \nu_4\ge\beta_4,\end{cases}$$

where

$$\beta_3=\left(\frac{\hbar_3(r)}{\hbar_2(r)}-1\right)\left(\frac{(\sigma-\zeta+[2]_{p,q}\zeta)^2[2]_{p,q}^{2n}}{(\sigma-\zeta+[3]_{p,q}\zeta)[3]_{p,q}^n\hbar_2(r)}\right)-\frac{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))[2]_{p,q}^{2n}+2(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}{2(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n},$$

$$\beta_4=\left(\frac{\hbar_3(r)}{\hbar_2(r)}+1\right)\left(\frac{(\sigma-\zeta+[2]_{p,q}\zeta)^2[2]_{p,q}^{2n}}{(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n\hbar_2(r)}\right)-\frac{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))[2]_{p,q}^{2n}+2(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}{2(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}$$

and

$$|d_2-\nu_4 d_1^2|\le\frac{\hbar_2(r)}{(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}\max\{1,|\eta_2|\},\tag{3.6}$$

where ν_4 is any complex number and

(3.7)

$$\eta_2=-\frac{\hbar_3(r)}{\hbar_2(r)}+\frac{(\sigma-1)(\sigma-2\zeta(1-[2]_{p,q}))[2]_{p,q}^{2n}+2(1-\nu)(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}{2(\sigma-\zeta+[2]_{p,q}\zeta)^2[2]_{p,q}^{2n}}\hbar_2(r),$$

where $\hbar_2(r)$ and $\hbar_3(r)$ are given by (1.8)*.*

Proof. Let $f(\varsigma)\in\mathfrak{N}_{p,q}^n(\zeta,\sigma,r)$. Using (2.12), (2.13) and (3.1), we have

$$d_1=\frac{-\hbar_2(r)\omega_1}{(\sigma-\zeta+[2]_{p,q}\zeta)\,[2]_{p,q}^n}$$

and

$$d_2 = \frac{\hbar_2^2(r)\omega_1^2}{\left(\sigma-\zeta+[2]_{p,q}\zeta\right)^2 [2]_{p,q}^{2n}} - \frac{\hbar_2(r)\omega_2 + \left(\hbar_3(r) - \frac{(\sigma-1)(\sigma-2\zeta+2[2]_{p,q}\zeta)}{2(\sigma-\zeta+[2]_{p,q}\zeta)^2}\hbar_2^2(r)\right)\omega_1^2}{\left(\sigma-\zeta+[3]_{p,q}\zeta\right)[3]_{p,q}^n}.$$

Therefore,

$$d_2 - \nu_4 d_1^2 = \frac{-\hbar_2(r)}{(\sigma-\zeta(1-[3]_{p,q}))[3]_{p,q}^n}\left\{\omega_2 - \eta_2\omega_1^2\right\},$$

where η_2 is given in (3.7). The results (3.5) are established by Lemma 1.1 and inequality (3.6) by Lemma 1.2. ☐

Theorem 3.3. *If $f(\varsigma) \in \mathfrak{L}_{p,q}^n(\rho,r)$ and $\Omega(\varsigma)$ is defined by (3.1), then*

(3.8)

$$|d_2-\nu_5 d_1^2| \le \begin{cases} -\frac{\hbar_3(r)}{\left([3]_{p,q}-1\right)[3]_{p,q}^n(1+e^{i\rho})} + \frac{\left\{(1-\nu)([3]_{p,q}-1)[3]_{p,q}^n-([2]_{p,q}-1)[2]_{p,q}^{2n}\right\}\hbar_2^2(r)}{\left([2]_{p,q}-1\right)^2\left([3]_{p,q}-1\right)(1+e^{i\rho})^2[3]_{p,q}^n[2]_{p,q}^{2n}}, & \nu_5 \le \beta_5, \\ \frac{\hbar_2(r)}{\left([3]_{p,q}-1\right)[3]_{p,q}^n(1+e^{i\rho})}, & \beta_5 \le \nu_5 \le \beta_6, \\ \frac{\hbar_3(r)}{\left([3]_{p,q}-1\right)[3]_{p,q}^n(1+e^{i\rho})} - \frac{\left\{(1-\nu)([3]_{p,q}-1)[3]_{p,q}^n-([2]_{p,q}-1)[2]_{p,q}^{2n}\right\}\hbar_2^2(r)}{([2]_{p,q}-1)^2([3]_{p,q}-1)(1+e^{i\rho})^2[3]_{p,q}^n[2]_{p,q}^{2n}}, & \nu_5 \ge \beta_6, \end{cases}$$

where

$$\beta_5 = \left(\frac{\hbar_3(r)}{\hbar_2(r)} - 1\right)\left(\frac{([2]_{p,q}-1)^2[2]_{p,q}^{2n}(1+e^{i\rho})}{([3]_{p,q}-1)[3]_{p,q}^n\hbar_2(r)}\right) + \frac{([2]_{p,q}-1)[2]_{p,q}^{2n} - ([3]_{p,q}-1)[3]_{p,q}^n}{([3]_{p,q}-1)[3]_{p,q}^n},$$

$$\beta_6 = \left(\frac{\hbar_3(r)}{\hbar_2(r)} + 1\right)\left(\frac{([2]_{p,q}-1)^2[2]_{p,q}^{2n}(1+e^{i\rho})}{([3]_{p,q}-1)[3]_{p,q}^n\hbar_2(r)}\right) + \frac{([2]_{p,q}-1)[2]_{p,q}^{2n} - ([3]_{p,q}-1)[3]_{p,q}^n}{([3]_{p,q}-1)[3]_{p,q}^n},$$

and

$$|d_2-\nu_5 d_1^2| \le \frac{\hbar_2(r)}{([3]_{p,q}-1)[3]_{p,q}^n(1+e^{i\rho})}\max\{1,|\eta_3|\}, \tag{3.9}$$

where ν_5 is any complex number and

$$\eta_3 = \frac{(1-\nu)([3]_{p,q}-1)[3]_{p,q}^n - ([2]_{p,q}-1)[2]_{p,q}^{2n}}{([2]_{p,q}-1)^2[2]_{p,q}^{2n}(1+e^{i\rho})}\hbar_2(r) - \frac{\hbar_3(r)}{\hbar_3(r)}, \tag{3.10}$$

where $\hbar_2(r)$ and $\hbar_3(r)$ are given by (1.8).

Proof. Let $f(\varsigma) \in \mathfrak{L}_{p,q}^n(\rho,r)$. Using (2.18), (2.19) and (3.1), we have

$$d_1 = \frac{-\hbar_2(r)\omega_1}{([2]_{p,q}-1)[2]_{p,q}^n(1+e^{i\rho})}$$

and

$$d_2 = \frac{\hbar_2^2(r)\omega_1^2}{([2]_{p,q}-1)[2]_{p,q}^n(1+e^{i\rho})} - \frac{\hbar_2(r)\omega_2 + \left(\hbar_3(r) + \frac{\hbar_2^2(r)}{([2]_{p,q}-1)(1+e^{i\rho})}\right)\omega_1^2}{([3]_{p,q}-1)[3]_{p,q}^n(1+e^{i\rho})}.$$

Therefore,

$$d_2 - \nu_5 d_1^2 = \frac{\hbar_2(r)}{([3]_{p,q} - 1)[3]_{p,q}^n (1 + e^{i\rho})} \Big\{ \omega_2 - \eta_3 \omega_1^2 \Big\},$$

where η_3 is defined by (3.10). The results (3.8) are established by Lemma 1.1 and the inequality (3.9) is established by Lemma 1.2. □

4. Conclusion

Using the concept of the subordination principle, we propose an approach that incorporates the Horadam polynomials together with the (p, q)-analogue of Sălăgean differential operator. These subclasses are characterized through the generalized fractional derivative operator $\mathfrak{D}_{p,q}^n$, which provides a flexible framework for studying various geometric properties. The main objective of this paper is to establish Fekete-Szegö inequalities associated with the k-th roots transformations $[f(\varsigma^k)]^{\frac{1}{k}}$ for functions in the subclasses of analytic univalent functions $\mathfrak{M}_{p,q}^n(\zeta, \sigma, r)$, $\mathfrak{N}_{p,q}^n(\zeta, \sigma, r)$ and $\mathfrak{L}_{p,q}^n(\rho, r)$. In addition, we consider similar problems for the reciprocal function $\frac{\varsigma}{f(\varsigma)}$ when $f(\varsigma)$ belongs to these classes.

References

[1] A. Amourah, B. A. Frasin and T. M. Seoudy, *An application of Miller-Ross-type Poisson distribution on certain subclasses of bi-univalent functions subordinate to Gegenbauer polynomials*, Mathematics **10** (2022), Article ID 2462. https://doi.org/10.3390/math10142462

[2] R. M. Ali, V. Ravichandran and N. Seenivasagan, *Coefficient bounds for p-valent functions*, Appl. Math. Comput. **187** (2007), 35–46. https://doi.org/10.1016/j.amc.2006.08.100

[3] E. E. Ali, H. M. Srivastava, W. Y. Kota, R. M. El-Ashwah and A. M. Albalah, *The second Hankel determinant and the Fekete-Szegö functional for a subclass of analytic functions by using the q-Sălăgean derivative operator*, Alexandria Engineering Journal **116** (2025), 141–146. https://doi.org/10.1016/j.aej.2024.12.019

[4] R. Chakrabarti and R. Jagannathan, *A (p, q)-oscillator realization of two-parameter quantum algebras*, J. Phys. A: Math. Gen. **24** (1991), 1711–1718.

[5] P. N. Chichra, *New subclasses of the class of close-to-convex functions*, Proc. Amer. Math. Soc. **62** (1977), 37–43.

[6] A. Chonweerayoot, D. K. Thomas and W. Upakamitikaset, *On the Fekete-Szegö theorem for close-to-convex functions*, Publ. Inst. Math. (Beograd) (N.S.) **52**(66) (1992), 18–26.

[7] H. E. Darwish, A. Y. Lashin, R. M. El-Ashwah and E. M. Madar, *Fekete-Szeqö type coefficient inequalities for certain subclasses of analytic functions*, South East Asian J. Math. Math. Sci. **16**(1) (2020), 1–14.

[8] S. Ding, Y. Ling and G. Bao, *Some properties of a class of analytic functions*, J. Math. Anal. Appl. **195** (1995), 71–81.

[9] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[10] R. M. El-Ashwah, *Subclasses of bi-univalent functions defined by convolution*, J. Egyptian Math. Soc. **22** (2014), 384–351. https://doi.org/10.1016/j.joems.2013.06.017

[11] R. M. El-Ashwah and D. K. Thomas, *Some coefficient, length, area results for a subclass of Bazilevič functions*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **33**(81)(4) (1989), 319–333.

[12] R. M. El-Ashwah and D. K. Thomas, *Growth results for a subclass of Bazilevič functions*, Int. J. Math. Math. Sci. **8**(4) (1985), 785–793.

[13] R. M. El-Ashwah and A. H. Hassan, *Fekete-Szeqö inequalities for certain subclass of analytic functions defined by using Sălăgean operator*, Miskolc Math. Notes **17**(2) (2017), 827–836. https://doi.org/10.18514/MMN.2017.1495

[14] H. Exton, *q-Hypergeometric Functions and Applications*, Hastead Press, New York, USA, 1983.

[15] E. Frenkel and E. Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, Commun. Math. Phys. **216** (2001), 23–57. https://doi.org/doi:10.1007/s002200000323

[16] A. W. Goodman, *Univalent Functions*, Vol. 1–2, Mariner, Tampa, Florida, 1983.

[17] M. Govindaraj and S. Sivasubramanian, *On a class of analytic functions related to conic domains involving q-calculus*, Analysis Math. **43**(3) (2017), 475–487. https://doi.org/doi:10.1007/s10476-017-0206-5

[18] V. Gupta, (p,q)*-Baskakov-Kantorovich operators*, Appl. Math. Inf. Sci. **10**(4) (2016), 1551–1556. http://dx.doi.org/10.18576/amis/100433

[19] A. F. Horadam and B. J. Mahon, *Pell and Pell-Lucas polynomials*, Fibonacci Quart. **23** (1985), 7–20.

[20] T. Horzum and E. G. Kocer, *On some properties of Horadam polynomials*, Int. Math. Forum **4**(25) (2009), 1243–1252.

[21] M. E. H. Ismail, E. Merkes and D. Styer, *A generalization of starlike functions*, Complex Variables, Theory and Application **14** (1990), 77–84.

[22] F. H. Jackson, *On q-definite integrals*, Quart. J. Pure Appl. Math. **41**(15) (1910), 193–203.

[23] F. H. Jackson, *q-Difference equations*, Amer. J. Math. **32**(4) (1910), 305–314.

[24] S. Kanas and D. Răducanu, *Some class of analytic functions related to conic domains*, Math. Slovaca **64** (2014), 1183–1196. https://doi.org/10.2478/s12175-014-0268-9

[25] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20**(1) (1969), 8–12.

[26] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*,in: Z. Li, F. Ren, L. Lang and S. Zhang (Eds.), *Proceddings of the Conference on Complex Analysis*, Nankai University, Tianjin, 1992, 157–169.

[27] T. H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc **104** (1962), 532–537.

[28] T. Panigrahi and R. M. El-Ashwah, *Coefficient inequalities for class of bi-univalent functions associated with* (p,q)*-derivative of Sălăgean operator*, Acta Univ. Apulensis **63** (2020), 109–122. https://doi.org/10.17114/j.aua.2020.63.09

[29] G. S. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Mathematics **1013**, Complex Analysis-Fifth Romanian-Finnish Seminar, Bucharest, Romania, 1981, 362–372.

[30] H. M. Srivastava, *Univalent functions, fractional calculus and associated generalized hypergeometric functions*, in: H. M. Srivastava and S. Owa (Eds.), *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press, Chichester, UK, John Wiley and Sons, New York, USA, 1989, 329–354.

[31] H. M. Srivastava and A. K. Wanas, *Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination*, Kyungpook Math. J. **59** (2019), 493–503. https://doi.org/10.5666/KMJ.2019.59.3.493

[32] H. M. Srivastava, *Operators of basic* (*or q-*) *calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), 327–344. https://doi.org/10.1007/s40995-019-00815-0

[33] H. M. Srivastava, *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations*, J. Nonlinear Convex Anal. **22** (2021), 1501–1520.

[34] H. M. Srivastava, *A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics*, Symmetry **13** (2021), Article ID 2294. https://doi.org/10.3390/sym13122294

[35] H. M. Srivastava, K. Mishra and M. K. Das, *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Var. **44** (2001), 145–163.

[36] H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, *Basic and fractional q-calculus and associated the Fekete-Szegö problem for p-valently q-starlike functions and p-valently q-convex functions of complex order*, Miskolc Math. Notes **20**(1) (2019), 489–509. https://doi.org/10.18514/MMN.2019.2405

[37] H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan and S. Hussain, *The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator*, Stud. Univ. Babes Bolyai Math. **63** (2018), 419–436. https://doi.org/10.24193/subbmath.2018.4.01

[38] S. M. Zagorodnyuk, *On a family of hypergeometric Sobolev orthogonal polynomials on the unit circle*, Constr. Math. Anal. **3** (2020), 75–84. https://doi.org/10.33205/cma.690236

[1]Department of Mathematics,
Faculty of Science,
Damietta University, Egypt
Email address: r_elashwah@yahoo.com or relashwah@du.edu.eg
ORCID iD: https://orcid.org/0000-0002-5490-3745
Email address: wafaa_kota@yahoo.com or wafaa_kota88@du.edu.eg
ORCID iD: https://orcid.org/0000-0002-5280-0822