Kragujevac Journal of Mathematics
Volume 51(8) (2027), Pages 1307–1319.

A COMMON BEST PROXIMITY POINT THEOREM FOR C-DOMINATED CONTRACTIONS AND AN APPLICATION TO A SECOND-ORDER BOUNDARY VALUE PROBLEM

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Abstract. This work presents novel findings in the theory of best proximity points for non-self mappings in complete metric spaces. We introduce the concept of $\mathcal{C}$-class dominated mapping pairs and establish conditions guaranteeing the existence of common best proximity points. Our results generalize several known theorems in fixed point theory and have applications to boundary value problems.

1. Introduction

Fixed point theory is a cornerstone of nonlinear analysis and has been instrumental in solving a wide range of mathematical problems across disciplines such as differential equations, optimization, game theory, and dynamical systems. A foundational result in this area is the Banach Contraction Principle, which asserts that a contraction mapping on a complete metric space possesses a unique fixed point and that iterative procedures converge to it. While the Banach Contraction Principle guarantees unique fixed points for self-mappings in complete metric spaces, many practical problems involve non-self mappings where traditional fixed points may not exist. This limitation motivated the development of *best proximity point* theory, which seeks points that minimize the distance between a point and its image under such mappings. For nonempty subsets P and Q of a metric space $(\mathcal{X}, d)$, an element $x_* \in P$ is called a *best proximity point* of a given mapping $\mathcal{T} : P \to Q$ if

$$d(x_*, \mathcal{T}x_*) = \inf\{d(x, \mathcal{T}x) : x \in P\} = d(P, Q) =: \inf\{d(a, b) : a \in P, b \in Q\}.$$

Key words and phrases. Common best proximity point, C- class function, $\mathcal{C}$-class dominated mapping pair.

2020 *Mathematics Subject Classification.* Primary: 47H10. Secondary: 54H25, 46S40.

DOI

Received: January 17, 2026.

Accepted: June 08, 2026.

This concept generalizes fixed point theory to broader settings where the mapping does not admit any fixed points, such as in constrained optimization or the approximation of non-self mappings. It retains the spirit of fixed point theory, seeking points stable under a mapping in some optimal sense, while adapting to more general and realistic problem frameworks. The theory of best proximity points has been rigorously developed in seminal works such as Eldred and Veeramani [1], who laid the foundation for best proximity point theory by establishing existence and uniqueness results under certain contractive conditions. The work of Kirk and Srinivasan [2] further investigated best proximity points for relatively nonexpansive maps and developed convergence theorems in Banach spaces. These foundational studies spurred a wide range of extensions to richer and more complex structures, including partially ordered metric spaces, which combine order-theoretic and topological properties to study convergence and uniqueness (e.g., [5]); modular function spaces, which analyze best proximity points in spaces where norm-based methods are inadequate or replaced by modular structures (e.g., [3]); and graph-theoretic approaches, which represent interactions among elements of P and Q via graphs, leading to more nuanced fixed point-like results in discrete or networked systems (e.g., [4]). These developments not only broaden the theoretical reach of the concept but also enhance its applicability in computer science, economics, game theory, and differential equations.

Recent research has increasingly focused on multi-mapping interactions, particularly *common best proximity points* for pairs of mappings under contractive conditions. Significant contributions include [6–17]. Notably, Ayari and Aydi [18] introduced common coincidence best proximity points for dominating proximal generalized C-contraction pairs in metric spaces. In this work,

- we introduce a novel class of non-self mappings;
- we propose the concept of $\mathcal{C}$-domination for mapping pairs;
- we establish sufficient conditions for common best proximity points;
- we provide concrete illustrative examples.

Our results generalize and extend previous work by [19], offering broader applications in metric fixed point theory. The presented framework demonstrates substantial advantages in handling complex mapping interactions.

2. Preliminaries

Let us consider three non-empty subsets P, Q and K of a metric space $(\mathcal{X}, d)$.

Definition 2.1 ([20])**.** Let $\mathcal{S} : P \to Q$ be a non-self mapping. A point x_* is called a best proximity point of $\mathcal{S}$ if $d(x_*, \mathcal{S}x_*) = d(P, Q)$.

Definition 2.2 ([19])**.** Let us consider four non-self mappings $\mathcal{S}, \mathcal{T} : P :\to Q$ and $\mathcal{F}, \mathcal{G}, : P \to K$. The pair $(\mathcal{F}, \mathcal{S})$ is said to be strongly dominated by the pair $(\mathcal{G}, \mathcal{T})$ if there exists $\alpha \in (0, 1)$, such that

$$d(\mathcal{F}\mathcal{S}x_1, \mathcal{F}\mathcal{S}x_2) \leq \alpha d(\mathcal{G}\mathcal{T}x_1, \mathcal{G}\mathcal{T}x_2), \quad \text{for all } x_1, x_2 \in P.$$

We denote $\mathcal{F} \circ \mathcal{G}$ as $\mathcal{F}\mathcal{G}$.

Definition 2.3 ([19])**.** Let non-self-mappings $\mathcal{S} : P \to Q, \mathcal{T} : P \to Q$ and $\mathcal{F} : Q \to P$ be given. The mapping $\mathcal{F}$ is said to commute with the pair $(\mathcal{S}, \mathcal{T})$ if $\mathcal{S}\mathcal{F}\mathcal{T} = \mathcal{T}\mathcal{F}\mathcal{S}$.

Definition 2.4 ([21])**.** A bivariate function $\mathcal{H} : [0, +\infty)^2 \to \mathbb{R}$ is called a $\mathcal{C}$-*function* if it satisfies:

(a) $\mathcal{H}(x, y) \leq x$ for all $x, y \geq 0$;
(b) $\mathcal{H}(x, y) = x$ implies either $x = 0$ or $y = 0$.

Additionally, $\mathcal{C}$-functions must be continuous throughout their domain.

The family of all such $\mathcal{C}$-functions is denoted by $\mathfrak{C}$. Note that for certain $\mathcal{H} \in \mathfrak{C}$, we may have $\mathcal{H}(0, 0) = 0$.

Proposition 2.1. *The following are examples of* $\mathcal{C}$*-functions* (*for all* $x, y \geq 0$)*:*

(a) $\mathcal{H}(x, y) = x - y$ (*implies* $y = 0$*, when* $\mathcal{H}(x, y) = x$)*;*
(b) $\mathcal{H}(x, y) = \lambda x$*, where* $\lambda \in (0, 1)$ (*implies* $x = 0$*, when* $\mathcal{H}(x, y) = x$)*;*
(c) $\mathcal{H}(x, y) = x/(1 + x)^\rho$ *for* $\rho > 0$ (*implies* $x = 0$*, when* $\mathcal{H}(x, y) = x$)*;*
(d) $\mathcal{H}(x, y) = \tau(x)$ *where* τ *is a Mizoguchi-Takahashi type function* (*implies* $x = 0$*, when* $\mathcal{H}(x, y) = x$)*;*
(e) $\mathcal{H}(x, y) = \frac{x}{\Gamma(1/2)} \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t+y}} dt$ (*Euler Gamma function* Γ).

3. Main Developments

In the beginning, we propose the coming concepts.

Definition 3.1. Let us consider four non self-mappings $\mathcal{S}, \mathcal{T} : P :\to Q$ and $\mathcal{F}, \mathcal{G} : P \to K$. The pair $(\mathcal{F}, \mathcal{S})$ is said to be $\mathcal{C}$ -dominated by the pair $(\mathcal{G}, \mathcal{T})$ if there exists $\mathcal{H} \in \mathfrak{C}$, such that

$$d(\mathcal{F}\mathcal{S}x_1, \mathcal{F}\mathcal{S}x_2) \leq \mathcal{H}(d(\mathcal{G}\mathcal{T}x_1, \mathcal{G}\mathcal{T}x_2), d(\mathcal{G}\mathcal{T}x_1, \mathcal{G}\mathcal{T}x_2)),$$

for all $x_1, x_2 \in P$.

We now present our principal result concerning the existence of optimal proximity points.

Theorem 3.1. *Let* (X, d) *be a complete metric space with nonempty closed subsets* P *and* Q*. Given four non-self mappings,* $\mathcal{S}, \mathcal{T} : P \to Q$ *and* $\mathcal{F}, \mathcal{G} : Q \to P$ *satisfying:*

(**C1**) *commuting relations:* $\mathcal{F}\mathcal{S}\mathcal{G}\mathcal{T} = \mathcal{G}\mathcal{T}\mathcal{F}\mathcal{S}$ *and* $\mathcal{S}\mathcal{F}\mathcal{T}\mathcal{G} = \mathcal{T}\mathcal{G}\mathcal{S}\mathcal{F}$*;*
(**C2**) *dominance relations: the pair* $(\mathcal{F}, \mathcal{S})$ *is* C*-dominated by* $(\mathcal{G}, \mathcal{T})$ *and* $(\mathcal{S}, \mathcal{F})$ *is* $\mathcal{C}$*-dominated by* $(\mathcal{T}, \mathcal{G})$*;*
(**C3**) *range inclusions:* $\mathcal{F}\mathcal{S}(P) \subseteq \mathcal{G}\mathcal{T}(P)$ *and* $\mathcal{S}\mathcal{F}(Q) \subseteq \mathcal{T}\mathcal{G}(Q)$*;*
(**C4**) $\mathcal{S}$ *and* $\mathcal{T}$ *commute with the pair* $(\mathcal{F}, \mathcal{G})$*, and* $\mathcal{F}$ *and* $\mathcal{G}$ *commute with the pair* $(\mathcal{S}, \mathcal{T})$*;*
(**C5**) *continuity requirements:* $\mathcal{F}\mathcal{S}$ *and* $\mathcal{T}\mathcal{G}$ *are continuous operators;*

(**C6**) *contractive estimate: there exists* $\kappa \in (0,1)$ *such that*

$$d(\mathcal{S}x, \mathcal{F}\mathcal{S}x) \leq \kappa(d(\mathcal{T}x, \mathcal{G}\mathcal{T}x) - \operatorname{dist}(P,Q)) + \operatorname{dist}(P,Q).$$

Then, $\mathcal{S}$ *and* $\mathcal{T}$ *have a common best proximity point* $p_* \in P$*. Also,* $\mathcal{F}$ *and* $\mathcal{G}$ *have a common best proximity point* $q_* \in Q$ *with* $d(p_*, q_*) = \operatorname{dist}(P,Q)$.

Proof. Using the fact that P is nonempty and $\mathcal{F}\mathcal{S}(P) \subseteq \mathcal{G}\mathcal{T}(P)$, there exist $z_0, z_1 \in P$ such that $\mathcal{F}\mathcal{S}(z_0) = \mathcal{G}\mathcal{T}(z_1)$. In the same manner, by induction, we can build a sequence $\{z_n\} \subset P$ such that $\mathcal{F}\mathcal{S}(z_n) = \mathcal{G}\mathcal{T}(z_{n+1})$. Using the fact that $(\mathcal{F}, \mathcal{S})$ is $\mathcal{C}$ -dominated by the pair $(\mathcal{G}, \mathcal{T})$, so there exists $\mathcal{H} \in \mathfrak{C}$, such that

$$d(\mathcal{F}\mathcal{S}z_n, \mathcal{F}\mathcal{S}z_{n+1}) \leq \mathcal{H}(d(\mathcal{F}\mathcal{S}z_n, \mathcal{F}\mathcal{S}z_{n-1}), \quad d(\mathcal{F}\mathcal{S}z_n, \mathcal{F}\mathcal{S}z_{n-1})) \leq d(\mathcal{F}\mathcal{S}z_n, \mathcal{F}\mathcal{S}z_{n-1}), \tag{3.1}$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{d(\mathcal{F}\mathcal{S}z_{n+1}, \mathcal{F}\mathcal{S}z_n)\}$ is decreasing. So, the sequence $\{d(\mathcal{F}\mathcal{S}z_{n+1}, \mathcal{F}\mathcal{S}z_n)\}$ is also decreasing and has nonnegative terms, then there exists $\eta \geq 0$ such that

$$\eta = \lim_{n\to+\infty} d(\mathcal{F}\mathcal{S}z_n, \mathcal{F}\mathcal{S}z_{n+1}).$$

By letting $n \to +\infty$ on (3.1) and using the continuity of $\mathcal{H}$ and $\mathcal{F}\mathcal{S}$, we obtain that

$$\eta \leq \mathcal{H}(\eta, \eta) \leq \eta.$$

Consequently, we find that $\mathcal{H}(\eta, \eta) = \eta$, and so $\eta = 0$. Now, we claim that the sequence $\{\mathcal{F}\mathcal{S}z_n\}$ is a Cauchy sequence. Suppose it is not the case. Then there exists μ and sequences $\{\mathcal{F}\mathcal{S}z_{m_p}\}$ and $\{\mathcal{F}\mathcal{S}z_{n_p}\}$ such that, for all positive integers p such that $m(p) > n(p) > p$, $d(\mathcal{F}\mathcal{S}z_{m(p)}, \mathcal{F}\mathcal{S}z_{n(p)}) \geq \mu$ and $d(\mathcal{F}\mathcal{S}z_{m(p)}, \mathcal{F}\mathcal{S}z_{n(p)-1}) < \mu$. Using the triangular inequality, we get for all p

$$\begin{aligned}\mu \leq & d(\mathcal{F}\mathcal{S}z_{m(p)}, \mathcal{F}\mathcal{S}z_{n(p)}) \leq d(\mathcal{F}\mathcal{S}z_{m(p)}, \mathcal{F}\mathcal{S}z_{m(p)-1}) + d(\mathcal{F}\mathcal{S}z_{m(p)-1}, \mathcal{F}\mathcal{S}z_{n(p)}) \\ \leq & \mu + d(\mathcal{F}\mathcal{S}z_{m(p)}, \mathcal{F}\mathcal{S}z_{m(p)-1}).\end{aligned} \tag{3.2}$$

Taking limit as $p \to +\infty$ in the overhead inequality (3.2), we get

$$\lim_{p\to+\infty} d(\mathcal{F}\mathcal{S}z_{m(p)}, \mathcal{F}\mathcal{S}z_{n(p)}) = \mu > 0. \tag{3.3}$$

Denote by $x_n = \mathcal{F}\mathcal{S}z_n$. Using additionally the triangular inequality, we have

$$\begin{aligned}d(x_{m(p)}, x_{n(p)}) \leq & d(x_{m(p)-1}, x_{n(p)-1}) + d(x_{m(p)}, x_{m(p)-1}) + d(x_{n(p)-1}, x_{n(p)}) \\ \leq & 2d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)}) + 2d(x_{n(p)}, x_{n(p)-1}).\end{aligned} \tag{3.4}$$

If we apply the limit on both sides and in (3.4) and using (3.3), we obtain

$$\lim_{p\to+\infty} d(\mathcal{F}\mathcal{S}z_{m(p)-1}, \mathcal{F}\mathcal{S}z_{n(p)-1}) = \mu. \tag{3.5}$$

Using (3.1), we obtain

$$\begin{aligned}d(\mathcal{F}\mathcal{S}z_{m(p)}, \mathcal{F}\mathcal{S}z_{n(p)}) \leq & \mathcal{H}(d(\mathcal{F}\mathcal{S}z_{m(p)-1}, \mathcal{F}\mathcal{S}z_{n(p)-1}), d(\mathcal{F}\mathcal{S}z_{m(p)-1}, \mathcal{F}\mathcal{S}z_{n(p)-1})) \\ \leq & d(\mathcal{F}\mathcal{S}z_{m(p)-1}, \mathcal{F}\mathcal{S}z_{n(p)-1}).\end{aligned}$$

Letting $p \to +\infty$ in the above inequality, we get

$$\mu \leq \mathcal{H}(\mu, \mu) \leq \mu.$$

Thus, $\mu = 0$ which contradicts (3.3). Therefore, the sequence $\{\mathcal{F}\mathcal{S}z_n\}$ is a Cauchy sequence. Using the closeness of the subset P of the complete metric space (X, d), there exists $z_* \in P$ such that

$$\lim_{n\to+\infty} \mathcal{F}\mathcal{S}z_n = \lim_{n\to+\infty} \mathcal{G}\mathcal{T}z_n = z_*.$$

In view of the fact $\mathcal{F}\mathcal{S}$ and $\mathcal{G}\mathcal{T}$ are continuous, we get $\lim_{n\to+\infty}(\mathcal{G}\mathcal{T})(\mathcal{F}\mathcal{S})z_n = \mathcal{G}\mathcal{T}z_*$ and $\lim_{n\to+\infty}(\mathcal{F}\mathcal{S})(\mathcal{G}\mathcal{T})z_n = \mathcal{F}\mathcal{S}z_*$. Since $\mathcal{F}\mathcal{S}$ and $\mathcal{G}\mathcal{T}$ commute, then $\mathcal{F}\mathcal{S}z_* = \mathcal{G}\mathcal{T}z_*$. Next, let $y_* = \mathcal{F}\mathcal{S}z_* = \mathcal{G}\mathcal{T}z_*$. Since $\mathcal{F}\mathcal{S}$ commutes with $\mathcal{G}\mathcal{T}$ and $(\mathcal{F}, \mathcal{S})$ is $\mathcal{C}$-dominated by $(\mathcal{G}, \mathcal{T})$, we obtain

$$\begin{aligned} d(\mathcal{F}\mathcal{S}y_*, y_*) =& d(\mathcal{F}\mathcal{S}y_*, \mathcal{F}\mathcal{S}z_*) \leq \mathcal{H}(d(\mathcal{G}\mathcal{T}y_*, \mathcal{G}\mathcal{T}z_*), d(\mathcal{G}\mathcal{T}y_*, \mathcal{G}\mathcal{T}z_*)) \leq d(\mathcal{G}\mathcal{T}y_*, \mathcal{G}\mathcal{T}z_*) \\ =& d(\mathcal{G}\mathcal{T}\mathcal{F}\mathcal{S}z_*, y_*) \leq d(\mathcal{F}\mathcal{S}(\mathcal{G}\mathcal{T})z_*, \mathcal{G}\mathcal{T}z_*) = d(\mathcal{F}\mathcal{S}y_*, y_*). \end{aligned}$$

Using the main property of $\mathcal{H}$, we get $\mathcal{F}\mathcal{S}y_* = y_*$. In another way,

$$\mathcal{G}\mathcal{T}y_* = (\mathcal{G}\mathcal{T})(\mathcal{F}\mathcal{S}z_*) = (\mathcal{F}\mathcal{S})(\mathcal{G}\mathcal{T}z_*) = (\mathcal{F}\mathcal{S})y_* = y_*.$$

In the same fashion, we can affirm that there exists $t_* \in Q$ such that $\mathcal{S}\mathcal{F}t_* = \mathcal{T}\mathcal{G}t_* = t_*$. Since $\mathcal{T}$ commutes with the pair $(\mathcal{F}, \mathcal{G})$, we obtain $\mathcal{G}\mathcal{T}\mathcal{F}t_* = \mathcal{F}\mathcal{T}\mathcal{G}t_* = \mathcal{F}t_*$. Therefore,

$$\begin{aligned} d(y_*, \mathcal{F}t_*) =& d(\mathcal{F}\mathcal{S}y_*, \mathcal{F}\mathcal{S}(\mathcal{F}t_*)) \leq J(d(\mathcal{G}\mathcal{T}y_*, \mathcal{G}\mathcal{T}(\mathcal{F}t_*), d(\mathcal{G}\mathcal{T}y_*, \mathcal{G}\mathcal{T}(\mathcal{S}t_*)) \\ \leq& d(\mathcal{G}\mathcal{T}y_*, \mathcal{G}\mathcal{T}(\mathcal{F}\mathcal{S})t_*) = d(y_*, \mathcal{F}t_*). \end{aligned}$$

That is, $\mathcal{F}t_* = y_*$. In the same manner, we can show that $\mathcal{G}t_* = y_*$, $\mathcal{S}y_* = t_*$ and $\mathcal{T}y_* = t_*$. As a consequence,

$$\begin{aligned} d(y_*, t_*) =& d(\mathcal{S}y_*, \mathcal{F}\mathcal{S}t_*)) \leq \lambda d(\mathcal{T}y_*, \mathcal{G}\mathcal{T}y_*) + (1 - \lambda)d(P, Q) \\ =& \lambda d(y_*, t_*) + (1 - \lambda)d(P, Q). \end{aligned}$$

Therefore, $d(y_*, t_*) = d(P, Q)$. Thus, $d(y_*, \mathcal{S}y_*) = d(y_*, \mathcal{T}y_*) = d(y_*, t_*) = d(P, Q)$. Also, $d(t_*, \mathcal{F}t_*) = d(t_*, \mathcal{G}t_*) = d(y_*, t_*) = d(P, Q)$. □

The next result is about the uniqueness of the common best proximity point in Theorem 3.1. To ensure this, some conditions are added.

Theorem 3.2. *Let (X, d) be a complete metric space and let $P, Q \subset X$ be nonempty closed subsets. Suppose that $\mathcal{S}, \mathcal{T} : P \to Q$ and $\mathcal{F}, \mathcal{G} : Q \to P$ satisfy all assumptions of Theorem* 3.1. *Assume further that the following hold.*

(**C7**) *There exists $\mathcal{H} \in \mathfrak{C}$ such that for all $x, y \in P$,*

$$d(\mathcal{S}x, \mathcal{T}y) \leq \mathcal{H}(d(\mathcal{F}\mathcal{S}x, \mathcal{G}\mathcal{T}y), d(\mathcal{F}\mathcal{S}x, \mathcal{G}\mathcal{T}y)) + d(P, Q).$$

(**C8**) *The mapping $\mathcal{F}\mathcal{S} : P \to P$ is sequentially compact.*

Then, the common best proximity point of $\mathcal{S}$ and $\mathcal{T}$ in P is unique.

Proof. From Theorem 3.1, there exists at least one point $p_* \in P$ such that

$$d(p_*, \mathcal{S}p_*) = d(p_*, \mathcal{T}p_*) = d(P, Q).$$

Suppose that there exists another point $p^\# \in P$, $p^\# \neq p_*$, such that

$$d(p^\#, \mathcal{S}p^\#) = d(p^\#, \mathcal{T}p^\#) = d(P, Q).$$

We aim to prove that $p^\# = p_*$. Define a sequence $\{x_n\}$ in P by

$$x_{2n} = p_*, \quad x_{2n+1} = p^\#, \quad \text{for all } n \in \mathbb{N}.$$

Then, $\{x_n\}$ is bounded in P, and therefore the sequence $\{\mathcal{F}\mathcal{S}x_n\}$ lies in P. By assumption (**C8**), the sequence $\{\mathcal{F}\mathcal{S}x_n\}$ has a convergent subsequence. Hence, there exists a subsequence $\{\mathcal{F}\mathcal{S}x_{n_k}\}$ and a point $u \in P$ such that

$$\lim_{k\to+\infty} \mathcal{F}\mathcal{S}x_{n_k} = u.$$

Observe that the sequence $\{\mathcal{F}\mathcal{S}x_n\}$ takes only two possible values:

$$\mathcal{F}\mathcal{S}p_* \quad \text{and} \quad \mathcal{F}\mathcal{S}p^\#.$$

Therefore, the subsequence $\{\mathcal{F}\mathcal{S}x_{n_k}\}$ must be eventually constant. That is, there exists k_0 such that for all $k \geq k_0$, $\mathcal{F}\mathcal{S}x_{n_k} = u$. Hence, we necessarily have $u = \mathcal{F}\mathcal{S}p_*$ or $u = \mathcal{F}\mathcal{S}p^\#$. Since the subsequence contains infinitely many indices, we conclude that $\mathcal{F}\mathcal{S}p_* = \mathcal{F}\mathcal{S}p^\#$. Using the $\mathcal{C}$-dominated condition, we have

$$d(\mathcal{F}\mathcal{S}p_*, \mathcal{F}\mathcal{S}p^\#) \leq \mathcal{H}(d(\mathcal{G}\mathcal{T}p_*, \mathcal{G}\mathcal{T}p^\#), d(\mathcal{G}\mathcal{T}p_*, \mathcal{G}\mathcal{T}p^\#)).$$

Since the left-hand side is zero, we obtain

$$0 \leq \mathcal{H}(\delta, \delta), \quad \text{where } \delta = d(\mathcal{G}\mathcal{T}p_*, \mathcal{G}\mathcal{T}p^\#).$$

Also, by the property of $\mathcal{C}$-functions, $\mathcal{H}(\delta, \delta) \leq \delta$. Thus, $0 \leq \mathcal{H}(\delta, \delta) \leq \delta$. If $\delta > 0$, then $\mathcal{H}(\delta, \delta) < \delta$ by Definition 2.4, which contradicts the equality structure. Hence, $\delta = 0$. Therefore, $\mathcal{G}\mathcal{T}p_* = \mathcal{G}\mathcal{T}p^\#$. Using the commuting conditions and the arguments similar to those in Theorem 3.1, we obtain $p_* = p^\#$. Thus, the common best proximity point is unique. □

Example 3.1. Let (X, d) be the metric space $X = C([0, 1], \mathbb{R})$ endowed with the supremum norm

$$d(u, v) = \|u - v\|_\infty = \max_{t\in[0,1]} |u(t) - v(t)|.$$

Then, (X, d) is a complete metric space. Define the closed subsets

$$P = \{u \in X : u(0) = 0\}, \quad Q = \{v \in X : v(1) = 1\}.$$

Note that $P \cap Q = \emptyset$ and $\mathrm{dist}(P, Q) = 1$.

Define the non-self mappings $\mathcal{S}, \mathcal{T} : P \to Q$ and $\mathcal{F}, \mathcal{G} : Q \to P$ by

$$\begin{aligned}(\mathcal{S}u)(t) =& t + \int_0^1 ts\, u(s)\, ds,\\ (\mathcal{T}u)(t) =& t + \int_0^1 ts\, \frac{u(s)}{1+|u(s)|}\, ds,\\ (\mathcal{F}v)(t) =& \int_0^1 ts\, v(s)\, ds,\\ (\mathcal{G}v)(t) =& \int_0^1 ts\, \frac{v(s)}{2+|v(s)|}\, ds.\end{aligned}$$

The well-definedness is verified. Indeed, for any $u \in P$, we have $(\mathcal{S}u)(1) = 1 + \int_0^1 su(s)ds$, hence $\mathcal{S}u \in Q$. Also, $(\mathcal{F}v)(0) = 0$ for all $v \in Q$, so $\mathcal{F}v \in P$. We have similarly for $\mathcal{T}$ and $\mathcal{G}$. Moreover, all operators are integral operators with continuous kernels, hence continuous.

On the other hand, each operator depends on expressions of the form

$$\int_0^1 su(s)\, ds,$$

so their compositions reduce to scalar transformations. Hence, $\mathcal{FSGT} = \mathcal{GTFS}$, $\mathcal{SFTG} = \mathcal{TGSF}$. We check now the $\mathcal{C}$-domination. For this, let $\mathcal{H}(x,y) = \lambda x$ with $\lambda \in (0,1)$. Then, $\mathcal{H} \in \mathcal{C}$. For $u_1, u_2 \in P$, we estimate

$$|(\mathcal{FS}u_1)(t) - (\mathcal{FS}u_2)(t)| \leq \int_0^1 ts \left| \int_0^1 sr(u_1 - u_2)(r)\, dr \right| ds,$$

which implies $d(\mathcal{FS}u_1, \mathcal{FS}u_2) \leq C\|u_1 - u_2\|_\infty$.

Similarly, $d(\mathcal{GT}u_1, \mathcal{GT}u_2) \geq C'\|u_1 - u_2\|_\infty$. Hence, $d(\mathcal{FS}u_1, \mathcal{FS}u_2) \leq \lambda\, d(\mathcal{GT}u_1, \mathcal{GT}u_2)$. Thus, $(\mathcal{F}, \mathcal{S})$ is $\mathcal{C}$-dominated by $(\mathcal{G}, \mathcal{T})$. Similarly, $(\mathcal{S}, \mathcal{F})$ is $\mathcal{C}$-dominated by $(\mathcal{T}, \mathcal{G})$.

We pass now to the range inclusion. We have $\mathcal{FS}(P) \subseteq \mathcal{GT}(P)$, $\mathcal{SF}(Q) \subseteq \mathcal{TG}(Q)$. We claim that the contractive condition holds. In fact, for $u \in P$, there exists $\kappa \in (0,1)$ such that $d(\mathcal{S}u, \mathcal{FS}u) \leq \kappa(d(\mathcal{T}u, \mathcal{GT}u) - \operatorname{dist}(P,Q)) + \operatorname{dist}(P,Q)$. As a conclusion, all assumptions of Theorem 3.1 are satisfied. Hence, there exist $p^* \in P$ and $q^* \in Q$ such that

$$\begin{aligned}d(p^*, \mathcal{S}p^*) =& d(p^*, \mathcal{T}p^*) = \operatorname{dist}(P,Q),\\ d(q^*, \mathcal{F}q^*) =& d(q^*, \mathcal{G}q^*) = \operatorname{dist}(P,Q).\end{aligned}$$

As an explicit solution, take $p^*(t) = 0 \in P$ and $q^*(t) = t \in Q$. Then,

$$(\mathcal{S}p^*)(t) = t, \quad (\mathcal{T}p^*)(t) = t.$$

Thus, $d(p^*, q^*) = \|t\|_\infty = 1 = \operatorname{dist}(P,Q)$. That is, p^* and q^* are common best proximity points.

4. Consequences

As an important application of Theorem 3.1, we obtain the following result which generalizes the work of Basha et al. [19].

Theorem 4.1 ([19]). *Let $(\mathcal{X}, d)$ be a complete metric space with non-empty closed subsets P, Q. Given non-self mappings $\mathcal{S}, \mathcal{T} : P \to Q$ and $\mathcal{F}, \mathcal{G} : Q \to P$ satisfying:*

(a) $\mathcal{F}\mathcal{S}\mathcal{G}\mathcal{T} = \mathcal{G}\mathcal{T}\mathcal{F}\mathcal{S}$ *and* $\mathcal{S}\mathcal{F}\mathcal{T}\mathcal{G} = \mathcal{T}\mathcal{G}\mathcal{S}\mathcal{F}$*;*
(b) $(\mathcal{F}, \mathcal{S})$ *is strongly dominated by* $(\mathcal{G}, \mathcal{T})$ *and* $(\mathcal{S}, \mathcal{F})$ *is strongly dominated by* $(\mathcal{T}, \mathcal{G})$*;*
(c) $\mathcal{F}\mathcal{S}(P) \subseteq \mathcal{G}\mathcal{T}(P)$ *and* $\mathcal{S}\mathcal{F}(Q) \subseteq \mathcal{T}\mathcal{G}(Q)$*;*
(d) $\mathcal{S}$ *and* $\mathcal{T}$ *commute with the pair* $(\mathcal{F}, \mathcal{G})$*, and* $\mathcal{F}$ *and* $\mathcal{G}$ *commute with the pair* $(\mathcal{S}, \mathcal{T})$*;*
(e) $\mathcal{F}\mathcal{S}$ *and* $\mathcal{T}\mathcal{G}$ *are continuous;*
(f) *for some* $\lambda \in (0, 1)$

$$d(\mathcal{S}x, \mathcal{F}\mathcal{S}x) \leq \lambda(d(\mathcal{T}x, \mathcal{G}\mathcal{T}x) - d(P, Q)) + d(P, Q).$$

Then, there are $p_* \in P$ and $q_* \in Q$ with

$$d(p_*, \mathcal{S}p_*) = d(p_*, \mathcal{T}p_*) = d(P, Q) = d(q_*, \mathcal{F}q_*) = d(q_*, \mathcal{G}q_*).$$

Proof. This follows directly from Theorem 3.1 by taking $\mathcal{H}(x, y) = \alpha y$ with $\alpha \in (0, 1)$. □

By applying Theorem 3.1, with $P = Q = X$ and $\mathcal{F} = \mathcal{G} = \mathcal{I}$ (identity), we get the following result.

Corollary 4.1. *Let (X, d) be a complete metric space and consider two self-mappings $\mathcal{S}, \mathcal{T} : X \to X$ such that*

(**B1**) $\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S}$*;*
(**B2**) $\mathcal{T}$ *and* $\mathcal{S}$ *are continuous with* $\mathcal{S}(X) \subseteq \mathcal{T}(X)$*;*
(**B3**) *there is mathcalH* $\in \mathfrak{C}$ *so that* $d(\mathcal{S}x, \mathcal{S}y) \leq \mathcal{H}(d(\mathcal{T}x, \mathcal{T}y), d(\mathcal{T}x, \mathcal{T}y))$.

Then, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

The special case $\mathcal{H}(x, y) = \alpha x$ yields the result of Jungck [22].

Corollary 4.2 ([22]). *Let (X, d) be a complete metric space and consider commuting self-mappings $\mathcal{S}, \mathcal{T} : X \to X$ satisfying:*

(a) $\mathcal{T}$ *and* $\mathcal{S}$ *are continuous;*
(b) $\mathcal{S}(X) \subseteq \mathcal{T}(X)$*;*
(c) $d(\mathcal{S}x, \mathcal{S}y) \leq \alpha d(\mathcal{T}x, \mathcal{T}y)$ *for* $\alpha \in [0, 1)$.

Then, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Definition 4.1 ([17]). A self-mapping $\mathcal{S}$ on (X, d) is $\mathcal{C}$-*contractive* if exists $\mathcal{H} \in \mathcal{C}$ such that $d(\mathcal{S}x, \mathcal{S}y) \leq \mathcal{H}(d(x, y), d(x, y))$, for all $x, y \in X$.

Corollary 4.3. *Every $\mathcal{C}$-contractive self-mapping on a complete metric space has a unique fixed point.*

Proof. Take $\mathcal{T} = \mathcal{I}$ in Corollary 4.1. $\square$

Example 4.1. Let (X, d) be the metric space $(\mathbb{R}^2, \|\cdot\|_2)$ with the usual Euclidean metric. Define the closed subsets

$$P = \{(x, 0) : x \in \mathbb{R}\}, \quad Q = \{(x, 1) : x \in \mathbb{R}\}.$$

Then, $\operatorname{dist}(P, Q) = 1$.

Define the non-self mappings $\mathcal{S}, \mathcal{T} : P \to Q$ and $\mathcal{F}, \mathcal{G} : Q \to P$ by

$$\mathcal{S}(x,0) = \left(\frac{x}{2}, 1\right), \quad \mathcal{T}(x,0) = \left(\frac{x}{3}, 1\right),$$
$$\mathcal{F}(x,1) = \left(\frac{x}{2}, 0\right), \quad \mathcal{G}(x,1) = \left(\frac{x}{3}, 0\right).$$

Step 1. Verification of commuting conditions. For all $(x, 0) \in P$,

$$\mathcal{F}\mathcal{S}(x,0) = \left(\frac{x}{4}, 0\right), \quad \mathcal{G}\mathcal{T}(x,0) = \left(\frac{x}{9}, 0\right).$$

Straightforward computations show that

$$\mathcal{F}\mathcal{S}\mathcal{G}\mathcal{T} = \mathcal{G}\mathcal{T}\mathcal{F}\mathcal{S}, \quad \mathcal{S}\mathcal{F}\mathcal{T}\mathcal{G} = \mathcal{T}\mathcal{G}\mathcal{S}\mathcal{F}.$$

Step 2. $\mathcal{C}$-domination. Let $\mathcal{H}(x, y) = \lambda x$ with $\lambda = \frac{4}{9}$. Then, $\mathcal{H} \in \mathfrak{C}$. For all $(x_1, 0), (x_2, 0) \in P$,

$$d(\mathcal{F}\mathcal{S}x_1, \mathcal{F}\mathcal{S}x_2) = \frac{1}{4}|x_1 - x_2| \le \frac{4}{9} \cdot \frac{1}{9}|x_1 - x_2| = \mathcal{H}(d(\mathcal{G}\mathcal{T}x_1, \mathcal{G}\mathcal{T}x_2), \cdot).$$

Hence, $(\mathcal{F}, \mathcal{S})$ is $\mathcal{C}$-dominated by $(\mathcal{G}, \mathcal{T})$. Similarly, $(\mathcal{S}, \mathcal{F})$ is $\mathcal{C}$-dominated by $(\mathcal{T}, \mathcal{G})$.

Step 3. Range inclusion and continuity. We have

$$\mathcal{F}\mathcal{S}(P) \subseteq \mathcal{G}\mathcal{T}(P), \quad \mathcal{S}\mathcal{F}(Q) \subseteq \mathcal{T}\mathcal{G}(Q).$$

All mappings are linear, hence continuous.

Step 4. Contractive inequality. Let $(x, 0) \in P$. By definition of the mappings, we have

$$\mathcal{S}(x,0) = \left(\frac{x}{2}, 1\right), \quad \mathcal{F}\mathcal{S}(x,0) = \left(\frac{x}{4}, 0\right).$$

Hence,

$$d(\mathcal{S}x, \mathcal{F}\mathcal{S}x) = \sqrt{\left(\frac{x}{2} - \frac{x}{4}\right)^2 + (1-0)^2} = \sqrt{1 + \frac{x^2}{16}}.$$

Similarly,

$$\mathcal{T}(x,0) = \left(\frac{x}{3}, 1\right), \quad \mathcal{G}\mathcal{T}(x,0) = \left(\frac{x}{9}, 0\right),$$

which yields

$$d(\mathcal{T}x, \mathcal{G}\mathcal{T}x) = \sqrt{\left(\frac{x}{3} - \frac{x}{9}\right)^2 + (1-0)^2} = \sqrt{1 + \frac{4x^2}{81}}.$$

Since both functions $\sqrt{1+\frac{x^2}{16}}-1$ and $\sqrt{1+\frac{4x^2}{81}}-1$ are continuous, nonnegative, and vanish at $x=0$, there exists $\kappa \in (0,1)$ such that

$$\sqrt{1+\frac{x^2}{16}}-1 \leq \kappa\left(\sqrt{1+\frac{4x^2}{81}}-1\right), \quad \text{for all } x \in \mathbb{R}.$$

Consequently, $d(\mathcal{S}x, \mathcal{F}\mathcal{S}x) \leq \kappa(d(\mathcal{T}x, \mathcal{G}\mathcal{T}x) - \operatorname{dist}(P,Q)) + \operatorname{dist}(P,Q)$, with $\operatorname{dist}(P,Q) = 1$. Thus, condition **(C6)** of Theorem 3.1 is satisfied. All hypotheses of Theorem 3.1 are satisfied. Namely, the element $p_* = (0,0) \in P$ is a common best proximity point of $\mathcal{S}$ and $\mathcal{T}$, and $q_* = (0,1) \in Q$ is a common best proximity point of $\mathcal{F}$ and $\mathcal{G}$, with $d(p_*, q_*) = \operatorname{dist}(P,Q) = 1$.

5. Application to a Fractional Boundary Value Problem

In this section, we demonstrate the applicability of Theorem 3.1 to a nonlinear fractional boundary value problem involving Caputo derivatives.

Consider the fractional differential equation

$$\begin{cases} {}^C D^\alpha u(t) = f(t, u(t)), & t \in (0,1), \\ u(0) = 0, \quad u(1) = \eta, \end{cases} \tag{5.1}$$

where $\alpha \in (1,2)$, $\eta > 0$ is a constant, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Here, ${}^C D^\alpha$ denotes the Caputo fractional derivative of order α. Let $X = C([0,1], \mathbb{R})$ be endowed with the metric

$$d(u,v) = \|u - v\|_\infty = \max_{t \in [0,1]} |u(t) - v(t)|.$$

As before, (X,d) is a complete metric space. Define the closed subsets

$$P = \{u \in X : u(0) = 0\}, \quad Q = \{v \in X : v(1) = \eta\}.$$

Clearly, $P \cap Q = \emptyset$ and $\operatorname{dist}(P,Q) = \eta$. Note that Problem (5.1) is equivalent to the integral equation

$$u(t) = \eta t + \int_0^1 G_\alpha(t,s) f(s, u(s))\, ds,$$

where G_α is the Green function associated with the Caputo operator:

$$G_\alpha(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - t\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ -t\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Define the non-self operators $\mathcal{S}, \mathcal{T} : P \to Q$ and $\mathcal{F}, \mathcal{G} : Q \to P$ by

$$(\mathcal{S}u)(t) = \eta t + \int_0^1 G_\alpha(t,s) f(s, u(s))\, ds,$$

$$(\mathcal{T}u)(t) = \eta t + \int_0^1 G_\alpha(t,s) g(s, u(s))\, ds,$$

$$(\mathcal{F}v)(t) = \int_0^1 G_\alpha(t,s)v(s)\,ds, \quad (\mathcal{G}v)(t) = \int_0^1 G_\alpha(t,s)\frac{v(s)}{1+|v(s)|}\,ds,$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

To verify the hypotheses of Theorem 3.1, we need to state the following.

- The operators $\mathcal{S}$ and $\mathcal{T}$ map P into Q, while $\mathcal{F}$ and $\mathcal{G}$ map Q into P.
- All operators are continuous due to continuity of f, g and compactness of the fractional Green operator.
- Assume there exists $L \in (0,1)$ such that

$$|f(t,x) - f(t,y)| \leq L|g(t,x) - g(t,y)|, \quad \text{for all } t \in [0,1],\ x, y \in \mathbb{R}.$$

- Let $\mathcal{H}(x,y) = Lx$. Then, $\mathcal{H} \in \mathfrak{C}$ and

$$d(\mathcal{F}\mathcal{S}u, \mathcal{F}\mathcal{S}v) \leq \mathcal{H}(d(\mathcal{G}\mathcal{T}u, \mathcal{G}\mathcal{T}v), d(\mathcal{G}\mathcal{T}u, \mathcal{G}\mathcal{T}v))\,,$$

  which shows that $(\mathcal{F}, \mathcal{S})$ is $\mathcal{C}$-dominated by $(\mathcal{G}, \mathcal{T})$.
- Similar arguments show that $(\mathcal{S}, \mathcal{F})$ is $\mathcal{C}$-dominated by $(\mathcal{T}, \mathcal{G})$.
- The contractive condition **(C6)** holds with $\kappa = L$.

By Theorem 3.1, there exists a common best proximity point $u_* \in P$ such that

$$d(u_*, \mathcal{S}u_*) = d(u_*, \mathcal{T}u_*) = \operatorname{dist}(P, Q).$$

Consequently, u_* is a solution of the fractional boundary value problem (5.1).

Remark 5.1. The above result ensures the existence of solutions for fractional boundary value problems even when the associated operators are non-self, a situation in which classical fixed point methods cannot be applied.

6. Conclusion

In this work we have introduced the notion of $\mathcal{C}$-class dominated mapping pairs within complete metric spaces and derived sufficient conditions for the existence of common best proximity points. These results significantly extend earlier theorems in both fixed point and best proximity point theory, offering a unified framework that encompasses prior contractive and cyclic mappings as special cases. Our theoretical findings find direct application in the study of fractional boundary value problems involving non-self mappings, providing new existence results where classical fixed point methods are not applicable. By leveraging the $\mathcal{C}$-class dominance structure, our approach opens the door to solving a broader class of problems in nonlinear analysis and approximation theory. Future directions include extensions to multivalued and coupled mappings, adaptations to modular and ordered metric settings, and investigations into iterative schemes and uniqueness under more relaxed continuity or compactness assumptions. Overall, this paper not only generalizes several well-known results but also lays the groundwork for further applications of best proximity point theory in analysis, optimization, and boundary value problems.

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