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A NEW CLASS OF FOUR-POINT FRACTIONAL SUM BOUNDARY VALUE PROBLEMS FOR NONLINEAR SEQUENTIAL FRACTIONAL DIFFERENCE EQUATIONS INVOLVING SHIFT OPERATORS

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Abstract. In this article, we study the existence result for a Riemann-Liouville fractional difference equation with four-point fractional sum boundary value conditions, by using the Sadovskii's fixed point theorem. Our problem contains the shift operators on fractional difference operators that are different orders. Finally, we present an example to show the importance of these result.

1. Introduction

In this paper we consider a Riemann-Liouville fractional difference equation with nonlocal four-point fractional sum boundary value conditions of the form

$$\begin{aligned}
(1.1)\qquad & \Delta_0^{\alpha}\left[E_{-\gamma}\left(\Delta_{\alpha+\gamma-1}^{\beta}u(t)\right)+E_{-\beta}\left(\Delta_{\alpha+\beta-1}^{\gamma}\phi(t)v(t)\right)\right]\\
&=f(t+\alpha+\beta+\gamma-1,u(t+\alpha+\beta+\gamma-1),v(t+\alpha+\beta+\gamma-1)),\\
&u(\alpha+\beta+\gamma-2)=\rho(u),\\
&u(\xi)=\left[\Delta_{\alpha+\beta+\gamma+\nu-3}^{-\nu}\,v(s+\nu)\,u(s+\nu)\right]_{s=\eta}^{T+\alpha+\beta+\gamma},
\end{aligned}$$

where $t\in\mathbb{N}_{0,T}:=\{0,1,\dots,T\}$, $0<\alpha+\beta<\alpha+\gamma<\alpha+\beta+\gamma\le 2$, $0<\nu\le 1$, $\xi,\eta\in\mathbb{N}_{\alpha+\beta+\gamma-1,T+\alpha+\beta+\gamma-1}$, $\xi<\eta$, functions $f\in C\left(\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}\times\mathbb{R}\times\mathbb{R}^{+},\mathbb{R}\right)$, $v,\phi\in C\left(\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma},\mathbb{R}^{+}\right)$, a functional $\rho: C\left(\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma},\mathbb{R}\right)\to\mathbb{R}$ are given, and the shift operator $E_{-\tau}u(t):=u(t-\tau)$.

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Fractional difference equations have been interested many mathematicians since they can use for describing many problems in the real-world phenomena such as physics, mechanics, chemistry, control systems, electrical networks, and flow in porous media. In recent years, mathematicians have used this fractional calculus to model and solve various related problems. In particular, fractional calculus is a powerful tool for the processes which appears in nature, e.g., biology, ecology and other areas, can be found in [29, 30], and the references therein.

Some good papers dealing with boundary value problems for fractional difference equations, which have helped to build up some of the basic theory of this field, see the textbooks [4, 16] and the papers [5]-[18] and references cited therein. For example, Goodrich [14] considered the discrete fractional boundary value problem

$$\begin{aligned} &-\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) = f(t+\mu_1+\mu_2+\mu_3-1, y(t+\mu_1+\mu_2+\mu_3-1)),\\ &y(0) = 0 = y(b+2), \end{aligned} \tag{1.2}$$

where $t \in \mathbb{N}_{2-\mu_1-\mu_2-\mu_3, b+2-\mu_1-\mu_2-\mu_3}$, $0 < \mu_1, \mu_2, \mu_3 < 1$, $1 < \mu_2 + \mu_3 < 2$, $1 < \mu_1 + \mu_2 + \mu_3 < 2$ and $f : \mathbb{N}_0 \times \mathbb{R} \to [0, +\infty)$ is a continuous function. Existence of positive solutions are obtained by the the Krasnosel'skii fixed point theorem.

Dong et al. [11] investigated the existence of solutions to the following fractional boundary value problem

$$\begin{aligned} &{}_T\Delta_{t-1}^{\nu}\left({}_t\Delta_{\nu-1}^{\nu}\right)x(t) = f(x(t+\nu-1)), \quad t \in \mathbb{N}_{0,T},\\ &x(\nu-2) = \left[{}_t\Delta_{\nu-1}^{\nu}x(t)\right]_{t=T} = 0, \end{aligned} \tag{1.3}$$

where $0 < \nu \le 1$, ${}_t\Delta_{\nu-1}^{\nu}$ are ${}_T\Delta_t^{\nu}$ are respectively, the left fractional difference and the right fractional difference operators and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Sitthiwirattham [23, 24] examined two fractional sum boundary value problems for fractional difference equations of the forms

$$\begin{aligned} &\Delta_C^{\alpha}[\phi_p(\Delta_C^{\beta}x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)),\\ &\Delta_C^{\beta}x(\alpha-1) = 0, \quad x(\alpha+\beta+T) = \rho\Delta^{-\gamma}x(\eta+\gamma), \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\Delta_{\alpha}^{\alpha}(\Delta_{\alpha+\beta-1}^{\beta} + \lambda E_{\beta})x(t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)),\\ &x(\alpha+\beta-2) = 0, \quad x(\alpha+\beta+T) = \rho\Delta_{\alpha+\beta-1}^{-\gamma}x(\eta+\gamma), \end{aligned} \tag{1.5}$$

where $t \in \mathbb{N}_{0,T}$, $0 < \alpha, \beta \le 1$, $1 < \alpha + \beta \le 2$, $0 < \gamma \le 1$, $\eta \in \mathbb{N}_{\alpha+\beta-1, \alpha+\beta+T-1}$, ρ, $f : \mathbb{N}_{\alpha+\beta-2, \alpha+\beta+T} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $E_{\beta}x(t) = x(t+\beta-1)$ and ϕ_p is the p-Laplacian operator.

The results mentioned above are the motivation for this research. The plan of this paper is as follows. In the next section, we recall some definitions and basic lemmas. In Section 3, we prove the existence of solutions to the boundary value problem (1.1) by the help of the Sadovskii's fixed point theorem. An illustrative example is presented in the last section.

2. Preliminaries

In the following, there are notations, definitions, and lemmas which are used in the main results.

Definition 2.1. The generalized falling function is defined by $t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, for any t and α for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

Lemma 2.1. [7] *Assume the following factorial functions are well defined.*

(i) $(t-\mu)\, t^{\underline{\mu}} = t^{\underline{\mu+1}}$, *where* $\mu \in \mathbb{R}$.
(ii) *If* $t \leq r$, *then* $t^{\underline{\alpha}} \leq r^{\underline{\alpha}}$ *for any* $\alpha > 0$.
(iii) $t^{\underline{\alpha+\beta}} = (t-\beta)^{\underline{\alpha}}\, t^{\underline{\beta}}$.

Definition 2.2. The Gauss hypergeometric function ${}_2F_1(a,b;c;x)$ is a function which can be defined in the form of a hypergeometric series

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a)(k+b)}{(k+c)}x, \quad c_0 = 1.$$

The resulting Gauss hypergeometric function is written by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!},$$

where $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$ is the Pochhammer symbol.

Theorem 2.1. [9] (The Gauss hypergeometric theorem) *Let* $2F_1(a,b;c;1)$ *be the Gauss hypergeometric function with* $x = 1$. *Then*

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},$$

where $c-a-b > 0$, $a,b,c \in \mathbb{R}$.

Definition 2.3. For $\alpha > 0$ and f defined on $\mathbb{N}_a := \{a, a+1, \ldots\}$, the α-order fractional sum of f is defined by

$$\Delta_{a+\alpha}^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.4. For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional difference of f is defined by

$$\Delta_{a+N-\alpha}^{\alpha} f(t) := \Delta^N \Delta_{a+N-\alpha}^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < \alpha \leq N$.

Lemma 2.2. [7] *Let $0 \leq N-1 < \alpha \leq N$ and y defined on $\mathbb{N}_{\alpha-N}$. Then*

$$\Delta_{\alpha-N}^{-\alpha}\Delta_0^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \cdots + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.

The following lemma deals with a linear variant of the boundary value problem (1.1) and gives a representation of the solution.

Lemma 2.3. *Let $0 < \alpha+\beta < \alpha+\gamma < 1$, $1 < \alpha+\beta+\gamma \leq 2$, $0 < \nu \leq 1$, $\xi < \eta$, $\xi, \eta \in \mathbb{N}_{\alpha+\beta+\gamma-1, T+\alpha+\beta+\gamma-1}$, $h \in C\left(\mathbb{N}_{\alpha+\beta+\gamma-1, T+\alpha+\beta+\gamma-1}, \mathbb{R}\right)$, $v, \phi \in C\left(\mathbb{N}_{\alpha+\beta+\gamma-2, T+\alpha+\beta+\gamma}, \mathbb{R}^+\right)$ and $\rho : C\left(\mathbb{N}_{\alpha+\beta+\gamma-2, T+\alpha+\beta+\gamma}, \mathbb{R}\right) \to \mathbb{R}$ be given. Then the problem*

$$\begin{aligned}
&\Delta_0^{\alpha}\left[E_{-\gamma}\left(\Delta_{\alpha+\gamma-1}^{\beta} u(t)\right) + E_{-\beta}\left(\Delta_{\alpha+\beta-1}^{\gamma}\phi(t)v(t)\right)\right] = h(t+\alpha+\beta+\gamma-1),\\
&u(\alpha+\beta+\gamma-2) = \rho(u),\\
&u(\xi) = \left[\Delta_{\alpha+\beta+\gamma+\nu-3}^{-\nu}\, v(s+\nu)\, u(s+\nu)\right]_{s=\eta}^{T+\alpha+\beta+\gamma},
\end{aligned} \tag{2.1}$$

has the unique solution

$$\begin{aligned}
u(t) =& \frac{\rho(u)\, t^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}} + \frac{\sum_{s=\alpha+\gamma-1}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}\,(s-\gamma)^{\underline{\alpha-1}}}{\sum_{s=\alpha+\gamma-1}^{\xi-\beta}(\xi-\sigma(s))^{\underline{\beta-1}}\,(s-\gamma)^{\underline{\alpha-1}}}\\
&\times\Bigg[-\frac{\rho(u)\,\xi^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}} + \mathcal{A}(u) + \frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\\
&\times\sum_{p=\alpha+\gamma-1}^{\xi-\beta}\ \sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\,\phi(s)v(s)\\
&-\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{\xi-\beta}\ \sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}\,h(s)\Bigg]\\
&-\frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\beta-1}^{t-\beta}\ \sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(t-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\,\phi(s)v(s)\\
&+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{t-\beta}\ \sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(t-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}\,h(s),
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\mathcal{A}(u) =& \frac{1}{\Gamma(\nu)\,\Theta(T,\xi,\eta)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}\,v(s)\\
&\times\left\{\Upsilon_\rho(s,\xi) + \psi(s,\xi)\Psi_{v,\phi}(\xi) - \psi(s,\xi)\mathfrak{I}_h(\xi) + \mathcal{J}_h(s) - \Phi_{v,\phi}(s)\right\},
\end{aligned} \tag{2.3}$$

and

$$\psi(s,\xi) = \frac{\sum_{p=\alpha+\gamma-1}^{s-\beta}(s-\sigma(p))^{\underline{\beta-1}}\,(p-\gamma)^{\underline{\alpha-1}}}{\sum_{p=\alpha+\gamma-1}^{\xi-\beta}(\xi-\sigma(p))^{\underline{\beta-1}}\,(p-\gamma)^{\underline{\alpha-1}}}, \tag{2.4}$$

$$\mathcal{P}_v\left[\psi(s,\xi)\right] = \frac{1}{\Gamma(\nu)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}\,v(s)\psi(s,\xi), \tag{2.5}$$

$$\Theta(T,\xi,\eta) = 1-\mathcal{P}_v\left[\psi(s,\xi)\right], \tag{2.6}$$

$$\Upsilon_\rho(s,\xi) = \left[s^{\underline{\beta-1}}-\psi(s,\xi)\,\xi^{\underline{\beta-1}}\right]\frac{\rho(u,v)}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}, \tag{2.7}$$

$$\begin{aligned}\Psi_{v,\phi}(\xi) =& \frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\gamma-1}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(\xi-\sigma(p))^{\underline{\beta-1}}\\ &\times(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\,\phi(s)v(s),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{I}_h(\xi) =& \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{\xi-\beta}\sum_{\omega=\alpha+\beta+\gamma-1}^{p+\beta-1}(\xi-\sigma(p))^{\underline{\beta-1}}\\ &\times(p+\alpha+\beta-1-\sigma(\omega))^{\underline{\alpha-1}}\,h(\omega),\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathcal{J}_h(s) =& \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{s-\beta}\sum_{\omega=\alpha+\beta+\gamma-1}^{p+\beta-1}(s-\sigma(p))^{\underline{\beta-1}}\\ &\times(p+\alpha+\beta-1-\sigma(\omega))^{\underline{\alpha-1}}\,h(\omega),\end{aligned} \tag{2.10}$$

$$\begin{aligned}\Phi_{v,\phi}(s) =& \frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\beta-1}^{s-\beta}\sum_{\omega=\alpha+\beta+\gamma-1}^{p+\beta}(s-\sigma(p))^{\underline{\beta-1}}\\ &\times(p+\beta-\gamma-\sigma(\omega))^{\underline{-\gamma-1}}\,\phi(\omega)v(\omega).\end{aligned} \tag{2.11}$$

Proof. Using Lemma 2.2 and the fractional sum of order $0<\alpha<1$ for (2.1), we obtain

$$\begin{aligned}&E_{-\gamma}\left(\Delta^{\beta}_{\alpha+\gamma-1}u(t)\right)+E_{-\beta}\left(\Delta^{\gamma}_{\alpha+\beta-1}\phi(t)v(t)\right)\\ =&C_1t^{\underline{\alpha-1}}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha+\beta+\gamma-1),\end{aligned}$$

for $t\in\mathbb{N}_{\alpha-1,T+\alpha}$. From $E_{-\tau}u(t)=u(t-\tau)$, we have

$$\begin{aligned}&\Delta^{\beta}_{\alpha+\gamma-1}\,u(t)+\Delta^{\gamma}_{\alpha+\beta-1}\,\phi(t+\beta-\gamma)v(t+\beta-\gamma)\\ =&C_1(t-\gamma)^{\underline{\alpha-1}}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\gamma-\alpha}(t-\gamma-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha+\beta+\gamma-1),\end{aligned} \tag{2.12}$$

for $t\in\mathbb{N}_{\alpha+\gamma-1,T+\alpha+\gamma}$.

Once again, using Lemma 2.2 and the fractional sum of order $0<\beta<1-\alpha$ for (2.12), we obtain

$$u(t) = C_2\,t^{\underline{\beta-1}}+\frac{C_1}{\Gamma(\beta)}\sum_{s=\alpha+\gamma-1}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}\,(s-\gamma)^{\underline{\alpha-1}}$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\gamma-1}^{t-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(t-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\phi(s)v(s)\\
&+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{t-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p-\beta-1}(t-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}h(s), \qquad (2.13)
\end{aligned}
$$

for $t\in\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$ and $1<\alpha+\beta+\gamma\leq 2$.

Applying the first boundary condition of (2.1) implies

$$
C_2=\frac{\rho(u)}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}. \qquad (2.14)
$$

The second condition of (2.1) implies

$$
\begin{aligned}
&\frac{\rho(u)\,\xi^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}+\frac{C_1}{\Gamma(\beta)}\sum_{s=\alpha+\gamma-1}^{\xi-\beta}(\xi-\sigma(s))^{\underline{\beta-1}}\,(s-\gamma)^{\underline{\alpha-1}}\\
&-\frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\gamma-1}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\phi(s)v(s)\\
&+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}h(s)\\
=&\frac{1}{\Gamma(\nu)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}v(s)\,u(s).
\end{aligned}
$$

The constant C_1 can be obtained by solving the above equation, so

$$
\begin{aligned}
C_1=&\frac{1}{\frac{1}{\Gamma(\beta)}\sum_{s=\alpha+\gamma-1}^{\xi-\beta}(\xi-\sigma(s))^{\underline{\beta-1}}\,(s-\gamma)^{\underline{\alpha-1}}}\Bigg[-\frac{\rho(u)\,\xi^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}\\
&+\frac{1}{\Gamma(\nu)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}v(s)\,u(s)+\frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\\
&\times\sum_{p=\alpha+\gamma-1}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\phi(s)v(s)\\
&-\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg]. \qquad (2.15)
\end{aligned}
$$

Substituting the constants C_1, C_2 into (2.14), we obtain

$$
\begin{aligned}
&u(t)\\
=&\frac{\rho(u)\,t^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}+\frac{\sum_{s=\alpha+\gamma-1}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}\,(s-\gamma)^{\underline{\alpha-1}}}{\sum_{s=\alpha+\gamma-1}^{\xi-\beta}(\xi-\sigma(s))^{\underline{\beta-1}}\,(s-\gamma)^{\underline{\alpha-1}}}
\end{aligned}
$$

$$\times\Bigg[-\frac{\rho(u)\,\xi^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}+\frac{1}{\Gamma(\nu)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}v(s)u(s)$$
$$+\frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\gamma-1}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\phi(s)v(s)$$
$$-\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg]$$
$$-\frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\beta-1}^{t-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(t-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}\phi(s)v(s)$$
$$+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{t-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(t-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}h(s).$$
(2.16)

Let

$$\mathcal{A}(u)=\frac{1}{\Gamma(\nu)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}v(s)\,u(s),$$

then we have

$$\mathcal{A}(u)=\frac{1}{\Gamma(\nu)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}v(s)$$
(2.17)
$$\times\Bigg\{\Upsilon_\rho(s,\xi)+\psi(s,\xi)\mathcal{A}(u)+\psi(s,\xi)\Psi_{v,\phi}(\xi)-\psi(s,\xi)\mathfrak{I}_h(\xi)+\mathfrak{J}_h(s)-\Phi_{v,\phi}(s)\Bigg\},$$

where $\psi(s,\xi)$, $\Upsilon_\rho(s,\xi)$, $\Psi_{v,\phi}(\xi)$, $\mathfrak{I}_h(\xi)$, $\mathfrak{J}_h(s)$ and $\Phi_{v,\phi}(s)$ are defined as (2.4), (2.7)-(2.11), respectively.

We simplify (2.17) into (2.3). Finally, substituting $\mathcal{A}(u)$ into (2.16), we obtain (2.2). This completes the proof. □

In the following, we shall give some definitions and lemma, which are associated with the Sadovskii fixed point theorem as follow.

Definition 2.5. Let M be a bounded set in metric space $(X;d)$, then Kuratowski measure of noncompactness, $\alpha(M)$ is defined as

$$\inf\{\epsilon : \text{M covered by a finitely many sets such that the diameter of each set} \le \epsilon\}.$$

Definition 2.6. Let $\Phi : D(\Phi)\subseteq X\to X$ be a bounded and continuous operator on Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B))<\alpha(B)$ for all bounded sets $B\subset D(\Phi)$, where α denotes the Kuratowski measure of noncompactness.

Lemma 2.4. [31] *The map $K+C$ is a k-set contraction with $0 \le k < 1$, and thus also condensing, if*

(i) *$K, C : D \subseteq X \to X$ are operators on the Banach space X;*

(ii) *K is k-contractive, i.e.,*

$$\|Kx - Ky\| \le k\|x - y\|$$

for all $x, y \in D$ and fixed $k \in [0,1)$;

(iii) *C is compact.*

Lemma 2.5. [17] (Arzelá-Ascoli theorem) *A set of function in $C[a,b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 2.6. [17] *If a set is closed and relatively compact then it is compact.*

Theorem 2.2. [22] (Sadovskii's fixed point theorem) *Let B be a convex, bounded and closed subset of a Banach space X and $\Phi : B \to B$ be a condensing map. Then Φ has a fixed point.*

3. The existence of solutions to the four-point fractional sum boundary value problem (1.1)

Now, we wish to establish the existence result for the problem (1.1). To accomplish this, we denote that $\mathcal{C} = C(\mathbb{N}_{\alpha+\beta+\gamma-2, T+\alpha+\beta+\gamma}, \mathbb{R})$ is a Banach space of all functions u with the norm defined by

$$|u\|_{\mathcal{C}} = \|u\|\|v\|,$$

where

$$\begin{aligned}\|u\| =& \max_{t\in\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}} \{|u(t)| : |u(t)| \ge 1\},\\ \|v\| =& \max\{|v(t)| : v \in C(\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}, \mathbb{R}^+) \text{ is a given function}\}.\end{aligned}$$

Also define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathcal{F}u)(t) = (\mathcal{F}_1 u)(t) + (\mathcal{F}_2 u)(t), \tag{3.1}$$

and

$$\begin{aligned}(\mathcal{F}_1 u)(t) =& \frac{\rho(u)\, t^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}} + \frac{\sum_{s=\alpha+\gamma-1}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} (s-\gamma)^{\underline{\alpha-1}}}{\sum_{s=\alpha+\gamma-1}^{\xi-\beta} (\xi-\sigma(s))^{\underline{\beta-1}} (s-\gamma)^{\underline{\alpha-1}}}\\ &\times \Bigg[-\frac{\rho(u)\, \xi^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}} + \widehat{\mathcal{A}}(u) + \frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\\ &\times \sum_{p=\alpha+\gamma-1}^{\xi-\beta} \sum_{s=\alpha+\beta+\gamma-1}^{p+\beta} (\xi-\sigma(p))^{\underline{\beta-1}} (p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}} \phi(s) v(s)\\ &- \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \sum_{p=\alpha+\gamma}^{\xi-\beta} \sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1} (\xi-\sigma(p))^{\underline{\beta-1}} (p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}\end{aligned}$$

$$\times f(s,u(s),v(s))\Bigg]$$

$$=\Upsilon_\rho(t,\xi)+\psi(t,\xi)\widehat{\mathcal{A}}(u)+\psi(t,\xi)\Psi_v(\xi)-\psi(t,\xi)\widehat{\mathfrak{I}}_f(\xi), \tag{3.2}$$

$$(\mathcal{F}_2u)(t)=-\frac{1}{\Gamma(\beta)\Gamma(-\gamma)}\sum_{p=\alpha+\beta-1}^{t-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta}(t-\sigma(p))^{\underline{\beta-1}}(p+\beta-\gamma-\sigma(s))^{\underline{-\gamma-1}}$$

$$\times\phi(s)v(s)+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{t-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(t-\sigma(p))^{\underline{\beta-1}}$$

$$\times(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))$$

$$=\widehat{\mathcal{J}}_f(t)-\Phi_v(t), \tag{3.3}$$

where

$$\widehat{\mathcal{A}}(u)=\frac{1}{\Gamma(\nu)\Theta(T,\xi,\eta)}\sum_{s=\eta}^{T+\alpha+\beta+\gamma}(T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}$$

$$\times v(s)\left\{\Upsilon_\rho(s,\xi)+\psi(s,\xi)\Psi_{v,\phi}(\xi)-\psi(s,\xi)\widehat{\mathfrak{I}}_f(\xi)+\widehat{\mathcal{J}}_f(s)-\Phi_{v,\phi}(s)\right\}, \tag{3.4}$$

$$\widehat{\mathfrak{I}}_f(\xi)=\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{\xi-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}$$

$$\times\ f(s,u(s),v(s))), \tag{3.5}$$

$$\widehat{\mathcal{J}}_f(s)=\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{t-\beta}\sum_{s=\alpha+\beta+\gamma-1}^{p+\beta-1}(t-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(s))^{\underline{\alpha-1}}$$

$$\times\ f(s,u(s),v(s)), \tag{3.6}$$

with $\psi(s,\xi)$, $\mathcal{P}_v(T,\eta)$, $\mathcal{Q}(\xi)$, $\Theta(T,\xi,\eta)$, $\Upsilon_\rho(s,\xi)$, $\Psi_{v,\phi}(s,\xi)$ and $\Phi_{v,\phi}(s,\xi)$ are defined as (2.4)-(2.7) and (2.11). The problem (1.1) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

In the following theorem, we shall give the existence result for the problem (1.1), by the help of Sadovskii's fixed point theorem.

Theorem 3.1. *Assume that functions* $f\in C\left(\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}\times\mathbb{R}\times\mathbb{R}^+,\mathbb{R}\right)$, $v,\phi\in C\left(\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma},\mathbb{R}^+\right)$ *and functional* $\rho: C\left(\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma},\mathbb{R}\right)\to\mathbb{R}$ *are given. Also, suppose* f, ρ *and* ϕ *satisfying the following conditions.*

(H_1) *There exists a constant* $L>0$ *such that*

$$|f(t,u_1,v)-f(t,u_2,v)|\le L|u_1-u_2||v|,$$

for each $t\in\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$ *and* $u_1,u_2\in\mathbb{R}$.

(H_2) *There exists a constant* $\lambda>0$ *such that*

$$|\rho(u_1)-\rho(u_2)|\le\lambda\|u_1-u_2\|\|v\|,$$

for each $u_1, u_2 \in \mathcal{C}$.

(H_3) $0 < n \le \phi(t) \le N$ *for each* $t \in \mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$.

Then the problem (1.1) *has at least one solution on* $\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$, *provided that*

$$\chi := \|v\|\Big\{\lambda\Omega_1 + L\Omega_2 + N\Omega_3\Big\} < 1,$$

where

$$\Omega_1 = \frac{|\Upsilon|}{|\Theta|}\Big(\mathcal{P} + |\Theta|\Big), \tag{3.7}$$

$$\Omega_2 = \frac{\psi_{\max}}{|\Theta|}\Big(\widehat{\mathcal{J}} + \widehat{\mathcal{I}}\Big)\Big(\mathcal{P} + |\Theta|\Big), \tag{3.8}$$

$$\Omega_3 = \frac{\psi_{\max}}{|\Theta|}\Big(\Phi + \Psi\Big)\Big(\mathcal{P} + |\Theta|\Big), \tag{3.9}$$

with $\psi_{\max}$, $\mathcal{P}$, $|\Theta|$, $|\Upsilon|$, Ψ, $\widehat{\mathcal{I}}$, $\widehat{\mathcal{J}}$ *and* Φ *are defined as* (3.10), (3.12)-(3.18), *respectively.*

Proof. Let $B_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \le R\}$ be a closed bounded and convex subset of $\mathcal{C}$, where R will be fixed later. We define a map $\mathcal{F} : B_R \to \mathcal{C}$ as

$$(\mathcal{F}u)(t) = (\mathcal{F}_1 u)(t) + (\mathcal{F}_2 u)(t),$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by (3.2) and (3.3) respectively. Notice that the problem (1.1) is equivalent to a fixed point problem $\mathcal{F}(u) = u$.

Step 1. $\mathcal{F}(B_R) \subset B_R$.

Set $\max\limits_{t \in \mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}} |f(t,0,0)| = K$, $\sup\limits_{u\in\mathcal{C}} |\rho(u)| = M$ and choose a constant R satisfied

$$R \ge \frac{M\Omega_1 + K\left(\Omega_2 + \widehat{\mathcal{J}}\right)}{\frac{1}{\|v\|} - \left[\lambda\Omega_1 + L\left(\Omega_2 + \widehat{\mathcal{J}}\right) + N\left(\Omega_3 + \Phi\right)\right]}.$$

We first consider that the values of $\psi(s,\xi)$, $\mathcal{P}_v$, $\mathcal{Q}$, $\Theta(s,T,\xi,\eta)$, $\Upsilon_\rho(s,\xi)$, $\Psi_{\rho,\phi}(\xi)$, $\widehat{\mathcal{I}}_f(s,\xi)$, $\widehat{\mathcal{J}}_f(s)$ and $\Phi_{v,\phi}(s)$, as follow

$$\psi(s,\xi) = \left|\frac{(s-\alpha-\gamma-1)^{\underline{\beta-1}}\,{}_2F_1(\alpha, \alpha+\beta+\gamma-s; \alpha+\gamma-s+1; 1)}{(\xi-\alpha-\gamma-1)^{\underline{\beta-1}}\,{}_2F_1(\alpha, \alpha+\beta+\gamma-\xi; \alpha+\gamma-\xi+1; 1)}\right|,$$

$$\begin{aligned}\psi_{\max} &= \psi(\alpha+\beta+\gamma, \xi)\\ &= \frac{(\beta-1)^{\underline{\beta-1}}\,{}_2F_1(\alpha, 0; 1-\beta; 1)}{(\xi-\alpha-\gamma-1)^{\underline{\beta-1}}\,{}_2F_1(\alpha, \alpha+\beta+\gamma-\xi; \alpha+\gamma-\xi+1; 1)},\end{aligned} \tag{3.10}$$

$$\psi_{\min} = \psi(\alpha+\beta+\gamma-1, \xi) = 0, \tag{3.11}$$

$$\begin{aligned}\mathcal{P}_v &\le \frac{\|v\|}{\Gamma(\nu)} \sum_{s=\eta}^{T+\alpha+\beta+\gamma} (T+\alpha+\beta+\gamma+\nu-\sigma(s))^{\underline{\nu-1}}\\ &= \frac{\|v\|\Gamma(T+\alpha+\beta+\gamma+\nu-\eta+1)}{\Gamma(\nu+1)\Gamma(T+\alpha+\beta+\gamma-\eta+1)} =: \|v\|\mathcal{P},\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\Big|\Theta(T,\xi,\eta)\Big| \geqslant & \Big|\varphi_{\min}\Big(\|v\|\mathcal{P}\Big)-1\Big| \;=\; |-1| = 1 \\
=&:|\Theta|, \qquad (3.13)\\
\Upsilon_{\rho}(s,\xi) \leq & \Big|\psi\xi^{\underline{\beta-1}} - s^{\underline{\beta-1}}\Big| \frac{\Big(|\rho(u)-\rho(0)|+|\rho(0)|\Big)}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}\\
\leq & \left(\lambda\|v\|\|u\|+M\right)\left|\frac{\psi_{\max}\xi^{\underline{\beta-1}}-s^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}\right|\\
\leq & \left(\lambda\|u\|_{\mathcal{C}}+M\right)\left|\frac{\psi_{\max}\xi^{\underline{\beta-1}}-(T+\alpha+\beta+\gamma)^{\underline{\beta-1}}}{(\alpha+\beta+\gamma-2)^{\underline{\beta-1}}}\right|\\
=&:\left(\lambda\|u\|_{\mathcal{C}}+M\right)|\Upsilon|, \qquad (3.14)\\
\Psi_{v,\phi}(\xi) \leqslant & \frac{\|v\|\|\phi\|\,(\xi-\alpha-\gamma)^{\underline{\beta-1}}}{\Gamma(\beta)}{}_2F_1(1-\gamma,\alpha+\beta+\gamma-\xi-1;\alpha+\gamma-\xi;1)\\
\leqslant & \frac{\|v\|\|u\|N(\xi-\alpha-\gamma)^{\underline{\beta-1}}}{\Gamma(\beta)}{}_2F_1(1-\gamma,\alpha+\beta+\gamma-\xi-1;\alpha+\gamma-\xi;1)\\
= & \frac{\|u\|_{\mathcal{C}}N(\xi-\alpha-\gamma)^{\underline{\beta-1}}}{\Gamma(\beta)}{}_2F_1(1-\gamma,\alpha+\beta+\gamma-\xi-1;\alpha+\gamma-\xi;1)\\
=&:\|u\|_{\mathcal{C}}N\Psi, \qquad (3.15)\\
\widehat{\mathfrak{I}}_f(\xi) \leq & \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{\xi-\beta}\sum_{\omega=\alpha+\beta+\gamma-1}^{p+\beta-1}(\xi-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(\omega))^{\underline{\alpha-1}}\\
& \times\Big(|f(\omega,u(\omega),v(\omega))-f(\omega,0,0)|+|f(\omega,0,0)|\Big)\\
\leqslant & \frac{(L\|v\|\|u\|+K)}{\Gamma(\beta)}(\xi-\alpha-\gamma-1)^{\underline{\beta-1}}\\
& \times{}_2F_1(\alpha+1,\alpha+\beta+\gamma-\xi;\alpha+\beta-\xi+1;1)\\
=&:\left(L\|u\|_{\mathcal{C}}+K\right)\widehat{\mathfrak{I}}, \qquad (3.16)\\
\widehat{\mathfrak{J}}_f(s) \leq & \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{p=\alpha+\gamma}^{s-\beta}\sum_{\omega=\alpha+\beta+\gamma-1}^{p+\beta-1}(s-\sigma(p))^{\underline{\beta-1}}(p+\alpha+\beta-1-\sigma(\omega))^{\underline{\alpha-1}}\\
& \times\Big(|f(\omega,u(\omega),v(\omega))-f(\omega,0,0)|+|f(\omega,0,0)|\Big)\\
\leqslant & \frac{(L\|v\|\|u\|+K)}{\Gamma(\beta)}(s-\alpha-\gamma-1)^{\underline{\beta-1}}\times\\
& {}_2F_1(\alpha+1,\alpha+\beta+\gamma-s;\alpha+\beta-s+1;1)\\
\leq & \frac{(L\|u\|_{\mathcal{C}}+K)}{\Gamma(\beta)}(\beta-1)^{\underline{\beta-1}}{}_2F_1(\alpha+1,1;1-\gamma;1)\\
=&:\left(L\|u\|_{\mathcal{C}}+K\right)\widehat{\mathfrak{J}}, \qquad (3.17)
\end{aligned}$$

$$\begin{aligned}
\Phi_{v,\phi}(s) \leqslant & \frac{\|v\|\|\phi\| \, (s-\alpha-\gamma)^{\underline{\beta-1}}}{\Gamma(\beta)} {}_2F_1(1-\gamma, \alpha+\beta+\gamma-s-1; \alpha+\gamma-s; 1) \\
\leq & \frac{\|v\| N \, (\beta-1)^{\underline{\beta-1}}}{\Gamma(\beta)} {}_2F_1(1-\gamma, 0; 1-\beta; 1) \\
= & \frac{\|u\|_{\mathcal{C}} N \, (\beta-1)^{\underline{\beta-1}}}{\Gamma(\beta)} {}_2F_1(1-\gamma, 0; 1-\beta; 1) \\
= & : \|u\|_{\mathcal{C}} N \Phi. \qquad (3.18)
\end{aligned}$$

Next, we consider

$$\left|\widehat{\mathcal{A}}(u)\right| \leqslant \left| \frac{\mathcal{P}_v}{\left|\Theta\right|} \left(\Upsilon_\rho(s,\xi) + \psi(s,\xi)\Psi_{v,\phi}(\xi) - \psi(s,\xi)\widehat{\mathfrak{I}}_f(\xi) + \widehat{\mathfrak{J}}_f(s) - \Phi_{v,\phi}(s) \right) \right|. \qquad (3.19)$$

Now, we will show that $\mathcal{F}(B_R) \subset B_R$. For each $t \in \mathbb{N}_{\alpha+\beta+\gamma-2, T+\alpha+\beta+\gamma}$ and $u \in B_R$, we have

$$\begin{aligned}
|(\mathcal{F}_1 u)(t)| = & \left| \Upsilon_\rho(t,\xi) + \psi(s,\xi)\widehat{\mathcal{A}}(u) + \psi(s,\xi)\Psi_{v,\phi}(\xi) - \psi(s,\xi)\widehat{\mathfrak{I}}_f(s,\xi) \right| \\
\leqslant & \left| \Upsilon_\rho(t,\xi) + \frac{\psi(s,\xi)\mathcal{P}_v}{|\Theta|} \right| + \left| \frac{\left[\widehat{\mathfrak{J}} - \psi(s,\xi)\widehat{\mathfrak{I}}_f(s,\xi)\right]\mathcal{P}_v}{|\Theta|} - \psi(s,\xi)\widehat{\mathfrak{I}}_f(s,\xi) \right| \\
& + \left| \frac{\left[\psi(s,\xi)\Psi_{v,\phi}(\xi) - \Phi\right]\mathcal{P}_v}{|\Theta|} - \psi(s,\xi)\Psi_{v,\phi}(\xi) \right| \\
\leqslant & (\lambda\|u\|_{\mathcal{C}} + M) \frac{|\Upsilon|}{|\Theta|} \Big[\mathcal{P} + |\Theta|\Big] + (L\|u\|_{\mathcal{C}} + K) \frac{\varphi_{\max}}{|\Theta|} \left[\left(\widehat{\mathfrak{J}} + \widehat{\mathfrak{I}}\right)\left(\mathcal{P} + |\Theta|\right) \right] \\
& + N\|u\|_{\mathcal{C}} \frac{\varphi_{\max}}{|\Theta|} \left[\left(\Phi + \Psi\right)\left(\mathcal{P} + |\Theta|\right) \right] \\
= & (\lambda \|u\|_{\mathcal{C}} + M) \, \mathbf{\Omega}_1 + (L \|u\|_{\mathcal{C}} + K) \, \mathbf{\Omega}_2 + N\|u\|_{\mathcal{C}} \mathbf{\Omega}_3, \qquad (3.20)
\end{aligned}$$

and

$$\begin{aligned}
\left|(\mathcal{F}_2 u)(t)\right| = & \left|\widehat{\mathfrak{J}}_f(t) - \Phi_{v,\phi}(t)\right| \\
\leq & (L\|u\|_{\mathcal{C}} + K) \, |\widehat{\mathfrak{J}} + N\|u\|_{\mathcal{C}} \Phi. \qquad (3.21)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|v\| \left|(\mathcal{F}u)(t)\right| = & \|v\| \left\{ \left|(\mathcal{F}_1 u)(t)\right| + \left|(\mathcal{F}_2 u)(t)\right| \right\} \\
= & \|v\| \left\{ (\lambda \|u\|_{\mathcal{C}} + M) \, \mathbf{\Omega}_1 + (L \|u\|_{\mathcal{C}} + K) \left[\mathbf{\Omega}_2 + \widehat{\mathfrak{J}}\right] + N\|u\|_{\mathcal{C}} \left[\mathbf{\Omega}_3 + \Phi\right] \right\} \\
\leq & R. \qquad (3.22)
\end{aligned}$$

Therefore $\left\|(\mathcal{F}u)(t)\right\|_{\mathcal{C}} \leq R$, it follows that $\mathcal{F}(B_R) \subset B_R$.

Step 2. $\mathcal{F}_1$ is continuous and χ-contractive.

Let $\epsilon > 0$ be given. Since ρ, f and v are continuous, so ρ, f and v are uniformly continuous on B_R. Therefore, there exists $\delta = \min\left\{\delta_1, \delta_2, \delta_3\right\} > 0$ such that

$$\begin{aligned}\left|\rho(u_1) - \rho(u_2)\right| &< \frac{\epsilon}{3\|v\|\lambda\Omega_1}, \quad \text{whenever } \left|u_1 - u_2\right| \cdot |v| < \delta_1,\\ \left|f(t, u_1, v) - f(t, u_2, v)\right| &< \frac{\epsilon}{3\|v\|L\Omega_2}, \quad \text{whenever } \max\left\{|u_1 - u_2| \cdot |v|\right\} < \delta_2,\\ \left\|u_1 - u_2\right\|_{\mathcal{C}} &< \frac{\epsilon}{3\|v\|N\Omega_3}, \quad \text{whenever } |u_1 - u_2| \cdot |v| < \delta_3.\end{aligned}$$

Thus, for all $t \in \mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$ and for all $u_1, u_2 \in B_R$, we obtain

$$\begin{aligned}&\|(\mathcal{F}_1u_1)(t) - (\mathcal{F}_1u_2)(t)\|_{\mathcal{C}}\\ &= \|v\|\|(\mathcal{F}_1u_1)(t) - (\mathcal{F}_1u_2)(t)\|\\ &\leq \|v\|\left\{\lambda\Omega_1\left|\rho(u_1) - \rho(u_2)\right| + L\Omega_2\left\|f(t, u_1, v) - f(t, u_2, v)\right\| + N\Omega_3\left\|u_1 - u_2\right\|_{\mathcal{C}}\right\}\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This means that $\mathcal{F}_1$ is continuous on B_R.

Next, we show that $\mathcal{F}_1$ is χ-contractive. For any $u_1, u_2 \in B_R$ and for each $t \in \mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$, we have

$$\begin{aligned}\|(\mathcal{F}_1u_1)(t) - (\mathcal{F}_1u_2)(t)\|_{\mathcal{C}} &= \|v\|\|(\mathcal{F}_1u_1)(t) - (\mathcal{F}_1u_2)(t)\|\\ &\leq \|v\|\left\{\lambda\Omega_1\left\|u_1 - u_2\right\|_{\mathcal{C}} + L\Omega_2\left\|u_1 - u_2\right\|_{\mathcal{C}} + N\Omega_3\left\|u_1 - u_2\right\|_{\mathcal{C}}\right\}\\ &= \chi\left\|u_1 - u_2\right\|_{\mathcal{C}}.\end{aligned}$$

By the given assumption: $\chi < 1$, it follows that $\mathcal{F}_1$ is χ-contractive.

Step 3. $\mathcal{F}_2$ is compact.

In Step 1. it has been shown that $\mathcal{F}_2$ is uniformly bounded. Now we show that $\mathcal{F}_2$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

For any $\epsilon > 0$, there exists $\widetilde{\delta} = \min\left\{\widetilde{\delta}_1, \widetilde{\delta}_2\right\} > 0$ such that

$$\begin{aligned}\left|\widehat{\mathcal{J}}_f(t_2) - \widehat{\mathcal{J}}_f(t_1)\right| &\leq \frac{\epsilon}{2\|v\|\|f\|}, \quad \text{whenever } |t_2 - t_1| < \widetilde{\delta}_1,\\ |\Phi_{v,\phi}(t_2) - \Phi_{v,\phi}(t_1)| &\leq \frac{\epsilon}{2\|v\|^2N}, \quad \text{whenever } |t_2 - t_1| < \widetilde{\delta}_2.\end{aligned}$$

Hence, for any $t_1, t_2 \in \mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$ any $u, v \in B_R$, we have

$$\begin{aligned}\|\mathcal{F}_2u(t_2) - \mathcal{F}_2u(t_1)\|_{\mathcal{C}} =&\|v\|\|\mathcal{F}_2u(t_2) - \mathcal{F}_2u(t_1)\|\\ \leq&\|v\|\left\{\|f\|\left|\widehat{\mathcal{J}}_f(t_2) - \widehat{\mathcal{J}}_f(t_1)\right| + \|v\|N\left|\Phi_{v,\phi}(t_2) - \Phi_{v,\phi}(t_1)\right|\right\}\\ \leq&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus $(\mathcal{F}_2)(B_R)$ is an equicontinuous set. Therefore it follows by Lemma 2.6 and the Arzelá-Ascoli theorem that $\mathcal{F}_2$ is compact on B_R.

Step 4. $\mathcal{F}$ is is condensing.

Since $\mathcal{F}_1$ is continuous, χ-contraction and $\mathcal{F}_2$ is compact, therefore, by Lemma 2.4, $\mathcal{F} : B_R \to B_R$ with $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ is a condensing map on B_R.

Consequently, by Theorem 3.1, the map $\mathcal{F}$ has a fixed point which, implies that the problem (1.1) has a solution. □

4. Example

In this section, in order to illustrate our result, we consider an example. Consider the following fractional sum boundary value problem

$$\Delta_0^{\frac{1}{4}} \left[E_{-\frac{1}{3}}\left(\Delta^{\frac{2}{3}}_{-\frac{5}{6}}u(t)\right) + E_{\frac{2}{3}}\left(\Delta^{\frac{1}{3}}_{-\frac{1}{12}}\right)\phi(t)v(t)\right] = \frac{e^{-(t+\frac{1}{4})}v(t+\frac{1}{4})}{\left(t+\frac{101}{4}\right)^2}\cdot\frac{|u|}{|u|+1}, \tag{4.1}$$

$$u\left(-\frac{3}{4}\right) = \sum_{i=0}^{7} C_i\, u(t_i), \quad t_i = i - \frac{3}{4},$$

$$u\left(\frac{13}{4}\right) = \left[\Delta_{\frac{7}{2}}^{-\frac{1}{2}} v\left(s+\frac{1}{2}\right) u\left(s+\frac{1}{2}\right)\right]_{s=\frac{21}{4}}^{\frac{25}{4}},$$

where $t \in \mathbb{N}_{0,5}$, $v(t) = e^{\sin(\pi t)}$, $\phi(t) = \frac{e^{-t}}{100\pi}$ are given functions, and C_i are given positive constants with $\sum_{i=0}^{7} C_i < \frac{e}{20}$.

Here $\alpha = \frac{1}{4}$, $T = 5$, $\beta = \frac{2}{3}$, $\gamma = \frac{1}{3}$, $\nu = \frac{1}{2}$, $\xi = \frac{13}{4}$, $\eta = \frac{21}{4}$, $f(t, u(t), v(t)) = \frac{e^{-t}}{(t+25)^2}\cdot\frac{|u|}{|u|+1}$ and $\rho(u) = \sum_{i=0}^{7} C_i\, u(t_i)$.

Let $t \in \mathbb{N}_{-\frac{3}{4},\frac{25}{4}}$ and $u_1, u_2 \in \mathcal{C}$, then

$$\frac{1}{e} \leqslant |v(t)| \leqslant e \quad \text{and} \quad |f(t, u_1(t), v(t)) - f(t, u_2(t), v(t))| \leq \frac{16\, e^{3/4}}{10201}|u_1 - u_2|.$$

So, (H_1) holds with $L = \frac{16}{10201\, e^{1/4}} \approx 0.0012$.

Also, we get

$$\begin{aligned} |\rho(u_1) - \rho(u_2)| &= \left|\sum_{i=0}^{7} C_i\, u_1(t_i) - \sum_{i=0}^{7} C_i\, u_2(t_i)\right| \\ &\leqslant \sum_{i=0}^{7} C_i\, |u_1 - u_2| < \frac{e}{20}|u_1 - u_2| = \frac{1}{20}\|v\||u_1 - u_2|. \end{aligned}$$

So, (H_2) holds with $\lambda = \frac{1}{20} \approx 0.05$. Since $6.1448 \times 10^{-6} \leq \phi(t) \leq 0.00674 = N$, then (H_3) is satisfied.

We can show that

$$\psi_{\max} = \frac{\left(-\frac{1}{3}\right)^{\overline{-1/3}}\ {}_2F_1(\frac{1}{4}, 0; \frac{1}{3}; 1)}{\left(\frac{5}{3}\right)^{\overline{-1/3}}\ {}_2F_1(\frac{1}{4}, -2; -\frac{5}{3}; 1)} = 1.1383,$$

$$
\begin{aligned}
\psi_{\min} =& 0,\\
\mathcal{P} =& \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})\Gamma(2)} = 1.5,\\
\left|\Theta\right| =& 1,\\
|\Upsilon| =& \left| \frac{1.1383\ \left(\frac{13}{4}\right)^{\underline{-1/3}} - \left(\frac{25}{4}\right)^{\underline{-1/3}}}{\left(-\frac{3}{4}\right)^{\underline{-1/3}}} \right| = 0.0830,\\
\Psi =& \frac{\left(\frac{8}{3}\right)^{\underline{-1/3}}}{\Gamma(\frac{2}{3})}\ {}_2F_1\left(\frac{2}{3}, -3; -\frac{8}{3}; 1\right) = 1.7284,\\
\widehat{\mathcal{I}} =& \frac{\left(\frac{5}{3}\right)^{\underline{-1/3}}}{\Gamma(\frac{2}{3})}\ {}_2F_1\left(\frac{5}{4}, -2; -\frac{5}{3}; 1\right) = 2.79514,\\
\widehat{\mathcal{J}} =& \frac{\left(\frac{14}{3}\right)^{\underline{-1/3}}}{\Gamma(\frac{2}{3})}\ {}_2F_1\left(\frac{5}{4}, -5; -\frac{13}{3}; 1\right) = 10.5789,\\
\Phi =& \frac{\left(\frac{17}{3}\right)^{\underline{-1/3}}}{\Gamma(\frac{2}{3})}\ {}_2F_1\left(\frac{2}{3}, -6; -\frac{17}{3}; 1\right) = 2.1082,
\end{aligned}
$$

and $\Omega_1 = 0.2075$, $\Omega_2 = 38.05917$ and $\Omega_3 = 10.9177$.
Therefore, we have

$$\chi = \|v\|\left\{\lambda\Omega_1 + L\Omega_2 + N\Omega_3\right\} = 0.1296e = 0.3523 < 1.$$

Hence, by Theorem 3.1, the boundary value problem (4.1) has at least one solution on $\mathbb{N}_{\alpha+\beta+\gamma-2,T+\alpha+\beta+\gamma}$.

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