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APPROXIMATION METHOD FOR HAMMERSTEIN INTEGRAL EQUATIONS

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Abstract. In this paper, we study a Mann-type algorithm combined with inertial effect for approximating solutions of Hammerstein integral equations $u + KFu = 0$, where K and F are maximal monotone maps defined on a real Hilbert space $\mathcal{H}$, with $D(K) = D(F) = \mathcal{H}$. The sequence of the algorithm is proved to converge strongly to a solution of the Hammerstein equation, without any *Lipschitz or boundedness condition* imposed on the maps K and F. The convergence analysis relies on geometric properties of Hilbert spaces and a key result concerning the *quasi-boundedness* of monotone operators whose domains contain the origin as an interior point. Furthermore, some numerical examples are given to study the *effect of the inertial term* in our algorithm in comparison with some algorithms without inertial term. Finally, the theorem proved, improves and complements some important related recent results in the literature

1. introduction

Let $\mathcal{X}^*$ be the dual space of a real normed space $\mathcal{X}$. A map $J : \mathcal{X} \to 2^{\mathcal{X}^*}$ defined by $J(u) = \{u^* \in \mathcal{X}^* : \langle u, u^* \rangle = ||u|| \cdot ||u^*||, ||u|| = ||u^*||\}$ is called the *normalized duality map.* It is well known that the normalized duality map is single-valued, one-to-one and onto, if $\mathcal{X}$ is a smooth, strictly convex and reflexive Banach space, respectively. For more properties of the normalized duality map (see e.g., [21, 22]).

A map $A : \mathcal{X} \to \mathcal{X}$ is called *accretive* if, for all $u, v \in \mathcal{X}$, there exists $j(u - v) \in J(u - v)$ such that $\langle Au - Av, j(u - v) \rangle \geq 0$. The map A is called *maximal accretive* if, A is accretive and the graph of A is not contained in the graph of any other accretive map. The map A is called *m-accretive* if, A is accretive and $R(I + sA) = \mathcal{X}$, for all

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$s > 0$, where $R(I + sA)$ denotes the range of $(I + sA)$. In real Hilbert spaces, accretive maps are called *monotone.* Accretive maps were introduced independently in 1967 by Browder [8] and Kato [25]. Interest in this class of maps stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces, i.e., equations of the form:

$$\begin{cases} u'(t) + Au(t) = h(u(t)), & t \geq 0, \\ u(0) = h_0, \end{cases} \tag{1.1}$$

where A is accretive. At equilibrium state, and setting $h \equiv 0$ in equation (1.1), we obtain the following equation:

$$Au = 0. \tag{1.2}$$

In many cases, where the map A is accretive, solutions of equation (1.2), represent the equilibrium state of the system described by equation (1.1).

An early fundamental theorem in the theory of accretive maps, due to Browder [8], states that the initial value problem (1.1) is solvable if A is locally Lipschitz and accretive on $\mathcal{X}$. Using the existence results for (1.1), Browder proved that if, A is locally Lipschitz and accretive on $\mathcal{X}$, then, A is m-accretive, i.e., $\mathcal{R}(I + A) = \mathcal{X}$, where $\mathcal{R}(I + A)$ denotes the range of $(I + A)$. Clearly, a consequence of this, is that the equation

$$u + Au = 0 \tag{1.3}$$

has a solution. One important generalization of equation (1.3) is the *equation of Hammerstein type.* Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable.

Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation of Hammerstein-type has the form:

$$u(x) + \int_\Omega k(x, y) f(y, u(y)) dy = w(x), \tag{1.4}$$

where the unknown function u and inhomogeneous function w lie in a Banach space $\mathcal{X}$ of measurable real-valued functions. Define a linear map K by

$$Kv(x) = \int_\Omega k(x, y) v(y) dy,$$

on Ω and denote by F the superposition or *Nemistskyi* operator corresponding to f, i.e., $Fu(y) = f(y, u(y))$. Then, Equation (1.4) can be put in the form: $u + KFu = 0$, where, without loss of generality, we have taken $w \equiv 0$.

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts posses Green's function can, as a rule, be put in the form (1.4) (see e.g., Pascali and Sburian [30], page 164). Several existence and uniqueness theorems have been proved for equations of Hammer-stein-type (see e.g., [5–7, 9, 23], and the references contained in them). In general, equations of Hammerstein type are nonlinear and there is no known method to find a closed form solutions for them.

Consequently, methods for approximating solutions of such equations are of interest. Chidume and Zegeye [13], introduced and studied the following coupled algorithm for approximating a solution of a Hammerstein equation in a real Hilbert space $\mathcal{H}$:

$$\begin{cases} u_0, \quad v_0 \in \mathcal{H}, \\ u_{n+1} = P_{D(F)}(u_n - \xi_n(Fu_n - v_n + \theta_n(u_n - w_1))), \quad n \geq 1, \\ v_{n+1} = P_{D(K)}(v_n - \xi_n(Kv_n + u_n + \theta_n(u_n - w_2))), \quad n \geq 1, \end{cases} \tag{1.5}$$

where F and K are monotone and bounded maps, $w_1 \in D(F)$, $w_2 \in D(K)$ are arbitrary but fixed and $D(F)$, $D(K)$ denote the domains of the maps F and K, respectively; $\{\xi_n\}$, $\{\theta_n\}$ are sequences in $(0,1)$ satisfying appropriate conditions and $P_{D(F)}$, $P_{D(K)}$ are metric projections on $\mathcal{H}$ onto $P_{D(F)}$ and $P_{D(K)}$, respectively. The sequence generated by the algorithm is proved to converge strongly to a solution of the Hammerstein equation. Recently, Ofoedu and Onyi [28] studied the following explicit algorithm for approximating solutions of Hammerstein equations in real Hilbert spaces:

$$\begin{cases} u_0, \quad v_0 \in \mathcal{H}, \\ u_{n+1} = (1 - \sigma_n)u_n + \sigma_n(u_n - Fu_n + v_n) - \sigma_n\xi_n\alpha_n u_n, \quad n \geq 1, \\ v_{n+1} = (1 - \sigma_n)v_n + \sigma_n(v_n - Kv_n - u_n) - \sigma_n\xi_n\alpha_n v_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\sigma_n\}$, $\{\xi_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$. In fact, they proved the following theorem.

Theorem 1.1. *Let $\mathcal{H}$ be a real Hilbert space. Let K, $F : \mathcal{H} \to \mathcal{H}$ be monotone and Lipschitz maps. Assume that $\{\xi_n\}$, $\{\theta_n\}$ and $\{\alpha_n\}$ satisfy the following:*

$$\lim_{n\to+\infty} \xi_n = \lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \frac{\sigma_n}{\xi_n\alpha_n} = \lim_{n\to+\infty} \frac{\xi_{n-1} - \xi_n}{\sigma_n\xi_n^2\alpha_n^2}$$
$$= \lim_{n\to+\infty} \frac{\alpha_{n-1} - \alpha_n}{\sigma_n\xi_n^2\alpha_n^2} = 0.$$

Then, the sequence $\{u_n\} \in \mathcal{H}$ generated by Algorithm (1.6)*, converges strongly to a solution u^* of $u + KFu = 0$.*

We observe that Algorithm (1.6) is equivalent to the following algorithm:

$$\begin{cases} u_0, v_0 \in \mathcal{H}, \\ u_{n+1} = u_n - \sigma_n(Fu_n - v_n) - \sigma_n\xi_n\alpha_n u_n, \quad n \geq 1, \\ v_{n+1} = v_n - \sigma_n(Kv_n + u_n) - \sigma_n\xi_n\alpha_n v_n, \quad n \geq 1. \end{cases} \tag{1.7}$$

Remark 1.1. These results of Chidume and Zegeye, and also, Ofoedu and Onyi do not include some important class of nonlinear operators such as the differential operator.

For earlier and more recent works on approximation of solutions of equations of Hammerstein-type, the reader may consult any of the following: [10–13, 15, 18–20, 26, 27, 37], and the references contained in them.

It is well known that the convergence of sequences for approximating zeros of monotone maps are generally *slow*. This is because monotone maps are not differentiable in

general. Hence, algorithms which improve the speed of convergent sequences such as the *Newton-Kantorovich* algorithm may not be used (see, e.g. [29]). Conscious efforts are now being put in constructing algorithms that improve and speed up convergent sequences. Such algorithms are desirable as they certainly reduce computational time, although, the computational cost may be slightly high. One of the methods that is now being studied extensively is to incorporate *inertial extrapolation term* in algorithms. In 1964, Polyak [31] introduced and studied the extrapolation inertial algorithm from the heavy ball experiment of two order time dynamical system, given by:

$$\zeta^{''}(t) + \gamma\zeta^{'}(t) + \nabla\phi(\zeta(t)) = 0, \tag{1.8}$$

where $\gamma > 0$ and $\phi : H \to \mathbb{R}$ is a differentiable functional. The dynamical system (1.8) is discretized using numerical computation such that, given the previous *two* iterates, ζ_n and ζ_{n-1}, the next iterate ζ_{n+1}, can be determined.

An *inertial-type algorithm* is a two-step iterative process in which the next iterate is defined by making use of the previous *two* iterates.The study of inertial algorithms, especially in connection with the PPA, forward and backward algorithm, double projection algorithm, extragradient and subgradient-extragradient algorithms, has become a flourishing area of research for numerous mathematicians. For more on the study of algorithms which speeds up convergence, see, for example,[1–4, 17, 34, 35].

Motivated by the ongoing research in this direction, we study a Mann-type algorithm combined with inertial effect for approximating solutions of Hammerstein integral equations $u + KFu = 0$, where K and F are maximal monotone maps defined on a real Hilbert space H. The sequence of this algorithm is proved to converge strongly to a solution of the Hammerstein equation, without any Lipschitz or boundedness condition imposed on the maps K and F or the computation of K^{-1}. The convergence analysis relies on geometric properties of Hilbert spaces and a key result concerning the *quasi-boundedness* of monotone operators whose domains contain the origin as an interior point. The boundedness condition imposed on K and F is dispensed with in our theorem. Furthermore, some numerical experiments are given to study the effect of inertial term in our algorithm over algorithms without inertial term. Finally, the theorem proved, complements and improves some important related recent results in the literature.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space. First, we recall some definitions and lemmas which will be needed in proving our main theorem. Let $D(\cdot)$ denote the domain of any given map.

Definition 2.1. A map $A : \mathcal{H} \to \mathcal{H}$ with $D(A)$ in $\mathcal{H}$ is called *monotone* if,

$$\langle Au - Av, u - v\rangle \geq 0, \quad \text{for all } u, v \in D(A).$$

Lemma 2.1 ([21]). *Let $\mathcal{X}$ be a real normed space. Then, for all $u, v \in \mathcal{X}$, there exists $j(u+v) \in J(u+v)$ such that:*

$$\|u+v\|^2 \leq \|u\|^2 + 2\langle v, j(u+v)\rangle,$$

where $J : \mathcal{X} \to 2^{\mathcal{X}^*}$ is the normalized duality map.

Lemma 2.2 ([36]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n b_n, \quad n \geq 1,$$

where $\{\sigma_n\} \subset [0,1]$, $\sum_{n=1}^{+\infty} \sigma_n = +\infty$ and $\limsup_{n\to+\infty} b_n \leq 0$. Then, $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.3 ([32]). *Let $\mathcal{H}$ be a Hilbert space. Let $A : \mathcal{H} \to \mathcal{H}$ be a monotone map with $D(A) = \mathcal{H}$ and suppose that A satisfies the range condition: $\mathcal{R}(I+\lambda A) = \mathcal{H}$, for all $\lambda > 0$. Let $J_t x := (I+tA)^{-1}x$, $t > 0$ be the resolvent of A, and assume that $A^{-1}(0)$ is nonempty. Then, for each $x \in \mathcal{H}$, $\lim_{t\to+\infty} J_t x \in A^{-1}(0)$.*

Lemma 2.4 ([16]). *Let $\mathcal{H}$ be a real Hilbert space and $F, K : \mathcal{H} \to \mathcal{H}$ be maps with $D(F) = D(K) = \mathcal{H}$. Let $\mathcal{Q} = \mathcal{H} \times \mathcal{H}$ and $A : \mathcal{Q} \to \mathcal{Q}$ be the map defined by*

$$Aw := (Fu - v, Kv + u), \quad \text{for all } w = (u, v) \in \mathcal{Q}.$$

Assume that F and K are monotone and satisfy the range condition. Then, A is monotone and also satisfies the range condition.

Lemma 2.5 ([33], [30]). *Let $\mathcal{X}^*$ be the dual space of a real normed space $\mathcal{X}$. Every monotone map A is locally bounded at the interior points of its domain.*

Lemma 2.6 ([30]). *Let $\mathcal{X}$ be a real normed space and $A : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a monotone map with $0 \in Int(D(A))$. Then, A is quasi-bounded, i.e., for any $M > 0$, there exists $C > 0$ such that:*
(i) $(y, v) \in G(A)$;
(ii) $\langle v, y\rangle \leq M\|y\|$ *and*
(iii) $\|y\| \leq M$, *imply* $\|v\| \leq C$.

3. Main Results

In Theorem 3.1 below, $\{\xi_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to+\infty}\theta_n = 0$, $\sum_{n=1}^{+\infty} \xi_n\theta_n = +\infty$;
(ii) $\lim_{n\to+\infty}\frac{\xi_n}{\theta_n} = 0$;
(iii) $\lim_{n\to+\infty}\frac{\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)}{\xi_n\theta_n} = 0$;
(iv) $\beta_n \leq \xi_n^3\theta_n\gamma_0$;
(v) $\xi_n \leq \theta_n\gamma_0$, for some $\gamma_0 > 0$.

Prototype of the parameters for the conditions required in Theorem 3.1 below are: $\theta_n = (n+1)^{-\frac{1}{9}}$, $\xi_n = (n+1)^{-\frac{1}{3}}$ and $\beta_n = (n+1)^{-\frac{10}{9}}$, $n \geq 1$.

With these choices, the conditions (i)-(v) given are easily satisfied. We prove the following Theorem.

Theorem 3.1. *Let $\mathcal{H}$ be a real Hilbert space. Let $F, K : \mathcal{H} \to \mathcal{H}$ be maximal monotone maps with $D(K) = D(F) = \mathcal{H}$. Let $\{u_n\}$ and $\{v_n\}$ be sequences in $\mathcal{H}$ defined by*

$$\begin{cases} (u_0, v_0), (u_1, v_1) \in \mathcal{H} \times \mathcal{H}, \\ y_n = u_n + \beta_n(u_n - u_{n-1}), \\ z_n = v_n + \beta_n(v_n - v_{n-1}), \\ u_{n+1} = (1 - \xi_n\theta_n)y_n - \xi_n(Fy_n - z_n), \quad n \geq 1, \\ v_{n+1} = (1 - \xi_n\theta_n)z_n - \xi_n(Kz_n + y_n), \quad n \geq 1. \end{cases} \tag{3.1}$$

Assume that the equation $u + KFu = 0$ has a solution, then, the sequence $\{u_n\}$ converges strongly to u^*, where u^* is a solution of $u + KFu = 0$, with $v^* = Fu^*$.

Proof. The proof is divided into three steps.

Step 1. We prove that the sequences $\{u_n\}$ and $\{v_n\}$ are bounded. Let $\mathcal{Q} := \mathcal{H} \times \mathcal{H}$ with $||w||^2_{\mathcal{Q}} := ||u||^2_{\mathcal{H}} + ||v||^2_{\mathcal{H}}$, where $w := (u, v) \in \mathcal{Q}$. Define a sequence $\{w_n\} \in \mathcal{Q}$ by $w_n := (u_n, v_n)$. Let $u^* \in \mathcal{H}$ be a solution of $u + KFu = 0$, $v^* := Fu^*$, $w^* := (u^*, v^*)$ and $w_1 := (u_1, v_1) \in \mathcal{Q}$. Then, there exists $r > 0$ sufficiently large such that

$$r \geq \max\{6||w^*||^2, ||w_1 - w^*||^2\}.$$

Define: $B := \overline{B(w^*, r)} = \{w \in \mathcal{Q} : ||w - w^*||^2 \leq r\}$ with $0 \in B$. Clearly, $B \subset \text{int}(D(A))$, where A is the map defined in Lemma 2.4. It is sufficient to show that $\{w_n\}$ is bounded. We proceed by induction. For $n = 1$, by construction, we have that $||w_1 - w^*||^2 \leq r$. Assume that $||w_n - w^*||^2 \leq r$, up to some $n \geq 1$, then, $||w_n|| \leq ||w^*|| + \sqrt{r}$. Now, we show that $||w_{n+1} - w^*||^2 \leq r$.

Let $s \in B$ be arbitrary and $(s, As) \in G(A)$. By Lemma 2.5, A is locally bounded at 0, i.e., there exist $\phi_0 > 0$ and $\varphi_0 > 0$ such that $||Ax|| \leq \varphi_0$, for all $x \in B(0, \phi_0) \subset B$. By the monotonicity of A, we have that:

$$\langle As, -s\rangle \leq \langle Ax, x - s\rangle + \langle As, -x\rangle, \quad \text{for all } x \in B(0, \phi_0).$$

Setting $g = -s$, we have that:

$$\langle As, g\rangle \leq \langle Ax, x + g\rangle + \langle As, -x\rangle \leq ||Ax||\,(||x|| + ||g||) + ||As|| \cdot ||x||.$$

Hence,

$$||As|| \leq \frac{\varphi_0\,(\phi_0 + ||w^*|| + \sqrt{r})}{||w^*|| + \sqrt{r} - \phi_0} := M_0, \quad \text{for all } s \in B.$$

Define $M := \max\{M_0, ||w^*|| + \sqrt{r}\}$. Then, $\langle As, s\rangle \leq M||s||$ and $||s|| \leq M$. By Lemma 2.6, there exists $C > 0$ such that $||As|| \leq C$ for all $s \in B$.

Define:

$$M_1 = \sup\left\{\|Fy - z + \theta y\|^2_H, (y, z) \in B, \theta \in (0, 1)\right\} + 1,$$
$$M_2 = \sup\left\{\|Kz + y + \theta z\|^2_H, (y, z) \in B, \theta \in (0, 1)\right\} + 1,$$

for some $K = \max\{M_1, M_2\}$. From the recursion formula and Lemma 2.1, we have that:

$$||y_n - u^*||^2 \leq ||u_n - u^*||^2 + K^*\beta_n \quad \text{and} \quad ||z_n - v^*||^2 \leq ||v_n - v^*||^2 + K^*\beta_n, \tag{3.2}$$

for some $K^* > 0$. Define: $\gamma_0 := \min\left\{1, \frac{r}{6(K^*+K)}\right\}$. Now, we compute as follows:

$$\begin{aligned} ||u_{n+1} - u^*||^2 &= ||y_n - u^* - \xi_n(Fy_n - z_n) - \xi_n\theta_n y_n||^2 \\ &= ||y_n - u^*||^2 - 2\xi_n\langle Fy_n - z_n + \theta_n y_n, y_n - u^*\rangle \\ &\quad + \xi_n^2||Fy_n - z_n + \theta_n y_n||^2 \\ &\leq ||y_n - u^*||^2 - 2\xi_n\langle Fy_n - z_n + \theta_n y_n, y_n - u^*\rangle + \xi_n^2 M_1. \end{aligned} \tag{3.3}$$

Observe that:

$$\begin{aligned} \langle Fy_n - z_n + \theta_n y_n, y_n - u^*\rangle =& \langle Fy_n - Fu^*, y_n - u^*\rangle - \langle z_n - v^*, y_n - u^*\rangle \\ &+ \theta_n||y_n - u^*||^2 + \theta_n\langle u^*, y_n - u^*\rangle. \end{aligned} \tag{3.4}$$

Applying inequalities (3.3) and (3.4) and the fact that F is monotone, we have that:

$$\begin{aligned} ||u_{n+1} - u^*||^2 \leq& (1 - 2\xi_n\theta_n)\,||y_n - u^*||^2 + 2\xi_n\langle z_n - v^*, y_n - u^*\rangle \\ &+ \xi_n\theta_n\left(||u^*||^2 + ||y_n - u^*||^2\right) + \xi_n^2 M_1. \end{aligned} \tag{3.5}$$

Following a similar argument, we also obtain that:

$$\begin{aligned} ||v_{n+1} - v^*||^2 \leq& (1 - 2\xi_n\theta_n)\,||z_n - v^*||^2 - 2\xi_n\langle y_n - u^*, z_n - v^*\rangle \\ &+ \xi_n\theta_n\left(||v^*||^2 + ||z_n - v^*||^2\right) + \xi_n^2 M_2. \end{aligned} \tag{3.6}$$

Thus, from inequalities (3.2), (3.5) and (3.6), we have that:

$$\begin{aligned} ||w_{n+1} - w^*||^2 &\leq (1 - \xi_n\theta_n)\left(||y_n - u^*||^2 + ||z_n - v^*||^2\right) + 2\xi_n\theta_n||w^*||^2 + 2\xi_n^2 K \\ &\leq (1 - \xi_n\theta_n)\,||w_n - w^*||^2 + 2\xi_n\theta_n||w^*||^2 + 2\xi_n^2 K + 2K^*\beta_n \\ &\leq (1 - \xi_n\theta_n)\,r + \xi_n\theta_n\frac{r}{3} + 2\xi_n\theta_n K\gamma_0 + 2K^*\xi_n\theta_n\gamma_0 = r. \end{aligned}$$

Thus, by induction, $\{w_n\} \subseteq B$. Hence, $\{u_n\}$ and $\{v_n\}$ are bounded.

Step 2. We show the existence of $f_n := (b_n, d_n) \in \mathcal{Q}$ such that

$$\begin{aligned} \theta_n b_n + Fb_n - d_n &= 0, \\ \theta_n d_n + Kd_n + b_n &= 0, \end{aligned}$$

and $\lim b_n = b^*$, $\lim d_n = d^*$, with $b^* + KFb^* = 0$ and $d^* = Fb^*$. Applying Lemmas 2.3 and 2.4, and the fact that $\theta_n \to 0$, we have:

$$\lim_{n\to+\infty}\left(I + \frac{1}{\theta_n}A\right)^{-1}(0,0) \in A^{-1}(0,0). \tag{3.7}$$

Setting $(b_n, d_n) := \left(I + \frac{1}{\theta_n}A\right)^{-1}(0,0)$ for each $n \geq 1$, we have that:

$$\begin{cases} \theta_n b_n + Fb_n - d_n = 0, & n \geq 1, \\ \theta_n d_n + Kd_n + b_n = 0, & n \geq 1. \end{cases} \tag{3.8}$$

By inclusion (3.7) and equation (3.8), we have that $f_n \to f^* := (b^*, d^*) \in A^{-1}(0,0)$. Moreover,

$$\begin{aligned} (b^*, d^*) \in A^{-1}(0,0) &\iff A(b^*, d^*) = (0,0) \\ &\iff Fb^* - d^* = 0 \text{ and } Kd^* + b^* = 0 \\ &\iff d^* = Fb^* \text{ and } b^* + Kd^* = 0. \end{aligned}$$

Step 3. We show that $(\{u_n\}, \{v_n\})$ converges to (u^*, v^*), where $u^* + KFu^* = 0$ and $v^* = Fu^*$. **Claim:** $\lim_{n\to+\infty} ||w_{n+1} - f_n|| = 0$, with $f_n := (b_n, d_n)$ and

$$||w_{n+1} - f_n||^2 = ||u_{n+1} - b_n||^2 + ||v_{n+1} - d_n||^2.$$

We compute as follows:

$$\begin{aligned} ||u_{n+1} - b_n||^2 &= ||y_n - b_n - \xi_n(Fy_n - z_n) - \xi_n\theta_n y_n||^2 \\ &\leq ||y_n - b_n||^2 - 2\xi_n\langle Fy_n - z_n + \theta_n y_n, y_n - b_n\rangle + \xi_n^2 M_1. \end{aligned} \tag{3.9}$$

From inequality (3.9) and the fact that F is monotone, we have that:

$$\begin{aligned} \langle Fy_n - z_n + \theta_n y_n, y_n - b_n\rangle =& \langle Fy_n - Fb_n, y_n - b_n\rangle + \langle Fb_n - z_n, y_n - b_n\rangle \\ &+ \theta_n||y_n - b_n||^2 + \theta_n\langle b_n, y_n - b_n\rangle \\ \geq& \theta_n||y_n - b_n||^2 + \langle d_n - z_n, y_n - b_n\rangle. \end{aligned} \tag{3.10}$$

From inequalities (3.9) and (3.10), we have that:

$$||u_{n+1} - b_n||^2 \leq (1 - 2\xi_n\theta_n)\,||y_n - b_n||^2 - 2\xi_n\langle d_n - z_n, y_n - b_n\rangle + \xi_n^2 M_1. \tag{3.11}$$

Following a similar argument, we also obtain that:

$$||v_{n+1} - d_n||^2 \leq (1 - 2\xi_n\theta_n)\,||z_n - d_n||^2 + 2\xi_n\langle b_n - y_n, z_n - d_n\rangle + \xi_n^2 M_2. \tag{3.12}$$

Thus, from inequalities (3.1), (3.2), (3.11) and (3.12), we have that:

$$\begin{aligned} ||w_{n+1} - f_n||^2 &\leq (1 - 2\xi_n\theta_n)\left(||y_n - b_n||^2 + ||z_n - d_n||^2\right) + 2\xi_n^2 K \\ &\leq (1 - \xi_n\theta_n)\left(||u_n - b_n||^2 + ||v_n - d_n||^2\right) + 2\xi_n^2 K + 2K^*\beta_n \\ &= (1 - \xi_n\theta_n)\,||w_n - f_n||^2 + 2\xi_n^2 K + 2K^*\beta_n. \end{aligned} \tag{3.13}$$

From Equation (3.8), we observe that:

$$\begin{aligned} b_{n-1} - b_n + \frac{1}{\theta_n}\left(Fb_{n-1} - d_{n-1} - Fb_n + d_n\right) &= \frac{\theta_n - \theta_{n-1}}{\theta_n} b_{n-1}, \\ d_{n-1} - d_n + \frac{1}{\theta_n}\left(Kd_{n-1} + b_{n-1} - Fb_n - b_n\right) &= \frac{\theta_n - \theta_{n-1}}{\theta_n} d_{n-1}. \end{aligned}$$

Therefore,

$$||f_{n-1} - f_n||_{\mathcal{Q}} \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right) ||f_{n-1}||_{\mathcal{Q}}. \tag{3.14}$$

By Lemma 2.1, we have

$$\begin{aligned} ||w_n - f_n||^2 &= ||(w_n - f_{n-1}) + (f_{n-1} - f_n)||^2 \\ &\leq ||w_n - f_{n-1}||^2 + 2||f_{n-1} - f_n|| \cdot ||w_n - f_n||. \end{aligned} \tag{3.15}$$

Using inequalities (3.13), (3.14), (3.15) and for some $C^* \geq \max\{||w_n - f_n||, ||f_{n-1}||\}$, we have:

$$\begin{aligned} ||w_{n+1} - f_n||^2 &\leq (1 - 2\xi_n\theta_n) ||w_n - f_n||^2 + 2\xi_n^2 K + 2K^*\beta_n \\ &\leq (1 - \xi_n\theta_n) ||w_n - f_{n-1}||^2 + 2C^* \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right) + 2\xi_n^2 K + 2K^*\beta_n \\ &\leq (1 - \xi_n\theta_n) ||w_n - f_{n-1}||^2 + 2\xi_n\theta_n \left[C^* \frac{\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)}{\xi_n\theta_n} + \frac{\xi_n}{\theta_n} K\right] + 2K^*\beta_n. \end{aligned}$$

Hence, this gives $a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n b_n + c_n,\ n \geq 1$, where $\sigma_n := \xi_n\theta_n$, $c_n := 2K^*\beta_n$ and

$$b_n := \left[C^* \frac{\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)}{\xi_n\theta_n} + \frac{\xi_n}{\theta_n} K\right].$$

It follows from Lemma 2.2 that $\lim_{n\to+\infty} ||w_n - f_{n-1}|| = 0$. Hence, $\lim_{n\to+\infty} ||u_n - u^*|| = 0$ and $\lim_{n\to+\infty} ||v_n - v^*|| = 0$, where $u^* = b^*$ and $v^* = d^*$. □

4. Numerical Experiment

In this section, we present numerical examples to study the effect of inertial extrapolation term in our algorithm, Algorithm (3.1) over Algorithm (1.6) without inertial extrapolation term. We construct these examples on $\mathbb{R}^2$ and $\mathcal{L}_2^{\mathbb{R}}([0,1])$. Numerical experiments are carried out on Matlab R2020a version. All programs are run on MacBook Pro with 2.6 GHz Dual-Core Intel Core i5 and 8GB 1600 MHz DDR3.

Example 4.1. In Theorems 1.1 and 3.1, we set $\mathcal{H} = \mathbb{R}^2$. Let $K, F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{aligned} Fu &:= \Big[2\cos u_1 + 2u_1 - u_2 - 2, 3\sin u_2 + 4u_2 + u_1\Big], \\ Kv &:= \Big[2\sin v_1 + 2v_1 - v_2, 3\sin v_2 + 3v_2 + v_1\Big]. \end{aligned}$$

Clearly, K and F are monotone and Lipschitz. Furthermore, $u^* = (0,0)$ is the solution of the equation $u + KFu = 0$. For our initial vectors, we take $(u_0, v_0) = (-1,-1)$ and $(u_1, v_1) = (1,1)$.

Example 4.2. In Theorems 1.1 and 3.1, we set $\mathcal{H} = \mathcal{L}_2^{\mathbb{R}}([0,1])$. Let $K,\ F : \mathcal{H} \to \mathcal{H}$ be maps defined by

$$Fu(t) = \frac{1}{1+t}u(t), \quad Kv(t) = (t+1)v(t), \quad \text{for all } t \in [0,1].$$

Clearly, $K,\ F$ are monotone and Lipschitz. Furthermore, the function $u^*(t) = 0$ is the solution of the equation $u + KFu = 0$. For our initial vectors, we take $u_0(t) = v_0(t) = t^2,\ u_1(t) = (1+t^2)^{-1},\ v_1(t) = e^{-t}$ for all $t \in [0,1]$. Recall that $||w||^2 = ||u||^2 + ||v||^2$. The numerical results and figures for Examples 4.1 and 4.2 are also presented.

Remark 4.1. Table 1 represents the sequence of parameters of the algorithms studied. Clearly, the parameters satisfy the given conditions in Theorems 1.1 and 3.1. Tables 2 and 3 represent computational results and CPU time of Algorithms (1.6) and (3.1). As observed in Figure 2, Algorithms (3.1) and (1.6) converged to the solution of the equation $u + KFu = 0$ after 5 iterations in 0.023870 seconds and after 40 iterations in 0.031928 seconds, respectively. Also, in Figure 2 Algorithm (3.1) converged to the solution of the equation $u + KFu = 0$ after 7 iterations in 4.018273 seconds, whereas, Algorithm (1.6) after 10 iterations is yet to converge. Finally, the Lipschitz and boundedness conditions imposed on the maps K and F, in the Theorems of [13, 28] has been dispensed with in Theorem 3.1.

Remark 4.2. Our theorem improves and complements the result in [28], which itself is an improvement in the result of [13, 14, 16, 24].

5. Tables and Figures

Table 1. Tolerance error and choice parameters.

$\mathbb{TOL}$	Algor. (1.6)	Algor. (3.1)
$\epsilon = 10^{-8}$	$\sigma_n = (n+1)^{-\frac{1}{3}}$	$\beta_n = (n+1)^{-\frac{10}{9}}$
	$\xi_n = (n+1)^{-\frac{1}{9}}$	$\xi_n = (n+1)^{-\frac{1}{3}}$
	$\alpha_n = (n+1)^{-\frac{1}{16}}$	$\theta_n = (n+1)^{-\frac{1}{9}}$

Table 2. Computational results and time for Example 4.1.

No. of iter.	Algor. (1.6)	Algor. (3.1)	Algor. (1.6)	Algor. (3.1)
	$\|\|w_N\|\|_2$	$\|\|w_N\|\|_2$	Compt. time	Compt. time
5	6.6232682×10	6.9060942	0.006937	0.007376
10	1.1509140×10^3	2.1597983	0.022811	0.011864
50	3.7037878	0.0000000	0.031928	0.023870

TABLE 3. Computational results and time for Example 4.2.

No. of iter.	Algor. (1.6)	Algor. (3.1)	Algor. (1.6)	Algor. (3.1)
	$\|\|w_N\|\|_2$	$\|\|w_N\|\|_2$	Compt. time	Compt. time
5	2.6615835	1.2535176×10^{-1}	0.157573	0.102801
10	3.7449517	6.1317460×10^{-3}	5.849995	4.018273

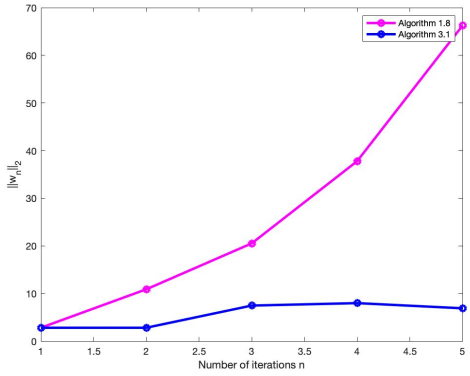


FIGURE 1. Figure of Example 4.1

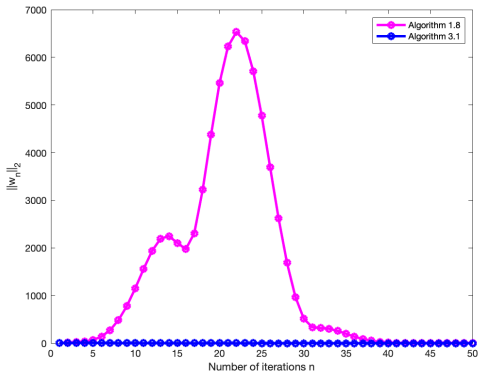


FIGURE 2. Figure of Example 4.1

6. CONCLUSION

In this paper, we studied a new extrapolation inertial algorithm for approximating solutions of Hammerstein equations $u + KFu = 0$, where K and F are maximal monotone maps defined on a real Hilbert space H. The sequence of the algorithm converged strongly to a solution of the Hammerstein equation, without any *Lipschitz or boundedness condition imposed on the maps K and F or the computation of K^{-1}*. Moreover, given the test examples, Algorithm (3.1) with *inertial extrapolation term* performed much better than Algorithm (1.6) without inertial term. Finally, the

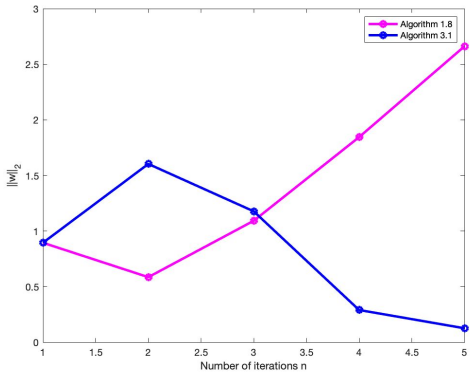


FIGURE 3. Figure of Example 4.2

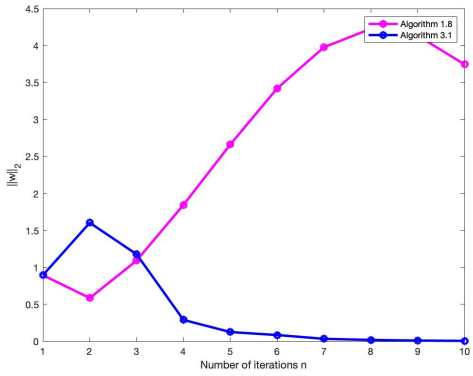


FIGURE 4. Figure of Example 4.2

theorem proved, improves and complements some important related recent results in the literature.

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