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SOLVABILITY OF GENERALIZED FRACTIONAL FUNCTIONAL-INTEGRAL EQUATIONS IN BANACH SPACES BY PETRYSHYN'S FIXED POINT THEOREM

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Abstract. We study the solvability of fractional functional-integral equations of product type in Banach spaces. We use Petryshyn's fixed point theorem associated with measure of non-compactness to obtain our results. Our results extend some existing results in the literature. We illustrate the applicability of our results with examples.

1. Introduction

Functional integral equations are a powerful tool for studying real-world problems. In recent years, there has been a growing interest in fractional integral equations. These equations are more general than traditional integral equations, and they can be used to model a wider range of phenomena. They have been used in a variety of fields, including mathematics, engineering, physics, mechanics, and modeling etc. [15, 33, 39].

Fixed point theorems are a powerful tool for studying fractional integral equations. They provide a solid theoretical foundation to prove the existence and uniqueness of solutions of these equations. This is supported by many books and articles that have been published on the subject, such as [20, 27, 40, 52]. In this paper, we discuss the

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solvability of the following fractional integral functional integral equation (FFIE):

$$\begin{aligned} u(\zeta) =& \Bigg(f(\zeta, u(\zeta), u(\alpha(\zeta)) + F\Big(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s))}{(\phi(\zeta) - s)^{1-h}} ds\Big)\Bigg) \\ & \times \Bigg(g(\zeta, u(\zeta), u(\eta(\zeta)) + G\Big(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))ds\Big)\Bigg), \end{aligned} \tag{1.1}$$

for $\zeta \in I_b = [0, b],\ 0 < h \leq 1$.

We found the following integral equations are particular types of the equation (1.1).

- Darwish [18] studied the existence of solutions for FFIE

$$\begin{aligned} u(\zeta) =& F\Big(\zeta, u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\zeta} \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds\Big) \\ & \times G\Big(t, u(\varphi(t)), u(\zeta) \int_0^1 q(\zeta, s, u(s))ds\Big), \quad \zeta \in [0, 1]. \end{aligned} \tag{1.2}$$

- Banaś and Rzepka [8] studied the existence of solutions for FFIE

$$u(\zeta) = A(\zeta) + \frac{\tilde{f}(\zeta, u(\zeta))}{\Gamma(h)} \int_0^{\zeta} \frac{p(s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, 1]. \tag{1.3}$$

- Banaś and Regan [10] studied the existence of solutions for FFIE

$$u(\zeta) = A(\zeta) + \frac{\tilde{f}(\zeta, u(\zeta))}{\Gamma(h)} \int_0^{\zeta} \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, +\infty). \tag{1.4}$$

- Darwish and Henderson [19] studied the existence of solutions for FFIE

$$u(\zeta) = f(\zeta, u(\zeta)) + \frac{\tilde{f}(\zeta, u(\zeta))}{\Gamma(h)} \int_0^{\zeta} \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, +\infty). \tag{1.5}$$

- Balachandran et al. [2] studied the FFIE

$$u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \frac{\tilde{f}(\zeta, u(\beta(\zeta)))}{\Gamma(h)} \int_0^{\zeta} \frac{p(\zeta, s, u(\mu(s)))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in [0, +\infty). \tag{1.6}$$

- Darwish [17] studied the existence of solutions for FFIE

$$u(\zeta) = A(\zeta) + \frac{u(\zeta)}{\Gamma(h)} \int_0^{\zeta} \frac{p(s, u(s))}{(\zeta - s)^{1-h}} ds, \quad \zeta \in I_b. \tag{1.7}$$

- Deep et al. [21] studied the existence of solution for FIE

$$\begin{aligned} u(\zeta) =& \Bigg(A(\zeta) + f(\zeta, u(\zeta)) + F\Big(\zeta, \int_0^{\zeta} p(\zeta, s, u(\mu(s))ds, u(\beta(\zeta))\Big)\Bigg) \\ & \times \Bigg(B(\zeta) + g(\zeta, u(\zeta)) + G\Big(\zeta, \int_0^{b} q(\zeta, s, u(\nu(s))ds, u(\varphi(\zeta))\Big)\Bigg), \quad \zeta \in I_b. \end{aligned} \tag{1.8}$$

• Deepmala [26] analyzed the following FIE

$$\begin{aligned} u(\zeta) =& \Bigg(f(\zeta, u(\zeta)) + F\Big(\zeta, \int_0^\zeta p(\zeta, s, u(s))ds, u(\beta(\zeta))\Big)\Bigg) \\ & \times \Bigg(G\Big(\zeta, \int_0^b q(\zeta, s, u(s))ds, u(\varphi(\zeta))\Big)\Bigg), \quad \zeta \in I_b. \end{aligned} \tag{1.9}$$

• Banaś [5] as well as Maleknejad et al. [42] analyzed the following FIE

$$u(\zeta) = F\Big(\zeta, \int_0^\zeta p(\zeta, s, u(\mu(s))ds, u(\beta(\zeta))\Big) \times G\Big(\zeta, \int_0^b q(\zeta, s, u(\nu(s))ds, u(\varphi(\zeta))\Big), \tag{1.10}$$

$\zeta \in I_b$.

• Caballero et al. [12] analyzed the following FIE

$$u(\zeta) = F\Big(\zeta, \int_0^\zeta p(\zeta, s, u(s))ds, u(\beta(\zeta))\Big) \times G\Big(\zeta, \int_0^b u(s)q(\zeta, s, u(s))ds, u(\varphi(\zeta))\Big), \tag{1.11}$$

$\zeta \in I_b$.

• Hu and Yan [32] examined the existence of solutions for FIE

$$u(\zeta) = F\left(\zeta, u(\zeta), \int_0^\zeta p(\zeta, s, u(s))ds\right), \quad \zeta \in [0, +\infty). \tag{1.12}$$

• Maleknejad et al. [43, 44] examined the existence result of FIEs

$$u(\zeta) = f(\zeta, u(\zeta)) + F\Big(\zeta, \int_0^\zeta p(\zeta, s, u(s))ds, u(\beta(\zeta))\Big), \tag{1.13}$$

$$u(\zeta) = \tilde{f}(\zeta, u(\beta(\zeta)) \int_0^\zeta p(\zeta, s, u(s))ds, \quad \zeta \in I_b. \tag{1.14}$$

• Banaś [6, 11] analyzed the following FIEs

$$u(\zeta) = \tilde{f}(\zeta, u(\zeta)) \int_0^\zeta p(\zeta, s, u(s))ds, \tag{1.15}$$

$$u(\zeta) = c(\zeta) + \tilde{f}(\zeta, u(\zeta)) \int_0^\zeta p(\zeta, s, u(s))ds, \quad \zeta \in [0, +\infty). \tag{1.16}$$

• Dhage [28] analyzed the following FIE

$$u(\zeta) = c(\zeta) \int_0^b q(\zeta, s, u(s))ds + \int_0^\zeta p(\zeta, s, u(s))ds \times \int_0^b q(\zeta, s, u(s))ds, \quad \zeta \in I_b. \tag{1.17}$$

• Çakan [13], Özdemir et al. [46], Özdemir [47] analyzed the following FIE

$$u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \tilde{f}(\zeta, u(\beta(\zeta))) \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, \tag{1.18}$$

$$u(\zeta) = f(\zeta, u(\alpha(\zeta))) + F\Big(\zeta, \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s))ds, u(\beta(\zeta))\Big), \quad \zeta \in I_b. \tag{1.19}$$

- Srivastava et al. [51], considered following FIE of two variables:

$$\begin{aligned} x(l,h) =& \Big(u(l,h,x(l,h)) + F\Big(l,h, \int_0^l \int_0^h P(l,h,r,u,x(r,u))drdu, x(l,h)\Big)\Big) \\ &\times G\Big(l,h,\int_0^a\int_0^a Q(l,h,r,u,x(r,u))drdu, x(l,h)\Big), \quad \text{for } l,h \in [0,a]. \end{aligned} \tag{1.20}$$

- Aghajani and Jalilian [1] have studied the equation

$$u(\zeta) = F\left(\zeta, u(\beta(\zeta)), \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds\right), \quad \zeta \in [0, +\infty). \tag{1.21}$$

- Cichon and Metwali [16] have studied the equation

$$u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \tilde{f}(\zeta, u(\zeta)) \int_0^1 p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0,1]. \tag{1.22}$$

- Vetro and Vetro [53] have studied the equation

$$u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0, +\infty). \tag{1.23}$$

- Hashem and Rwaily [29] have studied the equation

$$u(\zeta) = f(\zeta, u(\alpha(\zeta))) + \tilde{f}(\zeta, u(\beta(\zeta))) \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0, +\infty). \tag{1.24}$$

- Banaś and Dhage [9] have studied the equation

$$u(\zeta) = f(\zeta, u(\alpha(\zeta)) + \int_0^{\phi(\zeta)} p(\zeta, s, u(\mu(s)))ds, \quad \zeta \in [0, +\infty). \tag{1.25}$$

Further, some well-known first-order equations, such as the Volterra equation, Urysohn equation, Abel equation, and Chandrasekhar type [15], have the form

$$u(\zeta) = F(\zeta, u(\beta(\zeta))),$$

$$u(\zeta) = A(\zeta) + \frac{1}{\Gamma(h)} \int_0^\zeta \frac{p(\zeta, s, u(s))}{(\zeta - s)^{1-h}} ds, \tag{1.26}$$

$$u(\zeta) = A(\zeta) + \int_0^\zeta p(\zeta, s, u(s))ds, \tag{1.27}$$

$$u(\zeta) = A(\zeta) + \int_0^b p(\zeta, s, u(s))ds, \tag{1.28}$$

$$u(\zeta) = 1 + u(\zeta) \int_0^b \frac{\zeta}{\zeta + s} \hat{\phi(s)} u(s)ds. \tag{1.29}$$

It is worth noting that equation (1.1) is more general than equations (1.2)–(1.29). For example, equation (1.2) can be derived from equation (1.1) if

$$\begin{aligned} f(\zeta, u_1, u_2) =& g(\zeta, u_1, u_2) = 0, \\ F(\zeta, u_1, u_2, u_3) =& F(\zeta, u_2, u_3), \quad G(\zeta, u_1, u_2, u_3) = G(\zeta, u_2, u_1 u_3) \end{aligned}$$

and $\phi(\zeta) = \mu(\zeta) = \nu(\zeta) = \zeta$.

Similarly, if $f(\zeta, u_1, u_2) = A(\zeta)$, $g(\zeta, u_1, u_2) = 0$, $G(\zeta, u_1, u_2, u_3) = 1$, $\phi(\zeta) = \mu(\zeta) = \zeta$ and $F(\zeta, u_1, u_2, u_3) = \tilde{f}(\zeta, u_1)u_3$, then equation (1.1) will reduce to equation (1.4). Also, if

$$f(\zeta, u_1, u_2) = g(\zeta, u_1, u_2) = 0, \quad h = 1, \quad \phi(\zeta) = \zeta,$$

and

$$F(\zeta, u_1, u_2, u_3) = F(\zeta, u_3, u_2), \quad G(\zeta, u_1, u_2, u_3) = G(\zeta, u_3, u_2),$$

then equation (1.10) is obtained from equation (1.1).

For

$$f(\zeta, u_1, u_2) = A(\zeta), \quad g(\zeta, u_1, u_2) = 0, \quad \phi(\zeta) = \mu(\zeta) = \zeta,$$

and

$$F(\zeta, u_1, u_2, u_3) = u_3, \quad G(\zeta, u_1, u_2, u_3) = 1,$$

then equation (1.26) is reduced to equation (1.1). Also equation (1.1) reduces to equation (1.27) through $h = 1$, $f(\zeta, u_1, u_2) = A(\zeta)$, $g(\zeta, u_1, u_2) = 0$, $G(\zeta, u_1, u_2, u_3) = 1$, $\phi(\zeta) = \mu(\zeta) = \zeta$ and $F(\zeta, u_1, u_2, u_3) = u_3$ and so on.

To solve different types of functional integral equations, many authors have investigated these equations using the concepts of measure of noncompactness (MNC) with fixed point theorems in Banach function spaces. For more information, we refer to [4, 7, 14, 21–26, 30, 31, 34–38, 41–44, 49, 50] and references therein.

The paper is organized into 4 sections with the introduction. In Section 2, we recall preliminaries and specify the idea of MNC. Section 3 is involved to stating and proving a theorem for equation (1.1) including densifying operators using by Petryshyn's fixed point theorem. In the last section, we show some instances that demonstrate the validity of this class of FFIE.

2. Preliminaries

First, we recall the idea of fractional integral of order h for $u(\zeta)$.

Definition 2.1 ([40]). Let $u \in C[a, b]$ and $a < \zeta < b$. Then,

$$I^h_{a^+} u(\zeta) = \frac{1}{\Gamma(h)} \int_a^\zeta \frac{u(s)}{(\zeta - s)^{1-h}} ds, \quad h > 0,$$

is known as the Riemann-Liouville fractional integral of order h. Γ the gamma function is defined as

$$\Gamma(h) = \int_0^{+\infty} s^{h-1} e^{-s} ds.$$

Let E be a real Banach space and B_{r_θ} denote closed ball with center at θ and radius r_θ. $\partial B_{r_\theta} = \{u \in E : \|u\| = r_\theta\}$ for the sphere in E around θ with radius $r_\theta > 0$. The MNC is a helpful tool to apply fixed point theory in non-linear analysis in any Banach space E.

Definition 2.2 ([3])**.** Let H be a non-empty and bounded subset of E then the Kuratowski MNC is defined as:

$$\hat{\alpha}(H) = \inf\left\{\epsilon > 0 : H = \bigcup_{i=1}^{n} H_i \text{ with } \operatorname{diam} H_i \leq \epsilon,\ i = 1, 2, \ldots, n\right\}.$$

Definition 2.3 ([3])**.** The Hausdroff MNC is defined as:

$$\psi(H) = \inf\{\epsilon > 0 : \text{exists a finite } \epsilon\text{-net for } H \text{ in } E\},$$

where a finite ϵ-net for H in E is a set $\{u_1, u_2, \ldots, u_n\} \subset E$ such that the balls $B_\epsilon(E, u_1), B_\epsilon(E, u_2), \ldots, B_\epsilon(E, u_m)$ cover H.

These MNCs are respectively similar in the sense that

$$\psi(H) \leq \hat{\alpha}(H) \leq 2\psi(H),$$

for any bounded set $H \subset E$.

Theorem 2.1. *Let $H, \hat{H} \subset E$ be bounded sets and $\lambda \in \mathbb{R}$. Then,*

(i) $\psi(H \cup \hat{H}) = \max\{\psi(H), \psi(\hat{H})\}$;
(ii) $H \subseteq \hat{H}$ *implies* $\psi(H) \leq \psi(\hat{H})$;
(iii) $\psi(\bar{co}H) = \psi(H)$;
(iv) $\psi(H) = 0$ *if and only if H is relatively-compact;*
(v) $\psi(\lambda H) = |\lambda|\psi(H)$;
(vi) $\psi(H + \hat{H}) \leq \psi(H) + \psi(\hat{H})$.

Further, Banach space $C[0, b]$ is the set of all real valued continuous functions on $[0, b]$ with the sup norm

$$\|u\| = \max\{|u(\zeta)| : \zeta \in [0, b]\}.$$

Also, space $E = C[0, b]$ is the formation of Banach algebra.

Suppose $H \subset C[0, b]$ be a fix set. For $\epsilon > 0$ and $u \in H$, the modulus of continuity of u is defined by

$$\omega(u, \epsilon) = \sup\{|u(\zeta_2) - u(\zeta_1)| : \zeta_2, \zeta_1 \in [0, b], |\zeta_2 - \zeta_1| \leq \epsilon\}.$$

Further,

$$\omega(H, \epsilon) = \sup\{\omega(u, \epsilon) : u \in H\}, \quad \omega_0(H) = \lim_{\epsilon \to 0} \omega(H, \epsilon).$$

Theorem 2.2 ([34])**.** *The Hausdorff MNC is equivalent to*

$$\psi(H) = \lim_{\epsilon \to 0} \sup_{u \in H}\ \omega(u, \epsilon),$$

for all bounded sets $H \subset C[0, b]$.

Definition 2.4 ([45])**.** Let $T : E \to E$ be a continuous mapping of E. T is said to be a k-set contraction if for all $P \subset E$ with P bounded, $T(P)$ is bounded and $\hat{\alpha}(TP) \leq k\hat{\alpha}(P)$, $k \in (0, 1)$. If $\hat{\alpha}(TP) < \hat{\alpha}(P)$ for all $\hat{\alpha}(P) > 0$, then T is called the densifying or condensing mapping.

Theorem 2.3 ([48]). *Let $T : B_{r_0} \to E$ be a condensing mapping, which fulfills the boundary condition if $T(u) = ku$, for some $u \in \partial B_{r_0}$ then $k \leq 1$. Then, $\mathbf{F}(T)$ (the set of fixed points of T) in B_{r_0} is non-empty.*

Lemma 2.1 ([45]). *Let $\digamma$ be a Banach space. If the operators F and G each fulfil the Petryshyn's condition on a bounded set P of $\digamma$ with constants k_1 and k_2, respectively, then the operator $T = F.G$ fulfil the Petryshyn's condition (condensing map) on P with the constant $\|F(P)\|k_2 + \|G(P)\|k_1$. Particularly, if $\|F(P)\|k_2 + \|G(P)\|k_1 < 1$, then T is a contraction with respect to the measure ψ and it has at least one fixed point in the set P.*

3. Main Results

We analyze the equation (1.1) under the following conditions.

(1) $f, g \in C(I_b \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $F, G \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, C_1] \times \mathbb{R}, \mathbb{R})$, $q \in C(I_b \times [0, C_2] \times \mathbb{R}, \mathbb{R})$, and $\phi, \theta : I_b \to \mathbb{R}^+$, $\mu : [0, C_1] \to I_b$, $\nu : [0, C_2] \to I_b$, $\alpha, \beta, \eta, \varphi : I_b \to I_b$ are continuous, $\phi(\zeta) \leq C_1$, $\theta(\zeta) \leq C_2$ for every $\zeta \in I_b$.

(2) There exist the constants h_i, $i = 1, 2, \ldots, 10$, such that $h_1 + h_2 + h_5 + h_6 < 1$, $h_3 + h_4 + h_8 + h_9 < 1$ and

$$\begin{aligned}
|f(\zeta, u_1, u_2) - f(\zeta, v_1, v_2)| \leq & h_1|u_1 - v_1| + h_2|u_2 - v_2|,\\
|g(\zeta, u_1, u_2) - g(\zeta, v_1, v_2)| \leq & h_3|u_1 - v_1| + h_4|u_2 - v_2|,\\
|F(\zeta, u_1, u_2, u_3) - F(\zeta, v_1, v_2, v_3| \leq & h_5|u_1 - v_1| + h_6|u_2 - v_2| + h_7|u_3 - v_3|,\\
|G(\zeta, u_1, u_2, u_3) - G(\zeta, v_1, v_2, v_3| \leq & h_8|u_1 - v_1| + h_9|u_2 - v_2| + h_{10}|u_3 - v_3|.
\end{aligned}$$

(3) There exists $r_0 > 0$ such that the following bounded conditions are fullfiled: $\sup\{(f' + M_1) \times (g' + M_2)\} \leq r_0$ and $\{(f' + M_1)(h_3 + h_4 + h_8 + h_9) + (g' + M_2)(h_1 + h_2 + h_5 + h_6)\} < 1$. Here,

$$\begin{aligned}
f' =& \sup\{|f(\zeta, u_1, u_2)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0]\},\\
g' =& \sup\{|g(\zeta, u_1, u_2)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0]\},\\
M_1 =& \sup\Bigg\{|F(\zeta, u_1, u_2, u_3)| : \text{ for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \text{ and}\\
& u_3 \in \left[-\frac{H_1 C_1^h}{\Gamma(h+1)}, \frac{H_1 C_1^h}{\Gamma(h+1)}\right]\Bigg\},\\
M_2 =& \sup\Big\{|G(s, u_1, u_2, u_3)| : for\ all\ \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \text{ and}\\
& u_3 \in [-H_2 C_2, H_2 C_2]\Big\},\\
H_1 =& \sup\{|p(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_1] \text{ and } u \in [-r_0, r_0]\},\\
H_2 =& \sup\{|q(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_2] \text{ and } u \in [-r_0, r_0]\}.
\end{aligned}$$

Theorem 3.1. *Under the assumptions* (1)–(3)*, the equation* (1.1) *has at least one solution in* $E = C(I_b)$.

Proof. Define the operators $P, S : B_{r_0} \to E$ in the following form

$$(Pu)(\zeta)=\left(f(\zeta,u(\zeta),u(\alpha(\zeta)))+F\Big(\zeta,u(\zeta),u(\beta(\zeta)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta)}\frac{p(\zeta,s,u(\mu(s)))}{(\phi(\zeta)-s)^{1-h}}ds\Big)\right),$$

$$(Su)(\zeta)=\left(g(\zeta,u(\zeta),u(\eta(\zeta)))+G\Big(\zeta,u(\zeta),u(\varphi(\zeta)),\int_0^{\theta(\zeta)}q(\zeta,s,u(\nu(s)))ds\Big)\right),$$

for $\zeta \in [0,b]$.

Further, we set the operator T such that $Tu = (Pu)(Su)$. Now, we prove that P is continuous on B_{r_0}. To do this, let $\epsilon > 0$ and any $u, z \in B_{r_0}$ such that $\|u - z\| < \epsilon$. Then,

$$\begin{aligned}
&|(Pu)(\zeta)-(Pz)(\zeta)|\\
=&\Big|f(\zeta,u(\zeta),u(\alpha(\zeta)))+F\Big(\zeta,u(\zeta),u(\beta(\zeta)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta)}\frac{p(\zeta,s,u(\mu(s)))}{(\phi(\zeta)-s)^{1-h}}ds\Big)\\
&-f(\zeta,z(\zeta),z(\alpha(\zeta)))-F\Big(\zeta,z(\zeta),z(\beta(\zeta)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta)}\frac{p(\zeta,s,z(\mu(s)))}{(\phi(\zeta)-s)^{1-h}}ds\Big)\Big|\\
\leq& h_1|u(\zeta)-z(\zeta)|+h_2|u(\alpha(\zeta))-z(\alpha(\zeta))|+h_5|u(\zeta)-z(\zeta)|+h_6|u(\alpha(\zeta))-z(\alpha(\zeta))|\\
&+h_7\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta)}\frac{|p(\zeta,s,u(\mu(s)))-p(\zeta,s,z(\mu(s)))|}{(\phi(\zeta)-s)^{1-h}}ds\\
\leq&(h_1+h_2+h_5+h_6)|u(\zeta)-z(\zeta)|+\frac{h_7}{\Gamma(h+1)}C_1^h\omega(p,\epsilon),
\end{aligned}$$

where

$$\omega(p,\epsilon)=\sup\{|p(\zeta,s,u)-p(\zeta,s,z)| : \zeta\in I_b, s\in[0,C_1], u,z\in[-r_0,r_0], |u-z|\leq\epsilon\}.$$

By the uniform continuity of $p = p(\zeta, s, u)$ on $I_b \times [0, C_1] \times [-r_0, r_0]$, we indicate that $\omega(p, \epsilon) \to 0$ as $\epsilon \to 0$. From the above inequality, we prove that P is continuous on B_{r_0}. Further,

$$\begin{aligned}
&|(Su)(\zeta)-(Sz)(\zeta)|\\
=&\Big|g(\zeta,u(\zeta),u(\eta(\zeta)))+G\Big(\zeta,u(\zeta),u(\varphi(\zeta)),\int_0^{\theta(\zeta)}q(\zeta,s,u(\nu(s)))ds\Big)\\
&-g(\zeta,z(\zeta),z(\eta(\zeta)))-G\Big(\zeta,z(\zeta),z(\varphi(\zeta)),\int_0^{\theta(\zeta)}q(\zeta,s,z(\nu(s)))ds\Big)\Big|\\
\leq& h_3|u(\zeta)-z(\zeta)|+h_4|u(\eta(\zeta))-z(\eta(\zeta))|+h_8|u(\zeta)-z(\zeta)|+h_9|u(\varphi(\zeta))-z(\varphi(\zeta))|\\
&+h_{10}\int_0^{\theta(\zeta)}|q(\zeta,s,u(\nu(s)))-q(\zeta,s,z(\nu(s)))|ds\\
\leq&(h_3+h_4+h_8+h_9)\|u(\zeta)-z(\zeta)|+h_{10}C_2\omega(q,\epsilon),
\end{aligned}$$

where

$$\omega(q,\epsilon)=\sup\{|q(\zeta,s,u)-q(\zeta,s,z)|:\zeta,s\in I_b,u,z\in[-r_0,r_0],|u-z|\le\epsilon\}.$$

By the uniform continuity of $q=q(\zeta,s,u)$ on $I_b\times I_b\times[-r_0,r_0]$, we indicate that $\omega(q,\epsilon)\to 0$ as $\epsilon\to 0$. By the inequality above, we can prove that S is continuous on B_{r_0}. Hence, T is a continuous operator on B_{r_0}.

Next, we show that the P fulfill the condensing condition on B_{r_0} with respect to ψ. To do this, take any subset H of B_{r_0}. Select $\epsilon>0$ and $\zeta_1,\zeta_2\in I_b$ such that $|\zeta_1-\zeta_2|\le\epsilon$. We get

$$\begin{aligned}
&|(Pu)(\zeta_2)-(Pu)(\zeta_1)|\\
=&\bigg|f(\zeta_2,u(\zeta_2),u(\alpha(\zeta_2)))+F\bigg(\zeta_2,u(\zeta_2),u(\beta(\zeta_2)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_2)}\frac{p(\zeta_2,s,u(\mu(s)))}{(\phi(\zeta_2)-s)^{1-h}}ds\bigg)\\
&-f(\zeta_1,u(\zeta_1),u(\alpha(\zeta_1)))-F\bigg(\zeta_1,u(\zeta_1),u(\beta(\zeta_1)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\bigg)\bigg|\\
\le&|f(\zeta_2,u(\zeta_2),u(\alpha(\zeta_2))-f(\zeta_2,u(\zeta_1),u(\alpha(\zeta_1))|+|f(\zeta_2,u(\zeta_1),u(\alpha(\zeta_1))\\
&-f(\zeta_1,u(\zeta_1),u(\alpha(\zeta_1)))|+\bigg|F\left(\zeta_2,u(\zeta_2),u(\beta(\zeta_2)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_2)}\frac{p(\zeta_2,s,u(\mu(s)))}{(\phi(\zeta_2)-s)^{1-h}}ds\right)\\
&-F\left(\zeta_2,u(\zeta_2),u(\beta(\zeta_2)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right)\bigg|\\
&+\bigg|F\left(\zeta_2,u(\zeta_2),u(\beta(\zeta_2)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right)\\
&-F\left(\zeta_2,u(\zeta_2),u(\beta(\zeta_1)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right)\bigg|\\
&+\bigg|F\left(\zeta_2,u(\zeta_2),u(\beta(\zeta_1)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right)\\
&-F\left(\zeta_2,u(\zeta_1),u(\beta(\zeta_1)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right)\bigg|\\
&+\bigg|F\left(\zeta_2,u(\zeta_1),u(\beta(\zeta_1)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right)\\
&-F\left(\zeta_1,u(\zeta_1),u(\beta(\zeta_1)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right)\bigg|\\
\le&h_1|u(\zeta_2)-u(\zeta_1)|+h_2|u(\alpha(\zeta_2))-u(\alpha(\zeta_1))|+\omega_f(I_b,\epsilon)\\
&+\frac{h_7}{\Gamma(h)}\bigg|\int_0^{\phi(\zeta_2)}\frac{p(\zeta_2,s,u(\mu(s)))}{(\phi(\zeta_2)-s)^{1-h}}ds-\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\bigg|\\
&+h_5|u(\zeta_2)-u(\zeta_1)|+h_6|u(\beta(\zeta_2))-u(\beta(\zeta_1))|+\omega_F(I_b,\epsilon)\\
\le&h_1\omega(u,\epsilon)+h_2\omega(u,\omega(\alpha,\epsilon))+\omega_f(I_b,\epsilon)+h_5\omega(u,\epsilon)+h_6\omega(u,\omega(\beta,\epsilon))+\omega_F(I_b,\epsilon)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{h_7}{\Gamma(h)}\left[\left|\int_0^{\phi(\zeta_2)}\frac{p(\zeta_2,s,u(\mu(s)))}{(\phi(\zeta_2)-s)^{1-h}}ds-\int_0^{\phi(\zeta_2)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_2)-s)^{1-h}}ds\right|\right.\\
&+\left|\int_0^{\phi(\zeta_2)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_2)-s)^{1-h}}ds-\int_0^{\phi(\zeta_2)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right|\\
&+\left.\left|\int_0^{\phi(\zeta_2)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds-\int_0^{\phi(\zeta_1)}\frac{p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right|\right]\\
\leq& h_1\omega(u,\epsilon)+h_2\omega(u,\omega(\alpha,\epsilon))+\omega_f(I_b,\epsilon)+h_5\omega(u,\epsilon)+h_6\omega(u,\omega(\beta,\epsilon))+\omega_F(I_b,\epsilon)\\
&+\frac{h_7}{\Gamma(h)}\left[\int_0^{\phi(\zeta_2)}\frac{|p(\zeta_2,s,u(\mu(s)))-p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_2)-s)^{1-h}}ds\right.\\
&+\int_0^{\phi(\zeta_2)}|p(\zeta_1,s,u(\mu(s)))|[(\phi(\zeta_2)-s)^{h-1}-(\phi(\zeta_1)-s)^{h-1}]ds\\
&+\left.\int_{\phi(\zeta_1)}^{\phi(\zeta_2)}\frac{|p(\zeta_1,s,u(\mu(s)))}{(\phi(\zeta_1)-s)^{1-h}}ds\right]\\
\leq& h_1\omega(u,\epsilon)+h_2\omega(u,\omega(\alpha,\epsilon))+\omega_f(I_b,\epsilon)+h_5\omega(u,\epsilon)+h_6\omega(u,\omega(\beta,\epsilon))+\omega_F(I_b,\epsilon)\\
&+\frac{h_7}{\Gamma(h+1)}\omega_p(I_b,\epsilon)(\phi(\zeta_2))^h+\frac{h_7}{\Gamma(h+1)}H_1[(\phi(\zeta_2))^h-(\phi(\zeta_1))^h]\\
\leq& h_1\omega(u,\epsilon)+h_2\omega(u,\omega(\alpha,\epsilon))+\omega_f(I_b,\epsilon)+h_5\omega(u,\epsilon)+h_6\omega(u,\omega(\beta,\epsilon))+\omega_F(I_b,\epsilon)\\
&+\frac{h_7}{\Gamma(h+1)}\omega_p(I_b,\epsilon)(\phi(\zeta_2))^h+\frac{h_7}{\Gamma(h+1)}H_1[(\phi(\zeta_2))-(\phi(\zeta_2))]^h,
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_F(I_b,\epsilon)=&\sup\Bigg\{|F(\zeta_2,u_1,u_2,u_3)-F(\zeta_1,u_1,u_2,u_3)|:\zeta_2,\zeta_1\in I_b,\\
&u_3\in\left[-\frac{H_1C_1^h}{\Gamma(h+1)},\frac{H_1C_1^h}{\Gamma(h+1)}\right],u_1,u_2\in[-r_0,r_0],|\zeta_2-\zeta_1|\leq\epsilon\Bigg\},\\
\omega_p(I_b,\epsilon)=&\sup\{|p(\zeta_2,s,u)-p(\zeta_1,s,u)|:|\zeta_2-\zeta_1|\leq\epsilon,\zeta_2,\zeta_1\in I_b,u\in[-r_0,r_0],\\
&s\in[0,C_1]\},\\
\omega_f(I_b,\epsilon)=&\sup\{|f(\zeta_2,u_1,u_2)-f(\zeta_1,u_1,u_2)|:|\zeta_2-\zeta_1|\leq\epsilon,\zeta_2,\zeta_1\in I_b,\\
&u_1,u_2\in[-r_0,r_0]\},\\
\omega(\alpha,\epsilon)=&\sup\{|\alpha(\zeta_2)-\alpha(\zeta_1)|:\zeta_2,\zeta_1\in I_b,|\zeta_2-\zeta_1|\leq\epsilon\},\\
\omega(\beta,\epsilon)=&\sup\{|\beta(\zeta_2)-\beta(\zeta_1)|:\zeta_2,\zeta_1\in I_b,|\zeta_2-\zeta_1|\leq\epsilon\},\\
\omega(\phi,\epsilon)=&\sup\{|\phi(\zeta_2)-\phi(\zeta_1)|:\zeta_2,\zeta_1\in I_b,|\zeta_2-\zeta_1|\leq\epsilon\}.
\end{aligned}
$$

From the above estimate,

$$
\omega(FH,\epsilon)\leq h_1\omega(u,\epsilon)+h_2\omega(H,\omega(\alpha,\epsilon))+\omega_f(I_b,\epsilon)
$$

$$+ h_5\omega(u,\epsilon) + h_6\omega(H,\omega(\beta,\epsilon)) + \omega_F(I_b,\epsilon) + \frac{h_7}{\Gamma(h+1)}\omega_p(I_b,\epsilon)(C_1)^h + \frac{h_7}{\Gamma(h+1)}H_1[\omega(\phi,\epsilon)]^h.$$

Taking limit as $\epsilon \to 0$, we obtain $\omega_0(FH) \le (h_1+h_2+h_5+h_6)\omega_0(H)$. We get

$$\psi(FH) \le (h_1+h_2+h_5+h_6)\psi(H). \tag{3.1}$$

Hence, F is a condensing map.

Similarly,

$$\begin{aligned}
&|(Su)(\zeta_2) - (Su)(\zeta_1)| \\
=&\bigg|g(\zeta_2,u(\zeta_2),u(\eta(\zeta_2))) + G\Big(\zeta_2,u(\zeta_2),u(\varphi(\zeta_2)),\int_0^{\theta(\zeta_2)} q(\zeta_2,s,u(\nu(s)))dsds\Big) \\
&- g(\zeta_1,u(\zeta_1),u(\eta(\zeta_1))) - G\Big(\zeta_1,u(\zeta_1),u(\varphi(\zeta_1)),\int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))dsds\Big)\bigg| \\
\le&|g(\zeta_2,u(\zeta_2),u(\varphi(\zeta_2))) - g(\zeta_2,u(\zeta_1),u(\varphi(\zeta_1)))| + |g(\zeta_2,u(\zeta_1),u(\varphi(\zeta_1))) \\
&- g(\zeta_1,u(\zeta_1),u(\varphi(\zeta_1)))| + \bigg|G\Big(\zeta_2,u(\zeta_2),u(\varphi(\zeta_2)),\int_0^{\theta(\zeta_2)} q(\zeta_2,s,u(\nu(s)))ds\Big) \\
&- G\Big(\zeta_2,u(\zeta_2),u(\varphi(\zeta_2)),\int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))ds\Big)\bigg| + \bigg|G\Big(\zeta_2,u(\zeta_2),u(\varphi(\zeta_2)), \\
&\times \int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))ds\Big) - G\Big(\zeta_2,u(\zeta_2),u(\varphi(\zeta_1)),\int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))ds\Big)\bigg| \\
&+ \bigg|G\Big(\zeta_2,u(\zeta_2),u(\varphi(\zeta_1)),\int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))ds\Big) - G\Big(\zeta_2,u(\zeta_1),u(\varphi(\zeta_1)), \\
&\times \int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))ds\Big)\bigg| + \bigg|G\Big(\zeta_2,u(\zeta_1),u(\varphi(\zeta_1)),\int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))ds\Big) \\
&- G\Big(\zeta_1,u(\zeta_1),u(\varphi(\zeta_1)),\int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s)))ds\Big)\bigg| \\
\le&h_3|u(\zeta_2)-u(\zeta_1)| + h_4|u(\eta(\zeta_2)) - u(\eta(\zeta_1))| + \omega_g(I_b,\epsilon) \\
&+ h_{10}\bigg|\int_0^{\theta(\zeta_2)} q(\zeta_2,s,u(\nu(s))ds - \int_0^{\theta(\zeta_1)} q(\zeta_1,s,u(\nu(s))ds\bigg| \\
&+ h_8|u(\zeta_2)-u(\zeta_1)| + h_9|u(\varphi(\zeta_2)) - u(\varphi(\zeta_1))| + \omega_G(I_b,\epsilon) \\
\le&h_3\omega(u,\epsilon) + h_4(u,\omega(\eta,\epsilon)) + \omega_g(I_b,\epsilon) + h_8\omega(u,\epsilon) + h_9(u,\omega(\varphi,\epsilon)) + \omega_g(I_b,\epsilon) \\
&+ h_{10}\int_0^{\theta(\zeta_1)} |q(\zeta_2,s,u(\nu(s))) - q(\zeta_1,s,u(\nu(s)))|ds + h_{10}\int_{\theta(\zeta_1)}^{\theta(\zeta_2)} |q(\zeta_2,s,u(\nu(s))|ds \\
\le&h_3\omega(u,\epsilon) + h_4(u,\omega(\eta,\epsilon)) + \omega_g(I_b,\epsilon) + h_8\omega(u,\epsilon) + h_9(u,\omega(\varphi,\epsilon)) + \omega_g(I_b,\epsilon) \\
&+ h_{10}C_2\omega_q(I_b,\epsilon) + h_{10}H_2\omega(\theta,\epsilon),
\end{aligned}$$

where

$$\begin{aligned}
\omega_G(I_b,\epsilon) =& \sup\{|G(\zeta_2,u_1,u_2,u_3)-G(\zeta_1,u_1,u_2,u_3)| : \zeta_2,\zeta_1\in I_b,\\
& u_3\in[-C_2H_2,C_2H_2], u_1,u_2\in[-r_0,r_0], |\zeta_2-\zeta_1|\le\epsilon\},\\
\omega_q(I_b,\epsilon) =& \sup\{|q(\zeta_2,s,u)-q(\zeta_1,s,u)| : |\zeta_2-\zeta_1|\le\epsilon, \zeta_2,\zeta_1\in I_b, u\in[-r_0,r_0],\\
& s\in[0,C_2]\},\\
\omega_g(I_b,\epsilon) =& \sup\{|g(\zeta_2,u_1,u_2)-g(\zeta_1,u_1,u_2)| : |\zeta_2-\zeta_1|\le\epsilon, \zeta_2,\zeta_1\in I_b,\\
& u_1,u_2\in[-r_0,r_0]\},\\
\omega(\eta,\epsilon) =& \sup\{|\eta(\zeta_2)-\eta(\zeta_1)| : \zeta_2,\zeta_1\in I_b, |\zeta_2-\zeta_1|\le\epsilon\},\\
\omega(\varphi,\epsilon) =& \sup\{|\varphi(\zeta_2)-\varphi(\zeta_1)| : \zeta_2,\zeta_1\in I_b, |\zeta_2-\zeta_1|\le\epsilon\},\\
\omega(\theta,\epsilon) =& \sup\{|\theta(\zeta_2)-\theta(\zeta_1)| : \zeta_2,\zeta_1\in I_b, |\zeta_2-\zeta_1|\le\epsilon\}.
\end{aligned}$$

From above estimate

$$\begin{aligned}
\omega(GH,\epsilon) \le& h_3\omega(u,\epsilon)+h_4(H,\omega(\eta,\epsilon))+\omega_g(I_b,\epsilon)\\
& + h_8\omega(u,\epsilon)+h_9(H,\omega(\varphi,\epsilon))+\omega_g(I_b,\epsilon)+h_{10}C_2\omega_q(I_b,\epsilon)+h_{10}H_2\omega(\theta,\epsilon).
\end{aligned}$$

Taking limit as $\epsilon\to 0$, we obtain

$$\omega_0(GH)\le(h_3+h_4+h_8+h_9)\omega_0(H).$$

This gives the following estimate

$$\psi(GH)\le(h_3+h_4+h_8+h_9)\psi(H). \tag{3.2}$$

Hence, G is a condensing map. So, from (3.1), (3.2) and Lemma (2.1), we get T is a condensing mapping with constant $(f'+M_1)(h_3+h_4+h_8+h_9)+(g'+M_2)(h_1+h_2+h_5+h_6)<1$.

Now, let $u\in\partial B_{r_0}$ and if $Tu=ku$, then $\|Tu\|=k\|u\|=kr_0$ and by assumption (3),

$$\begin{aligned}
\|Tu(\zeta)\| =& \Bigg\|\Bigg(f(\zeta,u(\zeta),u(\alpha(\zeta)))+F\Big(\zeta,u(\zeta),u(\beta(\zeta)),\frac{1}{\Gamma(h)}\int_0^{\phi(\zeta)}\frac{p(\zeta,s,u(\mu(s)))}{(\phi(\zeta)-s)^{1-h}}ds\Big)\Bigg)\\
& \times\Bigg(g(\zeta,u(\zeta),u(\eta(\zeta)))+G\Big(\zeta,u(\zeta),u(\varphi(\zeta)),\int_0^{\theta(\zeta)}q(\zeta,s,u(\nu(s)))ds\Big)\Bigg)\Bigg\|\\
\le& r_0, \quad \text{for all } \zeta\in I_b.
\end{aligned}$$

Hence, $\|Tu\|\le r_0$, i.e., $k\le 1$. □

The following corollary is a direct consequence of Theorem 3.1 under the following assumptions $(1')-(3')$.

($1'$) $f\in C(I_b\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $F,G\in C(I_b\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $p\in C(I_b\times[0,C_1]\times\mathbb{R},\mathbb{R})$, $q\in C(I_b\times[0,C_2]\times\mathbb{R},\mathbb{R})$, $\phi,\theta: I_b\to\mathbb{R}^+$, $\mu:[0,C_1]\to I_b$, $\nu:[0,C_2]\to I_b$, $\alpha,\beta,\varphi: I_b\to I_b$ are continuous such that $\phi(\zeta)\le C_1, \theta(\zeta)\le C_2$ for every $\zeta\in I_b$.

(2′) There exist constants h_i, $i = 1, 2, \dots, 8$, such that $h_1 + h_2 + h_3 + h_4 < 1$, $h_6 + h_7 < 1$ and

$$\begin{aligned} |f(\zeta, u_1, u_2) - f(\zeta, v_1, v_2)| \leq& h_1|u_1 - v_1| + h_2|u_2 - v_2|, \\ |F(\zeta, u_1, u_2, u_3) - F(\zeta, v_1, v_2, v_3| \leq& h_3|u_1 - v_1| + h_4|u_2 - v_2| + h_5|u_3 - v_3|, \\ |G(\zeta, u_1, u_2, u_3) - G(\zeta, v_1, v_2, v_3| \leq& h_6|u_1 - v_1| + h_7|u_2 - v_2| + h_8|u_3 - v_3|. \end{aligned}$$

(3′) There exists $r_0 > 0$ such that the following bounded condition fullfiled

$$\sup\{(f' + M_1) \times (M_2)\} \leq r_0,$$

and $(f' + M_1)(h_6 + h_7) + (M_2)(h_1 + h_2 + h_3 + h_4) < 1$. Here,

$$\begin{aligned} f' =& \sup\{|f(\zeta, u_1, u_2)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0]\}, \\ M_1 =& \sup\Bigg\{|F(\zeta, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \\ & \text{and } u_3 \in \left[-\frac{H_1 C_1^h}{\Gamma(h+1)}, \frac{H_1 C_1^h}{\Gamma(h+1)}\right]\Bigg\}, \\ M_2 =& \sup\{|G(s, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0], u_3 \in [-H_2C_2, H_2C_2]\}, \\ H_1 =& \sup\{|p(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_1] \textit{ and } u \in [-r_0, r_0]\}, \\ H_2 =& \sup\{|q(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_2] \textit{ and } u \in [-r_0, r_0]\}. \end{aligned}$$

Corollary 3.1. *Under the assumptions* (1′)–(3′)*, the equation*

$$\begin{aligned} u(\zeta) =& \Bigg(f(\zeta, u(\zeta), u(\alpha(\zeta))) + F\Big(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds\Big)\Bigg) \\ & \times \Bigg(G\Big(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))) ds\Big)\Bigg) \end{aligned}$$

has at least one solution in $C(I_b)$.

Proof. We can omit the detailed proof here, as it closely resembles to the proof of Theorem 3.1. □

The next corollary is also derived from Theorem 3.1 under the following assumptions.

(1″) $F, G \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, C_1] \times \mathbb{R}, \mathbb{R})$, $q \in C(I_b \times [0, C_2] \times \mathbb{R}, \mathbb{R})$, $\phi, \theta : I_b \to \mathbb{R}^+$, $\mu : [0, C_1] \to I_b$, $\nu : [0, C_2] \to I_b$, $\beta, \varphi : I_b \to I_b$ are continuous, $\phi(\zeta) \leq C_1$, $\theta(\zeta) \leq C_2$ for every $\zeta \in I_b$.

(2″) There exist constants h_i, $i = 1, 2, \dots, 6$, such that $h_1 + h_2 < 1, h_4 + h_5 < 1$ and

$$|F(\zeta, u_1, u_2, u_3) - F(\zeta, v_1, v_2, v_3| \leq h_1|u_1 - v_1| + h_2|u_2 - v_2| + h_3|u_3 - v_3|,$$

$$|G(\zeta, u_1, u_2, u_3) - G(\zeta, v_1, v_2, v_3| \leq h_4|u_1 - v_1| + h_5|u_2 - v_2| + h_6|u_3 - v_3|.$$

(3′′) There exists $r_0 > 0$ such that the following bounded condition is fullfiled

$$\sup\{(M_1) \times (M_2)\} \leq r_0,$$

and $M_1(h_4 + h_5) + M_2(h_1 + h_2) < 1$, where

$$\begin{aligned}
M_1 =& \sup\Bigg\{ |F(\zeta, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0] \\
&\text{and } u_3 \in \left[-\frac{H_1 C_1^h}{\Gamma(h+1)}, \frac{H_1 C_1^h}{\Gamma(h+1)} \right] \Bigg\}, \\
M_2 =& \sup\{|G(s, u_1, u_2, u_3)| : \text{for all } \zeta \in I_b, u_1, u_2 \in [-r_0, r_0], u_3 \in [-H_2 C_2, H_2 C_2]\}, \\
H_1 =& \sup\{|p(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_1] \textit{ and } u \in [-r_0, r_0]\}, \\
H_2 =& \sup\{|q(\zeta, s, u)| : \text{for all } \zeta \in I_b, s \in [0, C_2] \textit{ and } u \in [-r_0, r_0]\}.
\end{aligned}$$

Corollary 3.2. *Under the above assumptions* (1′′)–(3′′)*, there exists at least one solution in* $C(I_b)$ *for the equation:*

$$\begin{aligned}
u(\zeta) =& \left(F\left(\zeta, u(\zeta), u(\beta(\zeta)), \frac{1}{\Gamma(h)} \int_0^{\phi(\zeta)} \frac{p(\zeta, s, u(\mu(s)))}{(\phi(\zeta) - s)^{1-h}} ds \right) \right) \\
&\times \left(G\left(\zeta, u(\zeta), u(\varphi(\zeta)), \int_0^{\theta(\zeta)} q(\zeta, s, u(\nu(s))ds \right) \right).
\end{aligned}$$

Proof. To avoid redundancy, we can skip the proof here, as it closely resembles the proof of Theorem 3.1. □

4. Examples

In this section, we present some examples of equations to illustrate the usefulness of our study.

Example 4.1. Let us consider following fractional integral equation

(4.1)
$$\begin{aligned}
u(\zeta) =& \Bigg(\frac{1}{4} e^{-\zeta} + \frac{\zeta^2}{3 + 3\zeta^2} \ln(1 + |u(\zeta^3)|) + \frac{1}{(5 + \sin(|u(\sqrt{\zeta})|))\Gamma(\frac{1}{3})} \\
&\times \int_0^{\zeta} \frac{e^{-3\zeta^2}(e^{\zeta} + \zeta \cos(1+s) + \frac{1}{3}(u(\sqrt{s})))}{(\zeta - s)^{\frac{2}{3}}} ds \Bigg) \times \Bigg(\frac{1}{3} \cos(u(1-\zeta)) + \frac{1}{4(e^t + |\cos(u(\zeta))|)} \\
&\times \int_0^{\zeta^3} \left[e^{-2\zeta^2} \left(e^{\zeta} + t\cos(s) + \sin\left(\frac{u(s)}{1 + u(s)} \right) \right) \right] ds \Bigg), \quad \zeta \in [0, 1].
\end{aligned}$$

Here $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F, G : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, \mu, \eta, \varphi, \theta, \phi, v : [0,1] \to [0,1]$, $p, q : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and comparing (4.1) with equation (1.1), we obtain

$$\alpha(\zeta) = \theta(\zeta) = \zeta^3, \quad \beta = \mu = \sqrt{\zeta}, \quad \varphi = \phi = v = \zeta, \quad \eta(\zeta) = 1 - \zeta,$$

$$h=\frac{1}{3},\quad C_1=C_2=1,$$
$$f(\zeta,u_1,u_2)=\frac{1}{4}e^{-\zeta}+\frac{\zeta^2}{3+3\zeta^2}\ln(1+|u_2|),\quad g(\zeta,u_1,u_2)=\frac{1}{3}\cos(u_2),$$
$$F(\zeta,u_1,u_2,z)=\frac{z}{5+\sin(|u_2|)},\quad z=\frac{1}{\Gamma(\frac{1}{3})}\int_0^{\zeta}\frac{p(\zeta,s,u(\mu(s)))}{(\zeta-s)^{\frac{2}{3}}}ds,$$
$$G(\zeta,u_1,u_2,w)=\frac{w}{4(e^t+|\cos(u_2)|)},\quad w=\int_0^{\zeta^3}q(\zeta,s,u(v(s)))ds,$$
$$p(\zeta,s,u(\mu(s)))=e^{-3\zeta^2}(e^{\zeta}+\zeta\cos(1+s)+\frac{1}{3}(u(\sqrt{s}))),\quad |p(\zeta,s,u)|\le e+1+\frac{1}{3}|u|,$$
$$q(\zeta,s,u(v(s)))=e^{-2\zeta^2}\left(e^{\zeta}+t\cos(s)+\sin\left(\frac{u(s)}{1+u(s)}\right)\right),\quad |q(\zeta,s,u)|\le e+2,$$

for all $\zeta\in[0,1]$.

Above functions fulfill the assumptions (1) and (2) of Theorem 3.1. We have

$$\begin{aligned}|u(\zeta)|=&\Bigg|\left(\frac{1}{4}e^{-\zeta}+\frac{\zeta^2}{3+3\zeta^2}\ln(1+|u(\zeta^3)|)+\frac{1}{(5+\sin(|u(\sqrt{\zeta})|))\Gamma(\frac{1}{3})}\right.\\&\left.\times\int_0^{\zeta}\frac{e^{-3\zeta^2}(e^{\zeta}+\zeta\cos(1+s)+\frac{1}{3}(u(\sqrt{s})))}{(\zeta-s)^{\frac{2}{3}}}ds\right)\times\left(\frac{1}{3}\cos(u(1-\zeta))\right.\\&\left.+\frac{1}{4(e^t+|\cos(u(\zeta))|)}\int_0^{\zeta^3}\left[e^{-2\zeta^2}(e^{\zeta}+t\cos(s)+\sin\left(\frac{u(s)}{1+u(s)}\right)\right]ds\right)\Bigg|\\&\le\left(\frac{1}{4}+\frac{1}{3}r_0+\frac{1}{5\Gamma(\frac{4}{3})}\left(\frac{1}{3}r_0+e+1\right)\right)\left(\frac{1}{3}+\frac{1}{4}(e+2)\right),\quad\text{for all }\zeta\in[0,1].\end{aligned}$$

Assume $\|u\|\le r_0$, $r_0>0$, then

$$\left(\frac{1}{4}+\frac{1}{3}r_0+\frac{1}{5\Gamma(\frac{4}{3})}\left(\frac{1}{3}r_0+e+1\right)\right)\left(\frac{1}{3}+\frac{1}{4}(e+2)\right)\le r_0.$$

Thus, assumption (3) of Theorem 3.1 holds if $r_0\ge 4.2796$. It is easy to see that $(f'+M_1)(h_3+h_4+h_8+h_9)+(g'+M_2)(h_1+h_2+h_5+h_6)<1$. Hence, from Theorem 3.1, the equation (4.1) has at least one solution in $C[0,1]$.

Example 4.2. Let us consider the following fractional integral equation

(4.2)

$$\begin{aligned}u(\zeta)=&\left(\frac{1}{2}\zeta e^{-(\zeta+4)}+\frac{\zeta^2(u(\zeta)+2u(\sqrt{\zeta})))}{9(1+\zeta^2)}+\frac{1}{4(1+\sin^2(u(\zeta^2)))\Gamma(\frac{1}{2})}\right.\\&\left.\times\int_0^{\zeta}\frac{\sin(u(1-s))+\frac{1}{3}\zeta\arctan\left(\frac{|u(1-s)|}{1+|u(1-s)|}\right)}{(\zeta-s)^{\frac{1}{2}}}ds\right)\times\left(\frac{\zeta}{4(1+\zeta)}\sin(u(\sqrt{\zeta}))\right.\end{aligned}$$

$$+\frac{\zeta^2}{2+2\zeta^2}\int_0^1\left[\left(\frac{\zeta}{1+\zeta+\ln(1+s)}\right)\sin\left(\frac{u(\sqrt{s})}{1+x(\sqrt{s})}\right)+\frac{u(\sqrt{s})}{2}\right]ds\right),\zeta\in[0,1].$$

Here,

$$\beta(\zeta)=\zeta^2,\quad \alpha(\zeta)=\eta(\zeta)=\sqrt{\zeta},\quad \varphi=0,\quad \phi(\zeta)=\zeta,\quad \theta(\zeta)=1,\quad v(s)=\sqrt{s},$$

$$\mu(s)=1-s,\quad h=\frac{1}{2},\quad C_1=C_2=1,\quad \text{for all } \zeta\in[0,1],$$

$$f(\zeta,u_1,u_2)=\frac{1}{2}\zeta e^{-(\zeta+4)}+\frac{\zeta^2(u_1+2u_2)}{9(1+\zeta^2)},\quad g(\zeta,u_1,u_2)=\frac{\zeta}{4(1+\zeta)}\sin(u_2),$$

$$F(\zeta,u_1,u_2,z)=\frac{z}{4(1+\sin^2(u_2))},\quad z=\frac{1}{\Gamma(\frac{1}{2})}\int_0^\zeta\frac{p(\zeta,s,u(\mu(s)))}{(\zeta-s)^{\frac{1}{2}}}ds,$$

$$G(\zeta,u_1,u_2,w)=\frac{w\zeta^2}{2+2\zeta^2},\quad w=\int_0^1 q(\zeta,s,u(v(s)))ds,$$

$$p(\zeta,s,u(\mu(s)))=\sin(u(1-s))+\frac{1}{3}\zeta\arctan\left(\frac{|u(1-s)|}{1+|u(1-s)|}\right),$$

$$|p(\zeta,s,u)|\le 1+\frac{1}{3}|u|,$$

$$q(\zeta,s,u(v(s)))=\left(\frac{\zeta}{1+\zeta+\ln(1+s)}\right)\sin\left(\frac{u(\sqrt{s})}{1+x(\sqrt{s})}\right)+\frac{u(\sqrt{s})}{2},$$

$$|q(\zeta,s,u)|\le\frac{1}{2}(1+|u|),\quad \text{for all } \zeta\in[0,1].$$

Above functions fulfill the assumptions (1) and (2) of Theorem 3.1. Now, we review that assumption (3) also holds. Assume $\|u\|\le r_0$, $r_0>0$, then

$$\begin{aligned}|u(\zeta)|=&\Bigg|\left(\frac{1}{2}\zeta e^{-(\zeta+4)}+\frac{\zeta^2(u(\zeta)+2u(\sqrt{\zeta}))}{9(1+\zeta^2)}+\frac{1}{4(1+\sin^2(u(\zeta^2)))\Gamma(\frac{1}{2})}\right.\\&\left.\times\int_0^\zeta\frac{\sin(u(1-s))+\frac{1}{3}\zeta\arctan\left(\frac{|u(1-s)|}{1+|u(1-s)|}\right)}{(\zeta-s)^{\frac{1}{2}}}ds\right)\times\left(\frac{\zeta}{4(1+\zeta)}\sin(u(\sqrt{\zeta}))\right.\\&\left.+\frac{\zeta^2}{2+2\zeta^2}\int_0^1\left[\left(\frac{\zeta}{1+\zeta+\ln(1+s)}\right)\sin\left(\frac{u(\sqrt{s})}{1+x(\sqrt{s})}\right)+\frac{u(\sqrt{s})}{2}\right]ds\right)\Bigg|\\&\le r_0,\quad \text{for all } \zeta\in[0,1].\end{aligned}$$

Hence, (3) holds if,

$$\left(\frac{1}{2}+\frac{1}{3}r_0+\frac{1}{4\Gamma(\frac{3}{2})}\left(1+\frac{1}{3}r_0\right)\right)\left(\frac{1}{4}+\frac{1}{4}(1+r_0)\right)\le r_0.$$

Thus, (3) holds if $r_0\in[0.76875, 4.7608]$. It is easy to verify that $(f'+M_1)(h_3+h_4+h_8+h_9)+(g'+M_2)(h_1+h_2+h_5+h_6)<1$. Hence, by Theorem 3.1, there exists at least one solution of (4.2) in $C[0,1]$.

5. Conclusions

We introduced a novel approach based on the MNC and Petryshyn's fixed point theorem for solving FFIEs. The method is shown to be effective in solving a variety of FFIEs, and the results are validated using numerical examples. The results have several implications, including the provision of a new tool for solving FFIEs, the demonstration of the effectiveness of the MNC in solving FFIEs, and the provision of a theoretical foundation for numerical methods for solving FFIEs. Interested researcher could extend it to other types of fractional equations, like those involving Caputo and Riemann-Liouville derivatives, which are commonly used in modeling physical processes. Another possibility is exploring higher-dimensional problems and developing numerical methods that could make our theoretical findings more applicable. Finally, applying our results to fields like control theory and optimization could lead to significant advancements, particularly in understanding the stability and control of systems described by FFIEs.

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