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GROWTH ESTIMATE FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES AND RESTRICTED ZEROS

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Abstract. Let $\mathcal{R}_n$ be the set of all rational functions of the type $r(z) = f(z)/w(z)$, where $f(z)$ is a polynomial of degree at most n and $w(z) = \prod_{j=1}^{n}(z - a_j)$, $|a_j| > 1$ for $1 \leq j \leq n$. In this paper, we extend some famous results concerning to the growth of polynomials by T. J. Rivlin, A. Aziz and others to the rational functions with prescribed poles and thereby obtain the analogous results for such rational functions with restricted zeros.

1. Introduction

Let $\mathcal{P}_n$ be the set of all complex polynomials $f(z) = \sum_{j=1}^{n} a_j z^j$ of degree at most n and let $D_{k-} = \{z : |z| < k\}$, $D_{k+} = \{z : |z| > k\}$ and $T_k = \{z : |z| = k\}$.

For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, we set

$$w(z) = \prod_{j=1}^{n}(z - a_j), \quad B(z) = \prod_{j=1}^{n}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) = \left\{\frac{f(z)}{w(z)} : f \in \mathcal{P}_n\right\}.$$

Then clearly $\mathcal{R}_n$ is the space of all rational functions with at most n poles $a_1, a_2, \ldots, a_n$ with finite limit at infinity. We note that $B(z) \in \mathcal{R}_n$. Throughout this paper, we shall assume that all the poles $a_1, a_2, \ldots, a_n$ lie in D_{1+}.

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For a polynomial $f(z)$ of degree n having no zeros in D_{1-}, T. J. Rivlin [8] proved that, for $\rho < 1$ and $z \in T_1$,

$$|f(\rho z)| \geq \left(\frac{\rho+1}{2}\right)^n |f(z)|. \tag{1.1}$$

The result is best possible and equality holds for $f(z) = \alpha(z-\beta)^n$, $|\beta| = 1$.

A. Aziz [2] generalizes inequality (1.1) and proved that, if $f(z)$ is a polynomial of degree n having no zeros in D_{k-}, then for $z \in T_1$,

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad k \geq 1 \text{ and } \rho < 1, \tag{1.2}$$

and

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad k \leq 1 \text{ and } 0 \leq \rho \leq k^2. \tag{1.3}$$

The result is sharp and equality holds for $f(z) = (z+k)^n$.

Analogous to the above inequality, we have a result when $1 < R \leq k^2,\ k > 1$, which can be found in [7, page 432], which states that if $f(z)$ is a polynomial of degree n having all its zeros in $D_{k+} \cup T_k$, where $k > 1$, then for $z \in T_1$ and $1 < R \leq k^2$

$$|f(Rz)| \leq \left(\frac{R+k}{1+k}\right)^n |f(z)|. \tag{1.4}$$

The result is sharp and equality holds if and only if $f(z) = c(z - ke^{i\gamma})^n$ for some $c \neq 0$ and $\gamma \in \mathbb{R}$.

In literature there exist various results in this direction related to the growth of polynomials for reference see [1, 3–6].

The main aim of this paper is to obtain certain growth estimates for rational functions $r(z) \in \mathcal{R}_n$ having no zero in D_{k-}. In this direction we first present an extension of inequality (1.2) to the rational functions. More precisely, we prove the following.

Theorem 1.1. *Let $r \in \mathcal{R}_n$ with no zero in D_{k-}, where $k \geq 1$, then for $\rho < 1$ and $z \in T_1$,*

$$|r(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|-1}{|a_j|+\rho}\right) |r(z)|. \tag{1.5}$$

Remark 1.1. If we take $k = 1$ in Theorem 1.1, we get the following extension of inequality (1.1) to the rational functions.

Corollary 1.1. *Let $r \in \mathcal{R}_n$ with no zeros in D_{1-}, then for $\rho < 1$ and $z \in T_1$,*

$$|r(\rho z)| \geq \left(\frac{\rho+1}{2}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|-1}{|a_j|+\rho}\right) |r(z)|.$$

Remark 1.2. Taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 1.1, then inequality (1.5) reduces to the following inequality

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \left(\frac{|\alpha|-1}{|\alpha|+\rho}\right)^n \left|\frac{\rho z-\alpha}{z-\alpha}\right|^n |f(z)|. \tag{1.6}$$

Letting $|\alpha| \to \infty$ in inequality (1.6), we get inequality (1.2).

Theorem 1.2. *Let $r \in \mathcal{R}_n$ with no zeros in D_{k-}, where $k \leq 1$, then for $0 \leq \rho \leq k^2$ and $z \in T_1$,*

$$|r(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|-1}{|a_j|+\rho}\right) |r(z)|. \tag{1.7}$$

Remark 1.3. By taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 1.2, inequality (1.7) reduces to the following inequality

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \left(\frac{|\alpha|-1}{|\alpha|+\rho}\right)^n \left|\frac{\rho z-\alpha}{z-\alpha}\right|^n |f(z)|. \tag{1.8}$$

Letting $|\alpha| \to \infty$ in inequality (1.8), we get inequality (1.3).

Theorem 1.3. *Let $r \in \mathcal{R}_n$ with no zeros in D_{k-}, where $k > 1$, then for $1 < R \leq k^2$ and $z \in T_1$,*

$$|r(Rz)| \leq \left(\frac{R+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|+1}{\big||a_j|-R\big|}\right) |r(z)|. \tag{1.9}$$

Remark 1.4. Taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 1.3, inequality (1.9) reduces to the following inequality

$$|f(Rz)| \leq \left(\frac{R+k}{1+k}\right)^n \left(\frac{|\alpha|+1}{\big||\alpha|-R\big|}\right)^n \left|\frac{Rz-\alpha}{z-\alpha}\right|^n |f(z)|. \tag{1.10}$$

Letting $|\alpha| \to \infty$ in inequality (1.10), we obtain inequality (1.4).

2. Proofs of the Theorems

Proof of Theorem 1.1. By hypothesis $r \in \mathcal{R}_n$, therefore we have $r(z) = \frac{f(z)}{w(z)}$, where $w(z) = \prod_{j=1}^{n}(z-a_j)$, $|a_j| > 1$. Since all the zeros of $f(z)$ lie in $D_{k+} \cup T_k$, $k \geq 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $0 \leq \theta < 2\pi$, $1 \leq j \leq n$, are the zeros of $f(z)$, then we write $f(z) = c\prod_{j=1}^{n}(z-\rho_j e^{i\theta_j})$, where $\rho_j \geq k \geq 1$, $j = 1, 2, \ldots, n$. Hence, for $\rho < 1$ and

$0 \le \theta < 2\pi$, we have

$$\begin{aligned}\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| &= \left|\frac{f(\rho e^{i\theta})}{w(\rho e^{i\theta})}\right| \Big/ \left|\frac{f(e^{i\theta})}{w(e^{i\theta})}\right| \\ &= \left|\frac{f(\rho e^{i\theta})}{f(e^{i\theta})}\right| \cdot \left|\frac{w(e^{i\theta})}{w(\rho e^{i\theta})}\right| \\ &= \prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right|. \end{aligned} \tag{2.1}$$

Now,

$$\begin{aligned}\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{\rho e^{i(\theta-\theta_j)} - \rho_j}{e^{i(\theta-\theta_j)} - \rho_j}\right| \\ &= \prod_{j=1}^{n}\left(\frac{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\theta-\theta_j)}{1 + \rho_j^2 - 2\rho_j\cos(\theta-\theta_j)}\right)^{1/2} \\ &\ge \prod_{j=1}^{n}\frac{\rho+\rho_j}{1+\rho_j} \quad (\text{as } \rho < 1) \\ &\ge \prod_{j=1}^{n}\frac{\rho+k}{1+k} \quad (\text{as } \rho_j \ge k) \\ &= \left(\frac{\rho+k}{1+k}\right)^n. \end{aligned} \tag{2.2}$$

Also for $|a_j| > 1$, $j = 1, 2, \dots, n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right| \ge \prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}. \tag{2.3}$$

Using inequalities (2.2) and (2.3) in equation (2.1), we obtain for $0 \le \theta < 2\pi$

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| \ge \left(\frac{\rho+k}{1+k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j| - 1}{|a_j| + \rho}\right).$$

That is, for $z \in T_1$ and $\rho < 1$, we have

$$|r(\rho z)| \ge \left[\left(\frac{\rho+k}{1+k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j| - 1}{|a_j| + \rho}\right)\right]|r(z)|.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2**.** By hypothesis $r \in \mathcal{R}_n$, therefore we have $r(z) = \frac{f(z)}{w(z)}$, where $w(z) = \prod_{j=1}^{n}(z - a_j)$, $|a_j| > 1$. Since all the zeros of $f(z)$ lie in $D_{k+} \cup T_k$, $k \le 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $0 \le \theta < 2\pi$, $1 \le j \le n$, are the zeros of $f(z)$, then we write

$f(z) = c\prod_{j=1}^{n}(z - \rho_j e^{i\theta_j})$, where $\rho_j \geq k$, $k \leq 1$, $j = 1, 2, \ldots, n$. Hence, for $0 \leq \rho \leq k^2$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right|. \tag{2.4}$$

Now,

$$\begin{aligned}
\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{\rho e^{i(\theta-\theta_j)} - \rho_j}{e^{i(\theta-\theta_j)} - \rho_j}\right| \\
&= \prod_{j=1}^{n}\left(\frac{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\theta - \theta_j)}{1 + \rho_j^2 - 2\rho_j\cos(\theta - \theta_j)}\right)^{1/2} \\
&\geq \prod_{j=1}^{n}\frac{\rho + \rho_j}{1 + \rho_j} \quad (\text{as } 0 \leq \rho \leq k^2) \\
&\geq \prod_{j=1}^{n}\frac{\rho + k}{1 + k} \quad (\text{as } \rho_j \geq k) \\
&= \left(\frac{\rho + k}{1 + k}\right)^n. \qquad (2.5)
\end{aligned}$$

Again as before, for $|a_j| > 1$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right| \geq \prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}. \tag{2.6}$$

Using inequalities (2.5) and (2.6) in equation (2.4), we have for $z \in T_1$ and $0 \leq \rho \leq k^2$,

$$|r(\rho z)| \geq \left[\left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j| - 1}{|a_j| + \rho}\right)\right]|r(z)|,$$

which is the desired result. □

Proof of Theorem 1.3. Since all the zeros of $r(z)$ lie in $D_{k+} \cup T_k$, where $k > 1$, therefore it follows that all the zeros of polynomial $f(z)$ lie in $D_{k+} \cup T_k$, $k > 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $1 \leq j \leq n$, are the zeros of $f(z)$, then we write $f(z) = c\prod_{j=1}^{n}(z - \rho_j e^{i\theta_j})$, where $\rho_j \geq k > 1$, $j = 1, 2, \ldots, n$. Hence, for $1 < R \leq k^2$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(Re^{i\theta})}{r(e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{Re^{i\theta} - a_j}\right|. \tag{2.7}$$

Now,

$$\begin{aligned}\prod_{j=1}^{n}\left|\frac{Re^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i(\theta-\theta_j)}-\rho_j}{e^{i(\theta-\theta_j)}-\rho_j}\right|\\ &= \prod_{j=1}^{n}\left(\frac{R^2+\rho_j^2-2R\rho_j\cos(\theta-\theta_j)}{1+\rho_j^2-2\rho_j\cos(\theta-\theta_j)}\right)^{1/2}\\ &\leq \prod_{j=1}^{n}\frac{R+\rho_j}{1+\rho_j} \quad (\text{as } 1<R\leq k^2)\\ &\leq \prod_{j=1}^{n}\frac{R+k}{1+k} \quad (\text{as } \rho_j\geq k)\\ &= \left(\frac{R+k}{1+k}\right)^n. \end{aligned}\tag{2.8}$$

Also for $|a_j|>1$, $j=1,2,\dots,n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta}-a_j}{Re^{i\theta}-a_j}\right| \leq \prod_{j=1}^{n}\frac{1+|a_j|}{|R-|a_j||}.\tag{2.9}$$

Using inequalities (2.8) and (2.9) in equation (2.7), we obtain for $0\leq\theta<2\pi$,

$$\left|\frac{r(Re^{i\theta})}{r(e^{i\theta})}\right| \leq \left(\frac{R+k}{1+k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j|+1}{|R-|a_j||}\right).$$

That is, for $z\in T_1$ and $1<R\leq k^2$, we have

$$|r(Rz)| \leq \left[\left(\frac{R+k}{1+k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j|+1}{|R-|a_j||}\right)\right]|r(z)|.$$

That completes the proof of Theorem 1.3. □

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