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WHEN ARE MULTIPLICATIVE (GENERALIZED)-(σ,τ)-DERIVATIONS ADDITIVE?

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Abstract. Let R be an associative ring. A multiplicative (generalized)-(σ,τ)-derivation F is a map on R satisfying $F(xy)=F(x)\sigma(y)+\tau(x)g(y)$ for all $x,y\in R$, where σ,τ are homomorphisms on R and g is any map on R. In this article, we have obtained some conditions on R, which make both F and g additive.

1. Introduction

The study of the additivity of mappings on rings as well as operator algebras has been an active area of research. Rickart [10] and Johnson [7] raised questions about when a multiplicative isomorphism becomes additive. Both imposed some sort of minimality conditions on ring R and answered it. Martindale [8] answered the above questions under some restriction on R which contains a family of idempotent elements. Daif et al. [1] introduced the definition of multiplicative derivation on R by choosing a mapping $d:R\to R$ such that $d(xy)=d(x)y+xd(y)$ for all $x,y\in R$ and proved that if R contains nontrivial idempotent elements then any multiplicative derivation is additive. Lu and Xie [3] established a condition on R, in the case where R may not contain any non-zero idempotents, that assures that a multiplicative isomorphism is additive, which generalizes Martindale's result. As an application, they showed that under a mild assumption, every multiplicative isomorphism from the radical of a nest algebra onto an arbitrary ring is additive.

Now let us recall the basic definition of Peirce decomposition. Let e in R be an idempotent element so that $e\neq 1, e\neq 0$ (R need not have an identity). We will

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formally put $e_1 = e$ and $e_2 = 1 - e$. Then, for $R_{ij} = e_iRe_j$, where $i, j = 1, 2$, one may write R in its Peirce decomposition as $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$, i.e., $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. An element of the subring R_{ij} will be denoted by a_{ij}. More transparently, e induces on R the generalized matrix ring structure

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix},$$

with the obvious matrix addition and multiplication. Here eRe, $eR(1-e)$, $(1-e)Re$ and $(1-e)R(1-e)$ are abelian subgroups of R.

A map $F : R \to R$ is called a multiplicative left centralizer if $F(xy) = F(x)y$ for all $x, y \in R$. In [12], M. S. Tammam El-Sayiad, M. N. Daif, and V. De Filippis proved especially the result for the additivity of the multiplicative left centralizers in prime and semiprime rings which contain an idempotent element. A map F on R is called a multiplicative generalized derivation of R if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ and some derivation d of R. Similarly, a map F on R is called a multiplicative semi-derivation of R if $F(xy) = F(x)g(y) + xF(y) = F(x)y + g(x)F(y)$ and $F(g(x)) = g(F(x))$ for all $x, y \in R$, where g is any map on R. Daif et al. [2, Theorem 2.1] and Siddeeque et al. [11, Theorem 2.1] proved the additivity of a multiplicative generalized derivation and multiplicative semi-derivation on an arbitrary ring under certain conditions, respectively.

Let $\Re$ be a ring and σ, τ be two endomorphisms on $\Re$. An additive mapping $F : \Re \to \Re$ is called a generalized (σ, τ)- derivation on $\Re$ if there exists a (σ, τ)-derivation $d : \Re \to \Re$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \Re$. A map on a ring $\Re$ defined as $x \to a\sigma(x) + \tau(x)b$, where a, b are fixed elements of $\Re$, called as generalized (σ, τ)-inner derivation, is an example of generalized-(σ, τ) derivation. More details about derivation, multiplicative derivation, and generalized derivation can seen in [4,5], and [9]. Hou et al. [6] proved that if R contains nontrivial idempotent elements, then any multiplicative (σ, τ)-derivation is additive and such map is called (σ, τ)-derivation. We give the notion of multiplicative (generalized)-(σ, τ) derivation as below.

A multiplicative (generalized)-(σ, τ)-derivation is a map satisfying $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ for all $x, y \in \Re$, where g is any map on $\Re$. Similarly a map $F : \Re \to \Re$ is called a reverse multiplicative (generalized)-(σ, τ)-derivation on $\Re$ if $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$ holds for all $x, y \in \Re$. Here σ and τ are endomorphisms and g is any map on $\Re$.

Now, we construct an example to support the definition of multiplicative (generalized) - (σ, τ)- derivation, which is not additive as follows.

Example 1.1. Let $C[0,1]$ be the ring of all complex-valued continuous functions defined on $[0,1]$. It can be easily verified that $\Re = C[0,1] \times C[0,1]$, forms a ring with regard to component wise operations. Define the maps F, g, σ and $\tau : \Re \to \Re$ such that:

$$F\left(h(x),k(x)\right) = \begin{cases} \left(\bar{h}(x)\log|h(x)|, \bar{k}(x)\log|k(x)|\right), & \text{if } h(x)\neq 0 \text{ and } k(x)\neq 0,\\ \left(0, \bar{k}(x)\log|k(x)|\right), & \text{if } h(x)=0 \text{ and } k(x)\neq 0,\\ \left(\bar{h}(x)\log|h(x)|, 0\right), & \text{if } h(x)\neq 0 \text{ and } k(x)=0,\\ (0,0), & \text{if } h(x)=0 \text{ and } k(x)=0,\end{cases}$$

$g(h(x),k(x)) = F(h(x),k(x))$ and $\sigma\left(h(x),k(x)\right) = \tau\left(h(x),k(x)\right) = \left(\bar{h}(x),\bar{k}(x)\right)$, where $\bar{h}$ denotes the conjugate of h. It can be easily proved that σ, τ are automorphisms of $\Re$ and F is a multiplicative (generalized)-(σ,τ)-derivation of $\Re$, i.e., $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ for all $x, y \in \Re$. But F is not additive on $\Re$.

Motivated by the above-cited results, we have proved the additivity of multiplicative (generalized)-(σ,τ)-derivation under some conditions on $\Re$ as follows.

Theorem 1.1. *Let $\Re$ be an associative ring with identity containing an idempotent $e(e\neq 0,1)$ which satisfies the following conditions:*

(*i*) $x\Re = (0) \implies x = 0$,
(*ii*) $\Re x = (0) \implies x = 0$.

If F is any multiplicative (generalized)-(σ,τ)-derivation on $\Re$, i.e., $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ holds for all $x, y \in \Re$, where σ,τ are endomorphisms on $\Re$ and g is any map on $\Re$ which is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.

For proof of Theorem 1.1, first we will prove some auxiliary results as follows.

Lemma 1.1. $F(0) = 0$.

Proof. By the definition of F, we have $F(0) = F(00) = F(0)\sigma(0) + \tau(0)g(0) = F(0)0 + 0g(0) = 0$, which completes the proof. ☐

Lemma 1.2. *For any $a_{11}\in\Re_{11}$, $a_{22}\in\Re_{22}$, $b_{12}\in\Re_{12}$ and $b_{21}\in\Re_{21}$, the following hold*

(*i*) $g(a_{11}+b_{21}) = g(a_{11}) + g(b_{21})$,
(*ii*) $g(a_{22}+b_{12}) = g(a_{22}) + g(b_{12})$.

Proof. We prove only (*i*), and the proof of (*ii*) goes similarly.

(*i*) For any $t_{n1}\in\Re_{n1}$ where $n\in\{1,2\}$, we have

$$\begin{aligned}\tau(t_{n1})(g(a_{11})+g(b_{21})) &= \tau(t_{n1})g(a_{11}) + \tau(t_{n1})g(b_{21})\\ &= F(t_{n1}a_{11}) - F(t_{n1})\sigma(a_{11}) + F(t_{n1}b_{21}) - F(t_{n1})\sigma(b_{21})\\ &= F(t_{n1}a_{11}) - F(t_{n1})\sigma(a_{11}) - F(t_{n1})\sigma(b_{21})\\ &= F(t_{n1}(a_{11}+b_{21})) - F(t_{n1})\sigma(a_{11}+b_{21})\\ &= \tau(t_{n1})g(a_{11}+b_{21}).\end{aligned}$$

This implies that

$$\tau(t_{n1})[(g(a_{11})+g(b_{21})) - g(a_{11}+b_{21})] = 0 \tag{1.1}$$

and

$$\begin{aligned}\tau(t_{n2})(g(a_{11})+g(b_{21})) &= \tau(t_{n2})g(a_{11})+\tau(t_{n2})g(b_{21})\\ &= F(t_{n2}a_{11})-F(t_{n2})\sigma(a_{11})+F(t_{n2}b_{21})-F(t_{n2})\sigma(b_{21})\\ &= F(t_{n2}b_{21})-F(t_{n2})\sigma(a_{11})-F(t_{n2})\sigma(b_{21})\\ &= F(t_{n2}(a_{11}+b_{21}))-F(t_{n2})\sigma(a_{11}+b_{21})\\ &= \tau(t_{n2})(a_{11}+b_{21}).\end{aligned}$$

Thus, we obtain that

$$\tau(t_{n2})[(g(a_{11})+g(b_{21}))-g(a_{11}+b_{21})]=0. \tag{1.2}$$

Using (1.1) and (1.2), we have $\Re[g(a_{11}+b_{21})-g(a_{11})-g(b_{21})]=(0)$. Using the hypothesis of Theorem 1.1, we get $g(a_{11}+b_{21})=g(a_{11})+g(b_{21})$. □

Lemma 1.3. *For any* $a_{11}\in\Re_{11},\ a_{22}\in\Re_{22},\ b_{12}\in\Re_{12},\ b_{21}\in\Re_{21}$ *and* $b_{22}\in\Re_{22}$, *we have the following:*

(*i*) $F(a_{11}+b_{12})=F(a_{11})+F(b_{12})$,
(*ii*) $F(a_{22}+b_{21})=F(a_{22})+F(b_{21})$,
(*iii*) $F(a_{11}+b_{22})=F(a_{11})+F(b_{22})$,
(*iv*) $F(a_{11}+b_{21})=F(a_{11})+F(b_{21})$,
(*v*) $F(a_{22}+b_{12})=F(a_{22})+F(b_{12})$.

Proof. Proofs of (*i*), (*ii*) and (*iii*) are similar to each other. Similarly the proofs of (*iv*) and (*v*) are on the same pattern. Therefore we prove only (*i*) and (*iv*).

(*i*) For any $t_{1n}\in\Re_{1n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{11})+F(b_{12}))\sigma(t_{1n}) &= F(a_{11})\sigma(t_{1n})+F(b_{12})\sigma(t_{1n})\\ &= F(a_{11}t_{1n})-\tau(a_{11})g(t_{1n})+F(b_{12}t_{1n})-\sigma(b_{12})g(t_{1n})\\ &= F((a_{11}+b_{12})t_{1n})-\tau(a_{11}+b_{12})g(t_{1n})\\ &= F(a_{11}+b_{12})\sigma(t_{1n}).\end{aligned}$$

Thus, we obtain that

$$[F(a_{11}+b_{12})-F(a_{11})-F(b_{12})]\sigma(t_{1n})=0. \tag{1.3}$$

For any $t_{2n}\in\Re_{2n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{11})+F(b_{12}))\sigma(t_{2n}) &= F(a_{11})\sigma(t_{2n})+F(b_{12})\sigma(t_{2n})\\ &= F(a_{11}t_{2n})-\tau(a_{11})g(t_{2n})+F(b_{12}t_{2n})-\tau(b_{12})g(t_{2n})\\ &= F((a_{11}+b_{12})t_{2n})-\tau(a_{11})g(t_{2n})-\tau(b_{12})g(t_{2n})\\ &= F((a_{11}+b_{12})t_{2n})-\tau(a_{11}+b_{12})g(t_{2n})\\ &= F(a_{11}+b_{12})\sigma(t_{2n}).\end{aligned}$$

This implies that

$$[F(a_{11}+b_{12})-F(a_{11})-F(b_{12})]\sigma(t_{2n})=0. \tag{1.4}$$

Using (1.3) and (1.4), we arrive at $[F(a_{11}+b_{12})-F(a_{11})-F(b_{12})]\Re=(0)$. Using the hypothesis of Theorem 1.1, we get $F(a_{11}+b_{12})=F(a_{11})+F(b_{12})$.

(*iv*) For any $t_{2n}\in\Re_{2n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{11})+F(b_{21}))\sigma(t_{2n}) &= F(a_{11})\sigma(t_{2n})+F(b_{21})\sigma(t_{2n})\\ &= F(a_{11}t_{2n})-\tau(a_{11})g(t_{2n})+F(b_{21}t_{2n})-\tau(b_{21})g(t_{2n})\\ &= -\tau(a_{11}+b_{21})g(t_{2n})\\ &= -F((a_{11}+b_{21})t_{2n})+F(a_{11}+b_{21})\sigma(t_{2n})\\ &= F(a_{11}+b_{21})\sigma(t_{2n}).\end{aligned}$$

This implies that

$$[F(a_{11}+b_{21})-F(a_{11})-F(b_{21})]\sigma(t_{2n})=0. \tag{1.5}$$

For any $t_{1n}\in\Re_{1n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}&F(a_{11}+b_{21})\sigma(t_{1n})=F((a_{11}+b_{21})t_{1n})-\tau(a_{11}+b_{21})g(t_{1n})\\ &=F((e_2+a_{11})(t_{1n}+b_{21}t_{1n}))-\tau(a_{11}+b_{21})g(t_{1n})\\ &=F(e_2+a_{11})\sigma(t_{1n}+b_{21}t_{1n})+\tau(e_2+a_{11})g(t_{1n}+b_{21}t_{1n})-\tau(a_{11}+b_{21})g(t_{1n}).\end{aligned}$$

Using Lemma 1.3 (*iii*) and Lemma 1.2, we have

$$\begin{aligned}&F(a_{11}+b_{21})\sigma(t_{1n})\\ &=F(e_2)\sigma(t_{1n})+F(e_2)\sigma(b_{21}t_{1n})+F(a_{11})\sigma(t_{1n})\\ &\quad+F(a_{11})\sigma(b_{21}t_{1n})+\tau(e_2)g(t_{1n})+\tau(e_2)g(b_{21}t_{1n})+\tau(a_{11})g(t_{1n})\\ &\quad+\tau(a_{11})g(b_{21}t_{1n})-\tau(a_{11})g(t_{1n})-\tau(b_{21})g(t_{1n})\\ &=F(e_2t_{1n})+F(e_2b_{21}t_{1n})+F(a_{11}t_{1n})+F(a_{11}b_{21}t_{1n})-\tau(a_{11})g(t_{1n})-\tau(b_{21})g(t_{1n})\\ &=F(b_{21}t_{1n})+F(a_{11}t_{1n})-\tau(a_{11})g(t_{1n})-\tau(b_{21})g(t_{1n})\\ &=F(b_{21})\sigma(t_{1n})+F(a_{11})\sigma(t_{1n})\\ &=(F(b_{21})+F(a_{11}))\sigma(t_{1n}).\end{aligned}$$

This shows that

$$[F(a_{11}+b_{21})-F(a_{11})-F(b_{21})]\sigma(t_{1n})=0. \tag{1.6}$$

Using (1.5) and (1.6), we have $[F(a_{11}+b_{21})-F(a_{11})-F(b_{12})]\Re=(0)$. With the help of hypothesis of Theorem 1.1, we obtain $F(a_{11}+b_{21})=F(a_{11})+F(b_{21})$. □

Lemma 1.4. *F is additive on $\Re_{12}$.*

Proof. Let $a_{12},b_{12}\in\Re_{12}$ and $t_{1n}\in\Re_{1n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}(F(a_{12})+F(b_{12}))\sigma(t_{1n}) &=F(a_{12})\sigma(t_{1n})+F(b_{12})\sigma(t_{1n})\\ &=F(a_{12}t_{1n})-\tau(a_{12})g(t_{1n})+F(b_{12}t_{1n})-\tau(b_{12})g(t_{1n})\\ &=-\tau(a_{12}+b_{12})g(t_{1n})\\ &=-F((a_{12}+b_{12})t_{1n})+F(a_{12}+b_{12})\sigma(t_{1n})\end{aligned}$$

$$=F(a_{12}+b_{12})\sigma(t_{1n}).$$

This gives us

$$[F(a_{12}+b_{12})-F(a_{12})-F(b_{12})]\sigma(t_{1n})=0. \tag{1.7}$$

For any $t_{2n}\in\Re_{2n}$, where $n\in\{1,2\}$, we have

$$\begin{aligned}
&F(a_{12}+b_{12})\sigma(t_{2n})\\
=&F((a_{12}+b_{12})t_{2n})-\tau(a_{12}+b_{12})g(t_{2n})\\
=&F((e+a_{12})(t_{2n}+b_{12}t_{2n}))-\tau(a_{12}+b_{12})g(t_{2n})\\
=&F(e+a_{12})\sigma(t_{2n}+b_{12}t_{2n})+\tau(e+a_{12})g(t_{2n}+b_{12}t_{2n})-\tau(a_{12}+b_{12})g(t_{2n}).
\end{aligned}$$

Using Lemma 1.3 (i) and Lemma 1.2, we obtain

$$\begin{aligned}
&F(a_{12}+b_{12})\sigma(t_{2n})\\
=&F(e)\sigma(t_{2n})+F(e)\sigma(b_{12}t_{2n})+F(a_{12})\sigma(t_{2n})+F(a_{12})\sigma(b_{12}t_{2n})+\tau(e)g(t_{2n})\\
&+\tau(e)g(b_{12}t_{2n})+\tau(a_{12})g(t_{2n})+\tau(a_{12})g(b_{12}t_{2n})-\tau(a_{12})g(t_{2n})-\tau(b_{12})g(t_{2n})\\
=&F(et_{2n})+F(eb_{12}t_{2n})+F(a_{12}t_{2n})+F(a_{12}b_{12}t_{2n})-\tau(a_{12})g(t_{2n})-\tau(b_{12})g(t_{2n})\\
=&F(b_{12}t_{2n})+F(a_{12}t_{2n})-\tau(a_{12})g(t_{2n})-\tau(b_{12})g(t_{2n})\\
=&F(b_{12})\sigma(t_{2n})+F(a_{12})\sigma(t_{2n})\\
=&(F(b_{12})+F(a_{12}))\sigma(t_{2n}).
\end{aligned}$$

This implies that

$$[F(a_{12}+b_{12})-F(a_{12})-F(b_{12})]\sigma(t_{2n})=0. \tag{1.8}$$

Using (1.7) and (1.8), we have $[F(a_{12}+b_{12})-F(a_{12})-F(b_{12})]\Re=(0)$. Using the hypothesis of Theorem 1.1, we get F is additive on $\Re_{12}$. □

Lemma 1.5. *F is additive on $\Re_{21}$.*

Proof. Proof is similar to Lemma 1.4. □

Lemma 1.6. *F is additive on $\Re_{11}$.*

Proof. Let $a_{11},b_{11}\in\Re_{11}$. We have,

$$F(a_{11}+b_{11})=F(e(a_{11}+b_{11}))=F(e)\sigma(a_{11}+b_{11})+\tau(e)g(a_{11}+b_{11}).$$

Since g is additive on $\Re_{11}$, we get $F(a_{11}+b_{11})=F(a_{11})+F(b_{11})$. □

Lemma 1.7. *F is additive on $\Re_{11}+\Re_{12}=e\Re$.*

Proof. Let $a_{11}+a_{12},b_{11}+b_{12}\in e\Re$. We have,

$$F((a_{11}+a_{12})+(b_{11}+b_{12}))=F((a_{11}+b_{11})+(a_{12}+b_{12})).$$

Using Lemma 1.3 (i), we have

$$F((a_{11}+a_{12})+(b_{11}+b_{12}))=F(a_{11}+b_{11})+F(a_{12}+b_{12}).$$

Lemma 1.4 and Lemma 1.6, provide us

$$F((a_{11}+a_{12})+(b_{11}+b_{12})) = F(a_{11})+F(a_{12})+F(b_{11})+F(b_{12}).$$

With the help of Lemma 1.3 (i), we get

$$F((a_{11}+a_{12})+(b_{11}+b_{12})) = F(a_{11}+a_{12})+F(b_{11}+b_{12}).$$

That is, F is additive on $e\Re$. □

Lemma 1.8. *F is additive on $\Re_{22}$.*

Proof. Proof is similar as Lemma 1.6. □

Lemma 1.9. *F is additive on $\Re_{21}+\Re_{22} = e_2\Re = (1-e)\Re$.*

Proof. Let $a_{21}+a_{22}, b_{21}+b_{22} \in (1-e)\Re$. We have

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F((a_{21}+b_{21})+(a_{22}+b_{22})).$$

Using Lemma 1.3 (ii), we get

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F(a_{21}+b_{21})+F(a_{22}+b_{22}).$$

Lemma 1.5 and Lemma 1.8 provide us

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F(a_{21})+F(a_{22})+F(b_{21})+F(b_{22}).$$

Lemma 1.3 (ii) provides us

$$F((a_{21}+a_{22})+(b_{21}+b_{22})) = F(a_{21}+a_{22})+F(b_{21}+b_{22}).$$

That is, F is additive on $(1-e)\Re$. □

Lemma 1.10. *F is additive on $\Re_{22}+\Re_{12} = \Re(1-e)$.*

Proof. Let $a_{22}+a_{12}, b_{22}+b_{12} \in \Re(1-e)$. We have

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F((a_{22}+b_{22})+(a_{12}+b_{12})).$$

Using Lemma 1.3 (v), we get

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F(a_{22}+b_{22})+F(a_{12}+b_{12}).$$

Lemma 1.4 and Lemma 1.8, provide us

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F(a_{22})+F(a_{12})+F(b_{22})+F(b_{12}).$$

With the help of Lemma 1.3 (v), we get

$$F((a_{22}+a_{12})+(b_{22}+b_{12})) = F(a_{22}+a_{12})+F(b_{22}+b_{12}).$$

Hence, F is additive on $\Re(1-e)$. □

Lemma 1.11. *F is additive on $\Re_{11}+\Re_{21} = \Re e$.*

Proof. Let $a_{11}+a_{21}, b_{11}+b_{21} \in \Re e$. We have

$$F((a_{11}+a_{21})+(b_{11}+b_{21})) = F((a_{11}+b_{11})+(a_{21}+b_{21})).$$

By Lemma 1.3 (iv), we get

$$F((a_{11}+a_{21})+(b_{11}+b_{21})) = F(a_{11}+b_{11})+F(a_{21}+b_{21}).$$

Lemma 1.5 and Lemma 1.6, provide us

$$F((a_{11}+a_{21})+(b_{11}+b_{21})) = F(a_{11})+F(a_{21})+F(b_{11})+F(b_{21}).$$

Finally, we conclude by Lemma 1.3 (iv)

$$F((a_{11}+a_{21})+(b_{11}+b_{21})) = F(a_{11}+a_{21})+F(b_{11}+b_{21}).$$

That is, F is additive on $\Re e$. □

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. (i) First we prove that g is additive.

Let $t \in e\Re = \Re_{11}+\Re_{12},\ x, y \in \Re$. We have $tx, ty \in e\Re$.

$$\begin{aligned}\tau(t)(g(x)+g(y)) &= \tau(t)g(x)+\tau(t)g(y)\\ &= F(tx)-F(t)\sigma(x)+F(ty)-F(t)\sigma(y)\\ &= F(tx)+F(ty)-F(t)\sigma(x)-F(t)\sigma(y).\end{aligned}$$

Lemma 1.7 provides us

$$\tau(t)(g(x)+g(y)) = F(t(x+y))-F(t)\sigma(x+y) = \tau(t)g(x+y).$$

This implies that

$$\tau(t)[(g(x)+g(y))-g(x+y)] = 0. \tag{1.9}$$

Also, let $m \in (1-e)\Re = \Re_{21}+\Re_{22}$. This shows that $mx, my \in (1-e)\Re$,

$$\begin{aligned}\tau(m)(g(x)+g(y)) &= \tau(m)g(x)+\tau(m)g(y)\\ &= F(mx)-F(m)\sigma(x)+F(my)-F(m)\sigma(y)\\ &= F(mx)+F(my)-F(m)\sigma(x)-F(m)\sigma(y).\end{aligned}$$

Using Lemma 1.9, we get

$$\tau(m)(g(x)+g(y)) = F(m(x+y))-F(m)\sigma(x+y) = \tau(m)(x+y).$$

This implies that

$$\tau(m)[(g(x)+g(y))-g(x+y)] = 0. \tag{1.10}$$

On adding (1.9) and (1.10), we have

$$\tau(t+m)[g(x+y)-g(x)-g(y)] = 0.$$

Since τ is onto on $\Re$, we get

$$\Re[g(x+y)-g(x)-g(y)] = (0).$$

Using hypothesis, we conclude that

$$g(x+y)=g(x)+g(y).$$

(ii) Now, we prove that F is additive.

Let $t\in\Re e=\Re_{11}+\Re_{21},\ m\in\Re(1-e)=\Re_{22}+\Re_{12}$, and $a,b\in\Re$. Then $at,bt\in\Re e$ and $am,bm\in\Re(1-e)$,

$$\begin{aligned}(F(a)+F(b))\sigma(t)&=F(a)\sigma(t)+F(b)\sigma(t)=F(at)-\tau(a)g(t)+F(bt)-\tau(b)g(t)\\&=F(at)+F(bt)-(\tau(a)+\tau(b))g(t).\end{aligned}$$

Using Lemma 1.11, we get

$$(F(a)+F(b))\sigma(t)=F((a+b)t)-\tau(a+b)g(t)=F(a+b)\sigma(t).$$

This implies that

$$[F(a+b)-F(a)-F(b)]\sigma(t)=0. \tag{1.11}$$

Also,

$$\begin{aligned}(F(a)+F(b))\sigma(m)&=F(a)\sigma(m)+F(b)\sigma(m)\\&=F(am)-\tau(a)g(m)+F(bm)-\tau(b)g(m)\\&=F(am)+F(bm)-(\tau(a)+\tau(b))g(m).\end{aligned}$$

Lemma 1.10 provides us

$$(F(a)+F(b))\sigma(m)=F((a+b)m)-\tau(a+b)g(m)=F(a+b)\sigma(m).$$

This implies that

$$[F(a+b)-F(a)-F(b)]\sigma(m)=0. \tag{1.12}$$

On adding (1.11)and (1.12), we have

$$[F(a+b)-F(a)-F(b)]\sigma(t+m)=0.$$

Since σ is onto on $\Re$, we conclude that

$$[F(a+b)-F(a)-F(b)]\Re=(0).$$

Using hypothesis, we get $F(a+b)=F(a)+F(b)$, i.e., F is additive. □

Corollary 1.1. *Let $\Re$ be a semi-prime ring with identity containing an idempotent $e\neq 0,1$. If F is any multiplicative (generalized)-(σ,τ)-derivation on $\Re$, i.e., $F(xy)=F(x)\sigma(y)+\tau(x)g(y)$ holds for all $x,y\in\Re$, where σ,τ are endomorphisms on $\Re$ and g is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.*

Now we construct an example to support the necessity of the condition that "g is additive on both $\Re_{11}$ and $\Re_{22}$" in Theorem 1.1 for additivity of F on $\Re$.

Example 1.2. Let S be an integral domain ring with the unity and

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in S \right\}$$

be the ring of upper triangular matrices over S. Let $C[0,1]$ be the ring of all complex-valued continuous functions defined on $[0,1]$. It can be easily shown that $\Re = C[0,1] \times M$ forms a ring concerning component-wise operations. Define the maps F, g, σ and $\tau : \Re \to \Re$ as follows

$$F\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{cases} \left(\bar{f}(x)\log|\bar{f}(x)|, \begin{bmatrix} 0 & a+c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) \neq 0, \\ \left(0, \begin{bmatrix} 0 & a+c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) = 0, \end{cases}$$

$$\sigma\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \left(\bar{f}(x), \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix}\right), \tau\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \left(\bar{f}(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right),$$

$$g\left(f(x), \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{cases} \left(\bar{f}(x)\log|\bar{f}(x)|, \begin{bmatrix} 0 & a-c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) \neq 0, \\ \left(0, \begin{bmatrix} 0 & a-c \\ 0 & 0 \end{bmatrix}\right), & \text{if } f(x) = 0, \end{cases}$$

where $\bar{f}$ denotes the conjugate of f. One can verify that $\Re$ satisfies both the conditions (i) and (ii) of Theorem 1.1 and also both σ, τ are automorphisms. F is a multiplicative (generalized)-(σ,τ) derivation, *i.e.*, $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ for all $x, y \in \Re$. Obviously a non trivial idempotent element of the ring $\Re$ is $e = \left(\mathbf{1}(x), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$ and then $1 - e = \left(\mathbf{0}(x), \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$, where $\mathbf{1}(x)$ and $\mathbf{0}(x)$ are the constant functions on $[0,1]$ defined as; $\mathbf{1}(x) = 1$ for all $x \in [0,1]$ and $\mathbf{0}(x) = 0$ for each $x \in [0,1]$. Then,

$$\Re_{11} = \left\{ \left(f(x), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \mid f(x) \in C[0,1], a \in S \right\}$$

and

$$\Re_{22} = \left\{ \left(\mathbf{0}(x), \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\right) \mid \mathbf{0}(x) \in C[0,1], a \in S \right\}.$$

Clearly, it can be proved that g is additive on $\Re_{22}$ but not additive on $\Re_{11}$. But F is not additive on $\Re$.

Similarly, if we choose another non-trivial idempotent element as $e_1 = \left(\mathbf{0}(x), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$, then $(1 - e_1) = \left(\mathbf{1}(x), \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$. In this case,

$$\Re_{11} = \left\{\left(\mathbf{0}(x), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \mid a \in S\right\}$$

and

$$\Re_{22} = \left\{\left(f(x), \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\right) \mid f(x) \in C[0,1], a \in S\right\}.$$

Here, one can observe that g is additive on $\Re_{11}$ but not additive on $\Re_{22}$. However, F is not additive on $\Re$.

Theorem 1.2. *Let $\Re$ be an associative ring with identity containing an idempotent $e(e \neq 0, 1)$ which satisfies the conditions (i) and (ii) of Theorem 1.1. If F is any reverse-multiplicative (generalized)-(σ,τ)-derivation on $\Re$, i.e., $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$ holds for all $x, y \in \Re$, where σ, τ are endomorphisms on $\Re$ and g is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.*

Proof. The proof is in the same pattern as done for multiplicative (generalized)-(σ,τ)-derivation in Theorem 1.1. □

Corollary 1.2. *Let $\Re$ be a semi-prime ring with identity containing an idempotent $e \neq 0, 1$. If F is any reverse-multiplicative (generalized)-(σ,τ)-derivation on $\Re$, i.e., $F(xy) = \sigma(x)F(y) + g(x)\tau(y)$ holds for all $x, y \in \Re$, where σ, τ are endomorphisms on $\Re$ and g is additive on $\Re_{11}$ and $\Re_{22}$, then F and g are additive.*

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