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ON SYSTEM OF VECTOR QUASI-EQUILIBRIUM PROBLEMS FOR MULTIVALUED MAPPINGS

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Abstract. In this paper, the notion of the vector quasiconcavity and lower vector continuity for multivalued mappings without using the algebraic structure are introduced. By applying these definitions and maximal element lemma, some existence theorems of the solution of the system of vector quasi-equilibrium problems for a family of multivalued mappings in the setting of topological order spaces are established. The results of this note improve and generalize the corresponding results in the literature, specially references [2, 6, 9–11, 23].

1. Introduction

In 2003, Fu [10] introduced the symmetric vector quasi-equilibrium problem that consists in finding $(\bar{x}, \bar{y}) \in C \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \notin -\operatorname{int} P, \quad \text{for all } x \in S(\bar{x}, \bar{y}),$$

$$g(\bar{x}, y) - g(x, y) \notin -\operatorname{int} P, \quad \text{for all } y \in T(\bar{x}, \bar{y}),$$

where X, Y and Z are real Hausdorff topological vector space, C and D are nonempty subsets of X and Y, respectively, $S : C \times D \to 2^C$ and $T : C \times D \to 2^D$ are set valued mappings, P is a convex cone subset of Z with $\operatorname{int} P \neq \emptyset$, and $f, g : C \times D \to Z$ are two mappings. The symmetric vector quasi-equilibrium problem is a generalization of the (scalar) symmetric quasi-equilibrium problem posed by Noor and Oettli [20] which this problem is a generalization of the equilibrium problem that, at the first, proposed by Blum and Oettli [7]. The equilibrium problem contains as special cases, for instance, optimization problems, problems of Nash equilibria, variational

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inequalities, and complementarity problems (see, for instance, [7]). A comprehensive bibliography on vector equilibrium problems, vector variational inequalities, vector variational-like inequalities and their generalizations can be found in a recent volume [13]. Ansari and Yao [5] and Chiang et al. [21] introduced and studied some vector quasi-equilibrium problems which generalized those quasi-equilibrium problems in [4, 16] and the references therein to the case of vector valued mapping. The system of vector quasi-equilibrium problems was introduced by Ansari et al. [1] with applications in Debreu-type equilibrium problem for vector-valued functions. The system of vector quasi-equilibrium problems (SVQEP, in short) is a unified way to research some nonlinear problems such as vector equilibrium problems (VEP), vector variational inequality [23], and vector complementarity problems [24] and so on. In all the above problems, the authors obtained some existence results in the setting of topological linear structure. As a generalization of the above models, we consider the SVQEP, in the setting of topological sup-semilattice where the linearity of the space is relaxed. Moreover, some existence theorems of a solution of the SVQEP, by applying maximal element lemma and introducing some new definitions, are established. The rest of this section section deals with introducing some definitions and preliminaries results which are needed in the sequel. For a nonempty set X, 2^X denotes the class of all nonempty subsets of X. A partially ordered set $(X, \leq)$ is called a sup-semilattice if any two elements x and y of X have a least upper bound, denoted by $x \vee y = \sup\{x, y\}$ (see, for instance, [18, 23]). If x and $x^{'}$ are elements in a partially ordered set $(X, \leq)$ with $x \leq x^{'}$, then the set

$$[x, x^{'}] = \{y \in X : x \leq y \leq x^{'}\}$$

is called an order interval. It is easy to check that if X is a sup-semilattice and A is a nonempty finite subset of X, then the set $\triangle A = \bigcup_{x\in A} [x, \sup A]$ is well defined and has the properties: $A \subseteq \triangle A$ and $\triangle A \subseteq \triangle B$ if $A \subseteq B$. Also the order interval $[x, y]$ is a subset of $\triangle A$. A sup-semilattice $(X, \leq)$ is called topological semilattice if X is equipped with such a topology where the mapping $\vee : X \times X \rightrightarrows X$ defined by

$$(x, y) \mapsto x \vee y$$

is continuous.

The following example shows that a sup-semilattice is not necessarily a topological semilattice.

Example 1.1. Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. It is easy to check that X together with following ordering

$$(a, b) \leq (c, d) \Leftrightarrow a < c \vee (a = c, b < d)$$

is sup-semilattice. It is not topological semilattice when X is endowed by the Euclidean topology. Because of

$$\left(\left(\frac{1}{n}, -1\right), \left(-\frac{1}{n}, 1\right)\right) \to ((0, -1), (0, 1)),$$

while

$$\sup\left\{\left(\frac{1}{n},-1\right),\left(-\frac{1}{n},1\right)\right\}=\left(\frac{1}{n},-1\right)\nrightarrow \sup\{(0,-1),(0,1)\}=(0,1).$$

The following examples illustrate the above definitions.

- It is easy to check that the real line (that is $X=\mathbb{R}$) with the usual topology and the usual ordering is a topological semilattice.
- Let $X=\mathbb{N}$, the positive integers numbers, together with the ordering $m\leq n$ if and only if there exists $k\in\mathbb{N}$ such that $n=km$ is sup-semilattice and with the discrete topology (i.e., $\tau=P(X)$) is a topological semilattice. Also, if we take $A=\{2,3,4\}$ then

$$\triangle A=\cup_{x\in A}[x,\sup A]=\{2,3,4,6,12\}.$$

- It is obvious that the set of all (real valued) continuously differentiable functions (denoted by $X=C^1([0,1])$) with the topology induced by the norm $\| f \|=\sup_{x\in[0,1]}|f(x)|+\sup_{x\in[0,1]}|f^{'}(x)|$ and the usual ordering is not sup-semilattice (note $\sup\{-x,x\}\notin C^1([0,1])$).

Definition 1.1. A subset E of a sup-semilattice X is called $\triangle$-convex, if for any nonempty finite subset $A\subseteq E$, we have $\triangle A\subseteq E$.

Let $(X,\leq)$ be a sup-semilattice and E be a subset of X. It is easy to verify that E fulfils in Definition 1.1 iff $(E,\leq)$ is sup-semilattice and $[x,y]\subset E$, for each $x,y\in E$ with $x\leq y$. It is worth noting that if E is an ordered topological vector space then the order interval $[x,y]$ is convex.

Some examples

- Let X equal to the real line with the usual ordering then a subset E of X is $\triangle$-convex if and only if it is convex and $\sup A\in E$, for each nonempty finite subset $A\subset E$.
- Let $X=\mathbb{N}$, the positive integers numbers, with the ordering $m\leq n$ if and only if there exists $k\in\mathbb{N}$ such that $n=km$, then a subset E of X is $\triangle$-convex if and only if it is closed under the lowest common multiple and contains all the divisors of its elements.
- Consider $\mathbb{R}^2$ with usual order defined by

$$(a,b)\leq(c,d)\Leftrightarrow a\leq c, b\leq d,\quad (a,b),(c,d)\in\mathbb{R}^2.$$

  Clearly,$(\mathbb{R}^2,\leq)$ is a topological sup-semilattice and the set

$$X=\{(0,y),0\leq y\leq 1\}\cup\{(x,0),0\leq x\leq 1\}$$

  is $\triangle$-convex but not convex (in the sense of linear structure). Also the set

$$X=\{(x,1-x),0\leq x\leq 1\}$$

  is convex but it not $\triangle$-convex.

Definition 1.2. Let S be a nonempty subset of a vector space X. The algebraic interior of S is denoted by by $\operatorname{cor} S$ and is defined as

$$\operatorname{cor} S = \{\bar{x} \in S : \text{ for all } x \in X, \text{ exists } \bar{\lambda} > 0, \ \bar{x} + \lambda x \in S, \text{ for all } \lambda \in [0, \bar{\lambda}]\}.$$

Lemma 1.1. [12,17] *Let C be a convex cone in a topological vector space X. Then*

- $\operatorname{cor}(C) = C + \operatorname{cor}(C)$;
- $\operatorname{int} C = C + \operatorname{int} C$.

Notice that the algebraic interior of a convex set is convex, while the converse may fail (for instance, consider the set of all rational numbers). Also if C is a subset of a topological vector space then $\operatorname{int} C \subseteq \operatorname{cor}(C)$ and the following example shows that it may be $\operatorname{int} C = \emptyset$ and $\operatorname{cor}(C) \neq \emptyset$.

Example 1.2. Let $X = C_{00}$ be the space of all the real sequences which have finite support, that is

$$X = C_{00} = \{x = (x(n)) : \text{ the set } \{n \in N : x(n) \neq 0\} \text{ is finite}\}$$

and $\| x \| = \max_{n \in \mathbb{N}} x(n)$, for all $x = (x(n)) \in C_{00}$. It easy to check that $(C_{00}, \| \, . \, \|)$ is a normed space. Let

$$C = \left\{ x = (x(n)) \in C_{00} : x(n) \leq \frac{1}{n}, \text{ for all } n) \right\}.$$

One can verify that $\operatorname{int} C = \emptyset$ and $(\alpha, 0, 0, \dots) \in \operatorname{cor}(C)$, where $0 < \alpha < 1$.

Definition 1.3. Let X be a sup-semilattice or a $\triangle$-convex set, Y a vector space and $C \subset Y$ be a subset of Y with $\operatorname{cor} C \neq \emptyset$. A multivalued mapping $F : X \to 2^Y \setminus \{\emptyset\}$ is said to be a $C_\triangle$-quasiconcave mapping if, for any pair $x_1, x_2 \in X$ and for any $x \in \triangle\{x_1, x_2\}$, we have either

$$F(x) \subset F(x_1) - C$$

or

$$F(x) \subset F(x_2) - C.$$

Recall that if K and C are convex subsets of the vector spaces X and Y, respectively, then $f : K \to Y$ is called C-quasiconcave if and only if for any pair $x_1, x_2 \in X$ and for any $x \in [x_1, x_2]$, we have either

$$f(x_1) - f(x) \in C \text{ or } f(x_2) - f(x) \in C.$$

Hence the C-quasiconcave is a special case of Definition 1.3 by taking $F(x) = \{f(x)\}$. Moreover, it is not difficult to verify that Definition 2.2 of [23] implies Definition 1.3 when the multivalued mapping F reduces to a single valued mapping.

Definition 1.4. Let X and Y be nonempty sets and $F : X \to 2^Y$ be a multivalued mapping. The domain of F is defined to be the set $\operatorname{dom} F = \{x \in X : F(x) \neq \emptyset\}$.

The following definition extends the definition of lower C-continuity given in [14] from single valued mappings to multivalued mappings.

Definition 1.5. Let X be a topological space, Y a topological vector space and C a subset of Y. A multi-valued mapping $F : X \to 2^Y$ is said to be lower C-continuous at $\bar{x} \in \operatorname{dom} F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that

$$F(\bar{x}) \subset F(x) + V - C, \quad \text{for all } x \in \operatorname{dom} F \cap U.$$

The following proposition is the multivalued version of Lemma 2.1 of [23] by relaxing the locally convexity of the space and replacing the topological interior of the cone C by the algebraic interior of C.

Proposition 1.1. *Let X be a sup-semilattice, Y a vector space and C a subset of Y. If the multivalued mapping $\phi : X \to 2^Y$ is $C_\triangle$-quasiconcave then the set $A = \{x \in X : \phi(x) \subset -\operatorname{cor} C\}$ is $\triangle$-convex.*

Proof. Let $x_1, x_2 \in A$ and $x^{'} \in \triangle\{x_1, x_2\}$. Then it follows from the $C_\triangle$-quasiconcavity of ϕ and the relation $-\operatorname{cor} C - C \subseteq -\operatorname{cor} C$ that

$$\phi(x^{'}) \subseteq \phi(x_1) - C \subseteq -\operatorname{cor} C - C \subseteq -\operatorname{cor} C$$

or

$$\phi(x^{'}) \subseteq \phi(x_2) - C \subseteq -\operatorname{cor} C - C \subseteq -\operatorname{cor} C.$$

Hence $x^{'} \in A$ and so the proof is completed. □

Now we are ready to introduce the main problem of the paper which is known as the system of vector quasi-equilibrium problem (SVQEP, for short). Let I be a nonempty set. For each $i \in I$, K_i is a topological sup-semilattice and Y_i is a topological vector space. Denote $K = \prod_{i \in I} K_i$, $K_{-i} = \prod_{j \in I \setminus \{i\}} K_j$, $C_i \subset Y_i$ is a closed, convex and pointed cone with $\operatorname{cor} C_i \neq \emptyset$. For each $i \in I$, $\phi_i : K_i \times K_{-i} \times K_i \rightrightarrows 2^{Y_i}$ and $G_i : K_{-i} \rightrightarrows 2^{K_i}$ are multivalued mappings.The system of vector quasi-equilibrium problem correspond to $(K_i, Y_i, C_i, \phi_i, G_i)_{i \in I}$ is to find $\bar{x} = (\bar{x}_i)_{i \in I} \in K$, such that for all $i \in I$

$$\overline{x}_i \in G_i(\overline{x}_{-i}) \text{ and } \phi_i(\overline{x}, y_i) \nsubseteq -\operatorname{cor} C_i, \quad \text{for all } y_i \in G_i(\overline{x}_{-i}).$$

If I is singleton then we can replace K_i, Y_i, C_i by K, Y and C, respectively. In this case the SVQEP reduces to the vector quasi-equilibrium problem studied (for instance) in [5, 9, 20, 22]. Moreover, the (SVQEP) will collapse to the problems given in [1, 4, 14, 23] when the algebraic interior of C_i is replaced by the interior of C_i. It is easy to present some examples in order to show that the converse of Proposition 1.1 may drop.

2. Main results

In this section we provide an existence theorem for a solution of the SVQEP and then we present an existence result for a solution of the VEP for a multivalued map which its domain is not necessarily convex and ordering induced by the convex cone C does not need to have a nonempty topological interior. The results of this section can be viewed as an extension of the corresponding results given in the literature, especially [23]. We need the following results which are needed in the sequel.

Theorem 2.1. [15] *Let X be a topological semilattice with path-connected (order) intervals, $X_0 \subseteq X$ be a nonempty subset of X and $R \subseteq X_0 \times X$ be a binary relation such that*

(i) *for each $x \in X_0$, the set $R(x) = \{y \in X : (x, y) \in R\}$ is nonempty and closed in $R(X_0) = \bigcup_{z \in X_0} R(z)$;*
(ii) *there exists $x_0 \in X_0$ such that the set $R(x_0)$ is compact;*
(iii) *for any nonempty finite subset $A \subseteq X_0$*

$$\bigcup_{x \in A} [x, \sup A] \subseteq \bigcup_{x \in X} R(x).$$

Then the set $\bigcap_{x \in X_0} R(x)$ is nonempty.

The following example shows that condition (ii) of Theorem 2.1 is essential.

Example 2.1. Let $X = X_0 = \mathbb{R}$. Define the binary relation R as follows

$$R = \{(0, x) : x \in \mathbb{R}\} \cup \{(x, y) : 0 \neq x \leq y\}.$$

It is clear that, for each nonzero element $x \in \mathbb{R}$, $R(x) = [x, +\infty)$ and $R(0) = \mathbb{R}$ which are closed but not compact. It is easy to check that the example satisfies all the conditions of Theorem 2.1 except (ii) and

$$\bigcap_{x \in X_0} R(x) = \bigcap_{x \in \mathbb{R}} [x, +\infty) = \emptyset.$$

Theorem 2.2. [18, 23] *Let X be a compact topological sup-semilattice with path connected order interval, $S : X \to 2^X$ a multivalued map on X. Assume that the following conditions are satisfied:*

(a) *for each $x \in X$, $S(x)$ is $\triangle$-convex;*
(b) *for each $y \in X$, $S^{-1}(y) = \{x \in X : y \in S(x)\} \subset X$ is open in X;*
(c) *for each $x \in X$, $x \notin S(x)$.*

Then there exists an $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$.

Notice that, one can deduce Theorem 2.2 from Theorem 2.1. Indeed, define the binary relation R as follows

$$(x, y) \in R \Leftrightarrow x \notin S(y).$$

It is obvious, for each $x \in X$, that $R(x) = X \backslash S^{-1}(x)$ which is closed by condition (b). It follows from condition (c) that $x \in R(x)$. Then, for each $x \in X$, the set $R(x)$ is nonempty and closed. Also if $A = \{x_1, x_2, \ldots, x_n\}$ is a finite subset of X, then

$$\bigcup_{i=1}^{n} [x_i, \sup A] \subseteq \bigcup_{i=1}^{n} R(x_i).$$

Because, otherwise there exists $z \in X$ such that

$$z \in \bigcup_{i=1}^{n} [x_i, \sup A] \setminus \bigcup_{i=1}^{n} R(x_i). \tag{2.1}$$

Hence, $z \notin R(x)$, for each $x \in A$. This means $x \in S(z)$, for each $x \in A$. Hence, it follows from condition (1) that, for each $i \in \{1, 2, \ldots, n\}$, $[x_i, \sup A] \subseteq S(z)$ and so by the relation 2.1 we have $z \in S(z)$ which is a contradiction by the condition (3). Consequently, the relation R satisfies all the assumptions of Theorem 2.1 and so there there exists $w \in X$ such that $w \in \bigcap\limits_{x \in X} R(x) = \bigcap\limits_{x \in X} S(x)$ and so specially $w \in S(w)$ which is contradicted by condition (3). This completes the proof.

Theorem 2.3. [15] *Let K be a nonempty compact $\triangle$-convex subset of a topological sup-semilattice with path-connected intervals, $F : K \to 2^K$ a multivalued mapping with nonempty $\triangle$-convex values, and let $F^{-1}(y) \subset K$ be open, for any $y \in K$. Then F has a fixed point.*

Note that, Theorem 2.3 can be deduce from Theorem 2.2 as an application. Because if we assume, on the contrary, that $y \notin F(y)$, for all $y \in K$. Then the mapping $S : K \to 2^K$ defined by $S(y) = F(y)$, for all $y \in K$ satisfies all the conditions of Theorem 2.2. Hence there exists $z \in K$ such that $F(z) = \emptyset$ which is contradicted by $F(y) \neq \emptyset$ for all $y \in K$. This completes the proof of Theorem 2.3. Now we are ready to provide an existence result for a solution of SVQEP.

Theorem 2.4. *Let $(K_i, Y_i, C_i, \phi_i, G_i)_{i \in I}$ be a system of vector quasi-equilibrium problems. For each $i \in I$ let K_i be a compact topological semilattice with path connected order intervals, Y_i be a Hausdorff locally convex topological vector space, and G_i be a multivalued mapping with non-empty and $\triangle$-convex values. Assume that for any $y_i \in K_i$ the set $G_i^{-1}(y_i) = \{x_{-i} \in K_{-i} : \ y_i \in G_i(x_{-i})\}$ is open in K_{-i} and the following conditions are satisfied:*

(i) *$\phi_i(z_i, x_{-i}, \cdot)$ is $C_{i\Delta}$-quasiconcave, for all $i \in I$, $x_{-i} \in K_{-i}$, $z_i \in G_i(x_{-i})$;*
(ii) *$\{z_i \in K_i : \phi_i(z_i, x_{-i}, x_i) \nsubseteq -\operatorname{cor} C_i\}$ is closed in $G_i(x_{-i})$, for all $i \in I$, $x_{-i} \in K_{-i}$, $y_i \in G_i(x_{-i})$;*
(iii) *for each $i \in I$, $x = (x_i) \in K$, if $x_i \in G_i(x_{-i})$ then $\phi(x_i, x_{-i}, x_i) \nsubseteq -\operatorname{cor} C_i$;*
(iv) *$\phi_i(\cdot, x_{-i}, y_i)$ is $C_{i\triangle}$-quasiconcave, for all $i \in I$, $x_{-i} \in K_{-i}$, $y_i \in G_i(x_{-i})$;*
(v) *$\{x_{-i} \in K_{-i} : \phi_i(z_i, x_{-i}, y_i) \nsubseteq -\operatorname{cor} C_i\}$ is closed for all $i \in I$, z_i, $y_i \in K_i$.*

Then the solution set of SVQEP *is nonempty.*

Proof. Define the multivalued mapping $B : K \to 2^K$ by $B(x) = \prod_{i \in I} B_i(x_{-i})$, for all $x \in K$, where

$$B_i(x_{-i}) = \{z_i \in G_i(x_{-i}) : \phi_i(z_i, x_{-i}, y_i) \nsubseteq -\operatorname{cor} C_i, \quad \text{for all } y_i \in G_i(x_{-i})\},$$

for all $i \in I$. We claim that the multivalued mapping B fulfils all the conditions of Theorem 2.2. To see this, we will complete our proof in the following three steps (a),

(b) and (c).
(a) For all $i \in I$, define multivalued mapping Q_i on $G_i(x_{-i})$ as follows

$$Q_i(z_i) = \{y_i \in G_i(x_{-i}) : \phi_i(z_i, x_{-i}, y_i) \subseteq -\operatorname{cor} C_i\}, \quad \text{for all } z_i \in G_i(x_{-i}).$$

Then for any $z_i \in G_i(x_{-i})$ we get

$$Q_i(z_i) = G_i(x_{-i}) \cap Y_i(z_i, x_{-i}),$$

where

$$Y_i(z_i, x_{-i}) = \{y_i \in K_i : \phi_i(z_i, x_{-i}, y_i) \subseteq -\operatorname{cor} C_i\}.$$

It follows from (i) that for all $i \in I$, the multivalued mapping $\phi_i(z_i, x_{-i}, \cdot)$ is $C_{i\triangle}$-quasiconvex. Hence, for each $i \in I$, the set $Y_i(z_i, x_{-i})$ is $\triangle$-convex by Theorem 1.1. Moreover, since for all $i \in I$ the set $G_i(x_{-i})$ is $\triangle$-convex we deduce that, for all $i \in I$, the set $Q_i(z_i)$ is $\triangle$-convex. Also it follows from (ii) that for each $y_i \in G_i(x_{-i})$, the set

$$Q_i^{-1}(y_i) = \{z_i \in G_{-i}(x_{-i}) : y_i \in Q_i(z_i)\} = \{z_i \in G_i(x_{-i}) : \phi_i(z_i, x_{-i}, y_i) \subseteq -\operatorname{cor} C_i\}$$

is open in $G_i(x_{-i})$. We assert that $x_i \notin Q_i(x_i)$. By contrary we suppose that $x_i \in Q_i(x_i)$. Hence by definition Q_i we have

$$x_i \in G_i(x_{-i}), \phi_i(x_i, x_{-i}, x_i) \subseteq -\operatorname{cor} C_i.$$

Which is contradiction with the condition (iii). So $x_i \notin Q_i(x_i)$. Therefore, for all $i \in I$, the multivalued mapping Q_i satisfies all the assumptions of Theorem 2.2 and so there exists an $x_i^* \in G_i(x_{-i})$ such that $Q_i(x_i^*) = \emptyset$. This means

$$\phi_i(x_i^*, x_{-i}, y_i) \nsubseteq -\operatorname{cor} C_i, \quad \text{for all } y_i \in G_i(x_{-i}).$$

Which implies that $B_i(x_{-i})$ is not empty and consequently $B(x)$ is also.
(b) For any $z_i^1, z_i^2 \in B_i(x_{-i})$, we have

$$z_i^j \in G_i(x_{-i} \text{ and } \phi_i(z_i^j, x_{-i}, y_i) \subseteq -\operatorname{cor} C_i, \quad \text{for all } y_i \in G_i(x_{-i}), j = 1, 2.$$

From the condition (iv), $\phi_i(\cdot, x_{-i}, y_i)$ is $C_{i\triangle}$-quasiconcave. Then for any $z_i \in \triangle\{z_i^1, z_i^2\}$ and $y_i \in G_i(x_{-i})$, without loss of generality, we have

$$\phi_i(z_i, x_{-i}, y_i) \in \phi_i(z_i^1, x_{-i}, y_i) + C_i.$$

Equivalently

$$\phi_i(z_i^1, x_{-i}, y_i) \in \phi_i(z_i, x_{-i}, y_i) - C_i.$$

Now if we assume there is $y_i^o \in G_i(x_{-i})$, such that $\phi_i(z_i, x_{-i}, y_i^o) \in -\operatorname{cor} C_i$, then

$$\phi_i(z_i^1, x_{-i}, y_i^o) \in \phi_i(z_i, x_{-i}, y_i^o) - C_i \subset -\operatorname{cor} C_i - C_i \subset -\operatorname{cor} C_i,$$

which is contradicted by $z_i^1 \in B_i(x_{-i})$. Therefore

$$z_i \in B_i(x_{-i}) \text{ and } \phi_i(z_i, x_i, y_i) \notin \operatorname{cor} C_i, \quad \text{for all } y_i \in G_i(x_{-i}),$$

that is $z_i \in B_i(x_{-i})$. Then $\triangle\{z_i^1, z_i^2\} \subseteq B_i(x_{-i})$ and so $B_i(x_{-i})$ is $\triangle$-convex. Consequently, we obtain $B(x)$ is $\triangle$-convex. For each $z \in K$, denote $D_i(z_i)$ as follows

$$D_i(z_i) = \cap_{y_i \in G_i(x_{-i})}\{x_{-i} : \phi_i(z_i, x_{-i}, y_i) \subseteq -\operatorname{cor} C_i\}$$

then we have

$$B_i^{-1}(z_i) = \{x_{-i} \in K_{-i} : z_i \in B_i(x_{-i})\} = \{x_{-i} \in K_{-i} : z_i \in G_i(x_{-i}) \cap D_i(z_i)\}.$$

The set $\{x_{-i} \in K_{-i} : \phi_i(z_i, x_{-i}, y_i) \nsubseteq -\operatorname{cor} C_i\}$ is closed according to (v). Thus D_i is open in K_{-i}. In addition, by assumption we have $G_i^{-1}(z_i) = \{x_{-i} \in K_{-i} : z_i \in G_i(x_{-i})\}$ is open. Consequently, we obtain $B_i^{-1}(z_i)$ is open, $B^{-1}(z)$ is also open. Up to now, $B(x)$ is non-empty and $\triangle$-convex, $B^{-1}(x)$ is open for any $x \in K$. If $x \notin B(x)$, for all $x \in K$, then there exists an $\tilde{x}$ such that $B(\tilde{x}) = \emptyset$ by Theorem 2.2. This is contradicted by $B(x)$ is non-empty for each $x \in K$. Therefore, we can find an $\bar{x} \in K$ such that $\bar{x} \in B(\bar{x})$. Obviously, $\bar{x}$ is a solution of the SVQEP. This completes the proof. □

As an application of the previous theorem we deduce the following existence result for a solution of the generalized vector equilibrium problem whose domain need not to have nonempty interior in the setting of topological semilattice. Moreover, it can be viewed as improvement of the corresponding result given in [2, 3, 6, 8, 11].

Corollary 2.1. *Let (K, Y, C, ϕ) be a vector equilibrium problem, where K is a compact topological sup-semilattice with path connected intervals, Y is a topological vector space and $\phi : K \times K \to 2^Y$. If the following conditions are satisfied:*

(i) *for each $x \in K$, $\phi(x, \cdot)$ is $C_\triangle$-quasiconcave;*
(ii) *for each $y \in K$, $\{x : \phi(x, y) \nsubseteq -\operatorname{cor} C\}$ is closed in K;*
(iii) *for each $x \in K$, $\phi(x, x) \nsubseteq -\operatorname{cor} C$;*

then there exists $x^* \in K$ such that

$$\phi(x^*, y) \nsubseteq -\operatorname{cor} C, \quad \textit{for all } y \in K.$$

Note that x^* is called a solution of generalized vector equilibrium problem (GVEP). Moreover, the solution set of (GVEP) is a closed subset of K.

Example 2.2. Let $P[0,1]$ denote the set of all polynomials on $K = [0,1]$ and let $P[0,1]$ denote the set of all polynomials over $K = [0,1]$ and C consist of all element $p \in P[0,1]$ which have degree less than or equal two with $p(t) \geq 0$ for all $t \in [0,1]$. The mapping $\phi : K \times K \to P[0,1]$ is defined as follows: $\phi(s,t) = p_{(s-t)}$ where

$$p_{(s-t)}(x) = (s-t)x^2, \quad \text{for all } s, t, x \in [0,1].$$

It is straightforward to see that C is closed , convex pointed cone and

$$\operatorname{int} C = \emptyset, \quad \operatorname{cor} C = \{p \in P[0,1] : \min_{x \in [0,1]} p(x) > 0\}.$$

It is obvious that the condition (i) of the previous corollary is satisfied and to check (ii) of it, we pick $s_0 \in K$. Then

$$\begin{aligned}\{t \in K : \phi(t, s_0) \notin \operatorname{cor} C\} =& \{t \in [0,1] : -p_{s_0 - t} \notin \operatorname{cor} C\} \\ =& \{t \in [0,1] : \max_{x \in [0,1]} (s_0 - t)x^2 \geq 0\} = [0,1],\end{aligned}$$

which is a closed in K. Also the condition (iii) is trivially fulfilled because

$$\phi(s,s) = p_0 = 0 \notin \operatorname{cor} C.$$

Hence all the assumptions of the aforementioned corollary is satisfied and one can check that the set of solution of the problem (GVEP) equal to K.

The next result is a multivalued version of the corresponding results given in [6, 9] for dual equilibrium problem in the setting of topological semilattice with mild assumptions.

Theorem 2.5. *Let K be a $C_\triangle$-convex with path connected intervals of a topological sup-semilattice X, C be a convex cone of a vector Y with $\operatorname{cor} C \neq \emptyset$. Assume $F : K \times K \to 2^Y$ satisfies the following conditions.*

(i) *for each $x \in K$, $F(x,x) \nsubseteq \operatorname{cor} C$;*
(ii) *for each $y \in K$, the set $\{x : F(x,y) \nsubseteq \operatorname{cor} C\}$ is closed in K;*
(iii) *for each $y \in K$, the set $\{x : F(x,y) \subseteq \operatorname{cor} C\}$ is $C_\triangle$-convex;*
(iv) *there exists $x_0 \in X$ such that $\{y \in K;\ F(x_0,y) \nsubseteq \operatorname{cor} C\}$ is compact.*

Then there exists $x^* \in K$ such that $F(x,x^*) \nsubseteq \operatorname{cor} C$ for all $x \in K$.

Proof. Let $W = \{(x,y) \in K \times K : F(x,y) \nsubseteq \operatorname{cor} C\}$. Then it follows from (ii) that, for each $x \in K$, the set $W(x) = \{y \in K : (x,y) \in W\}$ is closed and by (iv) the set $W(x_0)$ is compact. We claim that for all finite subset A of K the inclusion $\triangle A \subseteq \cup_{x\in A} W(x)$ is valid. Because otherwise there exists a finite subset $A = \{x_1, x_2, \dots, x_n\} \subseteq K$ such that $\triangle A \nsubseteq \cup_{x\in A} W(x)$. Hence, for some $y_0 \in \triangle A$ and all $x \in A$ we have $y_0 \notin W(x)$. Then, for all $i = 1, 2, \dots, n$, we get $F(x_i, y_0) \subseteq \operatorname{cor} C$ and it follows from (iii) that $F(\triangle A, y_0) \subseteq \operatorname{cor} C$. Consequently, since $y_0 \in \triangle A$ we have $F(y_0,y_0) \subseteq \operatorname{cor} C$ which is contradicted by (i). Therefore $\triangle A \subseteq \cup_{x\in A} W(x)$. Now, F satisfies all the conditions of Theorem 2.1 and so $\cap_{x\in A} W(x) \neq \emptyset$. Then there exists $x^* \in K$ such that for all $x \in K, x^* \in W(x)$. This means that,

$$F(x,x^*) \nsubseteq \operatorname{cor} C, \quad \text{for all } x \in K.$$

This completes the proof. □

Theorem 2.6. *Let K be a nonempty compact $\triangle$-convex subset of a topological sup-semilattice with path connected intervals, Y be a topological vector space, $F : K \times K \to 2^Y$, and C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(i) *for each $x \in K$, $F(x,x) \cap \operatorname{int} C = \emptyset$;*
(ii) *for each $x \in K$, $F(x,\cdot)$ is $C_\triangle$-quasiconcave;*
(iii) *for each $y \in K$, $F(\cdot,y)$ is lower C-continuous.*

Then there exists $x^* \in K$ such that $F(x^*,y) \cap \operatorname{int} C = \emptyset$ for all $y \in K$.

Proof. Define $P : K \to 2^K$ by

$$P(x) = \{y \in K : F(x,y) \cap \operatorname{int} C \neq \emptyset\}.$$

We are going to show that the multivalued mapping P fulfils all conditions of Lemma 1.1. (a) The values of P are $\triangle$-convex. Otherwise, there exist $x^{'} \in K$ and $y_1, y_2 \in P(x^{'})$ such that

$$\triangle(\{y_1, y_2\}) \nsubseteq P(x^{'}).$$

This means that there exists $z \in \triangle(\{y_1, y_2\})$ with $z \notin P(x^{'})$. Thus $F(x^{'}, z) \cap \operatorname{int} C = \emptyset$ and so by the Definition 1.3 (note, by (ii), F is $C_{\triangle}$-quasiconcave) and losing of generality we can assume that $F(x^{'}, y_1) \subset F(x^{'}, z) - C$. Since $F(x^{'}, y_1) \cap \operatorname{int} C \neq \emptyset$ (note $y_1 \in P(x^{'})$), we can take $u_1 \in F(x^{'}, y_1)$ with $u_1 \in \operatorname{int} C$. Then there exist $v_1 \in F(x^{'}, z)$ and $w_1 \in C$ such that $u_1 = v_1 - w_1$. Hence, by applying Lemma 1.1 we get $v_1 = u_1 + w_1 \in \operatorname{int} C + C \subseteq \operatorname{int} C$ which is contradicted by $F(x^{'}, z) \cap \operatorname{int} C = \emptyset$. Then, for all $x \in X$ we get $P(x)$ is $\triangle$-convex. Now we prove $P^{-1}(y)$ is open for each $y \in K$. We know that

$$\begin{aligned} K \setminus P^{-1}(y) =& \{x \in K; x \notin P^{-1}(y)\} = \{x \in K; y \notin P(x)\} \\ =& \{x \in K; F(x, y) \cap \operatorname{int} C = \emptyset\}. \end{aligned} \tag{2.2}$$

Put $D := K \setminus P^{-1}(y)$. The set D is closed. Because on the contrary if there exists $\bar{x} \in \bar{D} \setminus D$, then $F(\bar{x}, y) \cap \operatorname{int} C \neq \emptyset$. Therefore there exists $\bar{y} \in F(x, y) \cap \operatorname{int} C$. Hence there exists a neighborhood V of zero in Y such that $\bar{y} - V \subset \operatorname{int} C$. It follows from (iii) that there exists a neighborhood U of $\bar{x}$ such that

$$F(\bar{x}, y) \subset F(x, y) + V - C, \quad \text{for all } x \in U.$$

Then

$$0 \in F(x, y) - \bar{y} + V - C \subset F(x, y) - \operatorname{int} C - C \subset F(x, y) - \operatorname{int} C,$$

which implies

$$F(x, y) \cap \operatorname{int} C \neq \emptyset, \quad \text{for all } x \in U.$$

Since $\bar{x} \in \overline{D}$ there must be a net $\{x_\alpha\} \in D$ which is convergent to $\bar{x}$. Then there exists β such that $x_\alpha \in U$, for all $\alpha \geq \beta$ and then $F(x_\alpha, y) \cap \operatorname{int} C \neq \emptyset$, which contradicts $x_\alpha \in D$. Therefore $\bar{x} \in D$ and D is closed. Consequently we infer that $P^{-1}(y)$ is open for each $y \in K$. By Theorem 2.2 there exists $\overline{x} \in K$ such that $P(\overline{x}) = \emptyset$. This means, for all $y \in K$, $y \notin P(\overline{x})$ and so

$$F(\overline{x}, y) \cap \operatorname{int} C = \emptyset, \quad \text{for all } y \in K.$$

This completes the proof. □

It is clear from the proof of Theorem 2.6 that we can replace condition (iii) of the Theorem 2.6 by the lower semicontinuity of the multivalued mapping $x \mapsto F(x, y)$. There are examples which show the class of all lower semicontinuous multivalued mappings does not equal the class of all lower C-continuous. The following example illustrates Theorem 2.6.

Example 2.3. Let $K = [0,1] \times [0,1]$, $C = \mathbb{R}_+ = [0,\infty)$. The $(K,\preceq)$ is a sup-semilattice, in which $x \preceq y$ means that $x_1 \leq y_1$, $x_2 \leq y_2$ for any $x = (x_1, x_2)$, $y = (y_1, y_2)$ of K. Define multivalued mapping $F : [0,1] \times [0,1] \to 2^{\mathbb{R}}$ by $F(x,y) = (-\infty, y - x)$. It is easy to check F satisfies all the conditions of Theorem 2.6 and so there exists $x^* \in K$ such that

$$F(x^*, y) \geq 0, \quad \text{for all } y \in K,$$

in fact $x^* = 1$ is the unique solution.

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