

QUANTITATIVE UNCERTAINTY PRINCIPLE FOR STURM-LIOUVILLE TRANSFORM

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ABSTRACT. In this paper we consider the Sturm-Liouville transform $\mathcal{F}(f)$ on \mathbb{R}_+ . We analyze the concentration of this transform on sets of finite measure. In particular, Donoho-Stark and Benedicks-type uncertainty principles are given.

1. INTRODUCTION

The uncertainty principle says that a function and its transform cannot concentrate both on small sets. Depending on the precise way to measure “concentration” and “smallness” this principle can assume different forms. This paper focuses on studying different uncertainty principles for the Sturm-Liouville transform, by following the procedures for similar transforms, such as the Fourier transform (the classical setting) we refer to the book [10] and the surveys [4, 7] for further references. The concept of concentration has taken different interpretations in different contexts. For example: Benedicks [2], Slepian and Pollak [18], Landau and Pollak [13], and Donoho and Stark [6] paid attention to the supports of functions and gave quantitative uncertainty principles for the Fourier transforms. Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [11], Cowling and Price [5], Beurling [3], Miyachi [15] theorems enter within the framework of the quantitative uncertainty principles. The quantitative and qualitative uncertainty principles have been studied by many authors for various Fourier transforms, for examples (cf. [1, 11, 14, 16]).

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Our aim here is to consider uncertainty principles in which concentration is measured in sense of smallness of the support and when the transform under consideration is the Sturm-Liouville transform.

The first principle that is studied is a Donoho-Stark-type inequality. One can write the classical uncertainty principle in the following way: If a function $f(t)$ is essentially zero outside an interval of length Δt and its Fourier transform $f(w)$ is essentially zero outside an interval of length Δw , then $\Delta t \Delta w \geq 1$. In [6], Donoho and Stark show that it is not necessary to assume that the support and the spectrum are concentrated on intervals and one can replace intervals by measurable sets, and then the length of the interval is naturally replaced by the measure of the set. In Section 2, a version of this inequality for the Sturm-Liouville transform is given, and, as it appears in [6] it is explained how to reconstruct a signal f from a noisy measurement, knowing that the signal is supported on a set S .

The second principle, studied in Section 3, is a Benedicks-type result which shows that two measurable sets (S, Σ) with finite measure form a strong annihilating pair. This means that a function supported in S cannot have an spectrum in Σ giving a quantitative information of the mass of a function whose spectrum is contained in Σ . The approach is based on the corresponding version of this type of principle for the integral operators transform, studied in [8]. A version of Benedicks type-inequality for integral operators transform with bounded and homogeneous kernel has been proved in [8]. In this paper, we consider a transform of a different nature where in particular the kernel is not homogeneous.

We recall that, Soltani in [19] study what is the relation between the measure and the spectrum of a function f that is ε -concentrated in measurable sets giving. Concentration in support means that the part of the function that is not supported on a set is at least an ε part of the total mass. The analogous version for spectrum states that the part of the spectrum not supported on a set is an ε part of the total spectrum. It is shown that if a function is ε -concentrated in space and frequency, then the product of the measures of the support and spectrum is lower bounded by a number close to one.

In order to describe our results, we first need to introduce some facts about harmonic analysis related to Sturm-Liouville transform. We cite here, as briefly as possible, some properties. For more details we refer to [19].

The Sturm-Liouville operator Δ defined on \mathbb{R}_+ by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\mathcal{A}'(x)}{\mathcal{A}(x)} \frac{\partial}{\partial x} + \rho^2,$$

where ρ is a nonnegative real number and $\mathcal{A}(x) = x^{2\alpha+1}B(x)$, $\alpha > -\frac{1}{2}$, where B is a positive, even, infinitely differentiable function on \mathbb{R} such that $B(0) = 1$. Moreover, we assume that \mathcal{A} and B satisfy the following conditions:

- \mathcal{A} is increasing and $\lim_{x \rightarrow \infty} \mathcal{A}(x) = \infty$;

- $\frac{\mathcal{A}'}{\mathcal{A}}$ is decreasing and $\lim_{x \rightarrow \infty} \frac{\mathcal{A}'(x)}{\mathcal{A}(x)} = 2\rho$;
- there exists a constant $\delta > 0$ such that

$$\frac{\mathcal{A}'(x)}{\mathcal{A}(x)} = \begin{cases} 2\rho + D(x) \exp(-\delta x), & \text{if } \rho > 0, \\ \frac{2\alpha + 1}{x} + D(x) \exp(-\delta x), & \text{if } \rho = 0, \end{cases}$$

where D is an infinitely differentiable function on $]0, \infty[$, bounded and with bounded derivatives on all intervals $[x_0, \infty[$ for $x_0 > 0$. For all $\lambda \in \mathbb{C}$ the equation

$$\begin{cases} \Delta u = -\lambda^2 u, \\ u(0) = 1, \quad u'(0) = 0, \end{cases}$$

admits a unique solution denoted φ_λ , with the following properties:

- for $x \geq 0$ the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on \mathbb{C} ;
- for $\lambda \in \mathbb{C}$ the function $\lambda \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on \mathbb{R} ;
- $|\varphi_\lambda(x)| \leq 1$ for all $\lambda, x \in \mathbb{R}$.

For nonzero $\lambda \in \mathbb{C}$ the equation $\Delta u = -\lambda^2 u$ has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{1}{\sqrt{\mathcal{A}(x)}} \exp(i\lambda x) V(x, \lambda),$$

with $\lim_{x \rightarrow \infty} V(x, \lambda) = 1$. Consequently, there exists a function $\lambda \mapsto c(\lambda)$, such that

$$\varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}, \quad \text{for nonzero } \lambda \in \mathbb{C}.$$

Moreover, there exist positive constants k_1, k_2 and k such that

$$k_1 |\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1},$$

for all λ such that $\text{Im}\lambda \leq 0$ and $|\lambda| \geq k$.

Let us introduce the dilation operator D_ρ , $\rho > 0$, defined by

$$D_\rho f(x) = \frac{1}{\rho^{\alpha+1}} f\left(\frac{x}{\rho}\right).$$

We denote by $L^p(\mathbb{R}_+, \mu)$, $1 \leq p \leq \infty$, the space of measurable functions f on \mathbb{R}_+ such that

$$\|f\|_{L^p(\mathbb{R}_+, \mu)} = \left(\int_{\mathbb{R}_+} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$

$$\|f\|_\infty = \text{esssup}_{x \in \mathbb{R}_+} |f(x)| < +\infty, \quad \text{if } p = \infty,$$

where $d\mu(x) = \mathcal{A}(x)dx$.

The Sturm-Liouville transform \mathcal{F} is defined on $L^1(\mathbb{R}_+, \mu)$ by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}_+} f(x) \varphi_\lambda(x) d\mu(x), \quad \text{for all } \lambda \in \mathbb{R}.$$

Let ν the measure defined on $[0, \infty[$ by $d\nu(\lambda) = \frac{d\lambda}{2\pi|c(\lambda)|^2}$ and by $L^p(\nu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0, \infty[$, such that $\|f\|_{L^p(\mathbb{R}_+, \nu)} < \infty$.

For all $f \in L^1(\mathbb{R}_+, \mu)$, the function $\mathcal{F}(f)$ is continuous on \mathbb{R} and we have

$$(1.1) \quad \|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}_+, \nu)} \leq \|f\|_{L^1(\mathbb{R}_+, \mu)}.$$

Theorem 1.1 (Plancherel theorem). *The Sturm-Liouville transform \mathcal{F} extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}_+, \mu)$ onto $L^2(\mathbb{R}_+, \nu)$*

$$(1.2) \quad \int_{\mathbb{R}_+} |f(x)|^2 d\mu(x) = \int_{\mathbb{R}_+} |\mathcal{F}(f)(\lambda)|^2 d\nu(\lambda).$$

Theorem 1.2 (Inversion theorem). *Let $f \in L^1(\mathbb{R}_+, \mu)$ such that $\mathcal{F}(f) \in L^1(\mathbb{R}_+, \nu)$. Then*

$$f(x) = \int_{\mathbb{R}_+} \mathcal{F}(f)(\lambda) \varphi_\lambda(x) d\nu(\lambda) \text{ a.e. } x \in \mathbb{R}^+.$$

Theorem 1.3 (Riesz's interpolation theorem). *Let $f \in L^p(\mathbb{R}_+ > \mu)$. Then we get the Hausdorff-Young inequality (see [20]) $\|\mathcal{F}(f)\|_{L^q(\mathbb{R}_+, \nu)} \leq \|f\|_{L^p(\mathbb{R}_+, \mu)}$.*

Definition 1.1. Let S, Σ be two measurable subsets of \mathbb{R}_+^d . Then (S, Σ) is called a weak annihilating pair for the Sturm-Liouville transform if $\text{supp} f \subset S$ and $\text{supp} \mathcal{F}_{k,\alpha}(f) \subset \Sigma$, implies that $f = 0$, where $\text{supp} f = \{x : f(x) \neq 0\}$.

Definition 1.2. Let S, Σ be two measurable subsets of \mathbb{R}_+ . Then (S, Σ) is called a strong annihilating pair for the Sturm-Liouville transform if there exists a constant $C(S, \Sigma)$ such that for all function $f \in L^2(\mathbb{R}_+, \mu)$, with $\text{supp} \mathcal{F}(f) \subset \Sigma$,

$$(1.3) \quad \|f\|_{L^2(\mathbb{R}_+, \mu)} \leq C(S, \Sigma) \|f\|_{L^2(S^c, \mu)},$$

where $S^c = \mathbb{R}_+ \setminus S$ and $\text{supp} f = \{x : f(x) \neq 0\}$.

2. THE DONOHO-STRAK'S UNCERTAINTY PRINCIPLE

The classical uncertainty principle says that if a function $f(t)$ is essentially zero outside an interval of light Δt and its Fourier transform $\hat{f}(w)$ is essentially zero outside an interval of length Δw , then $\Delta t \Delta w \geq 1$. In this section we will prove a quantitative uncertainty inequality about the essential supports of a nonzero function $f \in L^2(\mathbb{R}_+, \mu)$ and its Sturm-Liouville transform.

The first such inequality for the usual Fourier transform was obtained by Donoho-Stark [6].

We consider a pair of orthogonal projections on $L^2(\mathbb{R}_+, \mu)$ defined by $P_S f = \chi_S f$, $Q_\Sigma f = \mathcal{F}^{-1}[\chi_\Sigma \mathcal{F}(f)]$, where S and Σ are measurable subsets of \mathbb{R}_+ , and χ_S denote the characteristic function of S .

Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$ and let $f \in L^2(\mathbb{R}_+, \mu)$ be a nonzero function. We say that f is ε_S -time-limited on S if $\|P_{S^c} f\|_{L^2(\mathbb{R}_+, \mu)} \leq \varepsilon_S \|f\|_{L^2(\mathbb{R}_+, \mu)}$. Similarly, we say that f is ε_Σ -band-limited on Σ for the Sturm-Liouville transform if $\|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+, \mu)} \leq \varepsilon_\Sigma \|f\|_{L^2(\mathbb{R}_+, \mu)}$.

We denote by $P_S \cap Q_\Sigma$ for the orthogonal projection onto the intersection of the ranges of P_S and Q_Σ , we will write $\text{Im} T$ for the range of a linear operator T . We

denote by $\|T\|_{HS}$ the Hilbert-Schmidt norm of the linear operator T . The definition of this norm [21, page 262] implies that for any pair of projections E, F one has

$$(2.1) \quad \dim(\text{Im}P_S \cap \text{Im}Q_\Sigma) = \|P_S \cap Q_\Sigma\|_{HS}^2 \leq \|P_S Q_\Sigma\|_{HS}^2.$$

Theorem 2.1. *Let $\Sigma, S \subset \mathbb{R}_+$ be a pair of measurable subsets and let $\varepsilon_S, \varepsilon_\Sigma > 0$ such that $\varepsilon_S^2 + \varepsilon_\Sigma^2 < 1$. Let $f \in L^2(\mathbb{R}_+, \mu)$ be a non function. If f is ε_S -time-limited on S and ε_Σ -band-limited on Σ for the Sturm-Liouville transform, then*

$$\mu(S)\nu(\Sigma) \geq \left(1 - \sqrt{\varepsilon_S^2 + \varepsilon_\Sigma^2}\right)^2.$$

We will need the following well-known lemma.

Lemma 2.1. *Let (S, Σ) be two measurable subsets of \mathbb{R}_+ . Then the following assertions are equivalent.*

- i) $\|P_S Q_\Sigma\| = \|P_S Q_\Sigma\|_{L^2(\mathbb{R}_+, \mu)} < 1$.
- ii) (S, Σ) is strongly annihilating pair for the Sturm-Liouville transform. Moreover, we have $\|f\|_{L^2(\mathbb{R}_+, \mu)}^2 \leq (1 - \|P_S Q_\Sigma\|)^{-2} \left(\|P_{S^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 + \|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 \right)$.

Proof. Firstly we show the following implication i) \Rightarrow ii). The identity operator I satisfies $I = P_S + P_{S^c} = P_S Q_\Sigma + P_S Q_{\Sigma^c} + P_{S^c}$, we have from the orthogonality of P_S and P_{S^c}

$$\begin{aligned} \|f - P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+, \mu)}^2 &= \|P_S Q_{\Sigma^c} f + P_{S^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 \\ &= \|P_S Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 + \|P_{S^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2. \end{aligned}$$

It follows, by $\|P_S\| = 1$, that

$$(2.2) \quad \|f - P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+, \mu)} \leq \left(\|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 + \|P_{S^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 \right)^{\frac{1}{2}}.$$

On the other hand, we have

$$\begin{aligned} \|f - P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+, \mu)} &\geq \|f\|_{L^2(\mathbb{R}_+, \mu)} - \|P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+, \mu)} \\ &\geq \|f\|_{L^2(\mathbb{R}_+, \mu)} - \|P_S Q_\Sigma\| \|f\|_{L^2(\mathbb{R}_+, \mu)}. \end{aligned}$$

It follows, from inequality (2.2),

$$(2.3) \quad (1 - \|P_S Q_\Sigma\|) \|f\|_{L^2(\mathbb{R}_+, \mu)} \leq \left(\|P_{S^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 + \|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 \right)^{\frac{1}{2}}.$$

As $\|P_S Q_\Sigma\| < 1$, then we obtain the desired result.

Let us now show the second implication ii) \Rightarrow i). Recall that

$$\begin{aligned}\|P_S Q_\Sigma\| &= \|Q_\Sigma P_S\| = \sup_{f \in L^2(\mathbb{R}_+, \mu)} \frac{\|Q_\Sigma P_S f\|_{L^2(\mathbb{R}_+, \mu)}}{\|f\|_{L^2(\mathbb{R}_+, \mu)}} \\ &= \sup_{f: f = P_S f} \frac{\|Q_\Sigma f\|_{L^2(\mathbb{R}_+, \mu)}}{\|f\|_{L^2(\mathbb{R}_+, \mu)}} \\ &= \sup_{f: f = Q_\Sigma f} \frac{\|P_S f\|_{L^2(\mathbb{R}_+, \mu)}}{\|f\|_{L^2(\mathbb{R}_+, \mu)}} \\ &< 1.\end{aligned}$$

We suppose that $\|P_S Q_\Sigma\| = 1$. Then we can find a bandlimited sequence $f_n \in L^2(\mathbb{R}_+, \mu)$ on Σ of norm 1 (in particular $f_n = Q_\Sigma f_n$) such that

$$\|P_S f_n\|_{L^2(\mathbb{R}_+, \mu)} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

By the orthogonality of S , we have

$$\|P_{S^c} f_n\|_{L^2(\mathbb{R}_+, \mu)}^2 = \|f_n\|_{L^2(\mathbb{R}_+, \mu)}^2 - \|P_S f_n\|_{L^2(\mathbb{R}_+, \mu)}^2 \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which contradicts (1.3). \square

Lemma 2.2. *If $0 < \mu(S)\nu(\Sigma) < 1$, then for all function $f \in L^2(\mathbb{R}_+, \mu)$ such that $\text{supp}\mathcal{F}(f) \subset \Sigma$ we have*

$$\|f\|_{L^2(\mathbb{R}_+, \mu)} \leq \left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1} \|f\|_{L^2(S^c, \mu)}.$$

Proof. A straightforward computation shows that $P_S Q_\Sigma$ is an integral operator with kernel $N(t, x) = \chi_S(t)\mathcal{F}^{-1}(\chi_\Sigma \varphi_\lambda(t))(x)$. Indeed, we have

$$\begin{aligned}P_S Q_\Sigma f(t) &= \chi_S(t) \int_{\mathbb{R}_+} \chi_\Sigma(\xi) \mathcal{F}(f)(\xi) \varphi_\lambda(t) d\nu(\xi) \\ &= \chi_S(t) \int_{\mathbb{R}_+} \chi_\Sigma(\xi) \varphi_\lambda(t) \left(\int_{\mathbb{R}_+} f(x) \varphi_\lambda(x) d\mu(x) \right) d\nu(\xi) \\ &= \int_{\mathbb{R}_+} f(x) N(t, x) d\mu(x),\end{aligned}$$

where

$$N(t, x) = \chi_S(t) \int_{\mathbb{R}_+} \chi_\Sigma(\xi) \varphi_\lambda(t) \varphi_\lambda(x) d\nu(\xi).$$

Since, $\nu(\Sigma) < \infty$ and φ_λ is bounded, then for all $t \in \mathbb{R}_+$, $\chi_\Sigma \varphi_\lambda(t) \in L^2(\mathbb{R}_+, \nu)$. Then $P_S Q_\Sigma$ is an integral operator with kernel $N(t, x) = \chi_S(t)\mathcal{F}^{-1}(\chi_\Sigma \varphi_\lambda(t))(x)$. As $\|P_S Q_\Sigma\|_{HS} = \|N\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \mu \otimes \mu)}$, it follows from Plancherel's theorem 1.2 that

$$\begin{aligned}\|P_S Q_\Sigma\|_{HS}^2 &= \int_{\mathbb{R}_+} |\chi_S(t)|^2 \left(\int_{\mathbb{R}_+} |\mathcal{F}^{-1}(\chi_\Sigma \varphi_\lambda(t))(x)|^2 d\mu(x) \right) d\nu(t) \\ &= \int_{\mathbb{R}_+} \chi_S(t) \int_{\mathbb{R}_+} \chi_\Sigma(\xi) |\varphi_\lambda(t)|^2 d\nu(\xi) d\mu(t).\end{aligned}$$

We can deduce from $|\varphi_\lambda(t)| < 1$ that

$$(2.4) \quad \|P_S Q_\Sigma\| \leq \|P_S Q_\Sigma\|_{HS} \leq \sqrt{\mu(S)\nu(\Sigma)}.$$

Since $\mu(S)\nu(\Sigma) < 1$, then we have from inequality (2.4) and Lemma 2.1

$$\|f\|_{L^2(\mathbb{R}_+, \mu)}^2 \leq \left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-2} \left(\|P_{S^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 + \|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2\right).$$

Since $\text{supp}\mathcal{F}(f) \subset \Sigma$, it follows from Plancherel's theorem 1.2 that

$$\|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+, \mu)}^2 = \int_{\Sigma^c} |\mathcal{F}(\xi)|^2 d\nu(\xi) = \|\mathcal{F}(f)\|_{L^2(\Sigma^c, \nu)}^2 = 0,$$

which shows the desired result. \square

Proof of Theorem 2.1. The result follows from inequalities (2.3) and (2.4). Indeed, f is ε_S -time-limited on S , then $\|P_{S^c} f\|_{L^2(\mathbb{R}^2, \mu)} \leq \varepsilon_S \|f\|_{L^2(\mathbb{R}^2, \mu)}$. f is ε_Σ -band-limited on Σ for the Sturm-Liouville transform, then $\|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}^2, \mu)} \leq \varepsilon_\Sigma \|f\|_{L^2(\mathbb{R}^2, \mu)}$. It follows that

$$(2.5) \quad \|P_{S^c} f\|_{L^2(\mathbb{R}^2, \mu)}^2 + \|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}^2, \mu)}^2 \leq (\varepsilon_S^2 + \varepsilon_\Sigma^2) \|f\|_{L^2(\mathbb{R}^2, \mu)}^2,$$

from (2.3) we deduce that $(1 - \|P_S Q_\Sigma\|)^2 \leq \varepsilon_S^2 + \varepsilon_\Sigma^2$. It follows, from (2.4), that $1 - \sqrt{\varepsilon_S^2 + \varepsilon_\Sigma^2} \leq \|P_S Q_\Sigma\| \leq \sqrt{|S||\Sigma|}$, which proves the desired result. \square

Remark 2.1. From inequalities (2.1) and (2.4) it follows that

$$(2.6) \quad \dim(\text{Im}P_S \cap \text{Im}Q_\Sigma) \leq \|P_S Q_\Sigma\|_{HS}^2 < \infty.$$

The following example is prototypical. A signal f is transmitted to a receiver who know that f is bandlimited on S for the Sturm-Liouville transform, meaning that f is synthesized using only frequency on S ; equivalently $f = Q_\Sigma f$. Suppose that the observation of f is corrupted by a noise $n \in L^2(\mathbb{R}_+, \mu)$ (which is nonetheless assumed to be small) and an unregistered values on S . Thus, the observable function r satisfies

$$r(x) = \begin{cases} f(x) + n(x), & x \in S^c, \\ 0, & x \in S. \end{cases}$$

Here, we have assumed without loss of generality that $n = 0$ on S . Equivalently, $r = (I - P_S)f + n$. We say that f can be stably reconstructed from r , if there exists a linear operator K and a constant C such that

$$(2.7) \quad \|f - Kr\|_{L^2(\mathbb{R}_+, \mu)} \leq C \|n\|_{L^2(\mathbb{R}_+, \mu)}.$$

The estimate (2.7) shows that the noise n is at most amplified by a factor C .

Corollary 2.1. *If S and Σ are arbitrary measurable sets of \mathbb{R}_+ with $0 < \mu(S)\nu(\Sigma) < 1$, then f can be stably reconstructed from r . The constant C in equation (2.7) is not larger than $(1 - \sqrt{\mu(S)\nu(\Sigma)})^{-1}$.*

Proof. If $\mu(S)\nu(\Sigma) < 1$, using (2.4), $\|P_S Q_\Sigma\| < 1$. Hence, $I - P_S Q_\Sigma$ is invertible. Let $K = (I - P_S Q_\Sigma)^{-1}$. Since f is bandlimited on Σ , then $(I - P_S)f = (I - P_S Q_\Sigma)f$. Therefore,

$$\begin{aligned} f - Kr &= f - K((I - P_S)f + n) \\ &= f - K(I - P_S Q_\Sigma)f - Kn \\ &= f - (I - P_S Q_\Sigma)(I - P_S Q_\Sigma)^{-1}f - Kn \\ &= 0 - Kn. \end{aligned}$$

So, that

$$\begin{aligned} \|f - Kr\|_{L^2(\mathbb{R}_+, \mu)} &= \|Kn\|_{L^2(\mathbb{R}_+, \mu)} \\ &\leq \|(I - P_S Q_\Sigma)^{-1}\| \|n\|_{L^2(\mathbb{R}_+, \mu)} \\ &\leq \sum_{k=0}^{\infty} \|P_S Q_\Sigma\|^k \|n\|_{L^2(\mathbb{R}_+, \mu)} \\ &\leq \sum_{k=0}^{\infty} (\mu(S)\nu(\Sigma))^{\frac{k}{2}} \|n\|_{L^2(\mathbb{R}_+, \mu)} \\ &= \left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1} \|n\|_{L^2(\mathbb{R}_+, \mu)}. \end{aligned}$$

The constant C in equation (2.7) is therefore not larger than $\left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1}$. \square

The identity $K = (I - P_S Q_\Sigma)^{-1} = \sum_{k=0}^{\infty} (P_S Q_\Sigma)^k$ suggests an algorithm for computing Kr . Put $f^{(n)} = \sum_{k=0}^n (P_S Q_\Sigma)^k r$, then

$$f^{(0)} = r, \quad f^{(n+1)} = r + P_S Q_\Sigma f^{(n)} \quad \text{and} \quad f^{(n)} \rightarrow Kr \text{ as } n \rightarrow \infty.$$

As f is bandlimited on Σ we deduce that

$$(2.8) \quad f^{(n+1)} - f = P_S Q_\Sigma (f^{(n)} - f).$$

Algorithms of this type have applied to a host of problems in signal recovery (see for examples [12, 17]).

3. UNCERTAINTY PRINCIPLES

In this section we will give some remarks about annihilating sets.

Proposition 3.1. *Let $f \in L^2(\mathbb{R}_+, \mu)$ has non empty support, then*

$$\nu(\text{supp}\mathcal{F})\mu(\text{supp}f) \geq 1.$$

In particular, if $\mu(\text{supp}f)\nu(\text{supp}\mathcal{F}) < 1$, then $f = 0$.

Proof. If the function $f \in L^2(\mathbb{R}_+, \mu)$ has non empty support, by the Cauchy-Schwartz inequality and (1.1), we have

$$\begin{aligned} \|\mathcal{F}\|_{L^2(\mathbb{R}_+, \nu)}^2 &\leq \nu(\text{supp } \mathcal{F}(f)) \|\mathcal{F}(f)\|_\infty^2 \\ &\leq \nu(\text{supp } \mathcal{F}(f)) \|f\|_{L^1(\mathbb{R}_+, \mu)}^2 \\ &\leq \nu(\text{supp } \mathcal{F}(f)) \mu(\text{supp } f) \|f\|_{L^2(\mathbb{R}_+, \mu)}^2. \end{aligned}$$

Using Plancherel's theorem 1.2 we have the following quantitative uncertainty inequality connecting the support of f and the support of its Sturm-Liouville transform \mathcal{F}

$$(3.1) \quad \nu(\text{supp } \mathcal{F}) \mu(\text{supp } f) \geq 1.$$

It follows that if $\mu(\text{supp } f) \nu(\text{supp } \mathcal{F}) < 1$, then $f = 0$. \square

Proposition 3.2. *Let $f \in L^1(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then*

$$\|\mathcal{F}\|_{L^q(\mathbb{R}_+, \nu)} \leq \nu(\text{supp } \mathcal{F}(f))^{1/q} \mu(\text{supp } f)^{1/q} \|f\|_{L^p(\mathbb{R}_+, \mu)},$$

where $q = \frac{p}{p-1}$.

Proof. Let $f \in L^1(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then by Hölder's inequality and (1.1), we get

$$\begin{aligned} \|\mathcal{F}(f)\|_{L^q(\mathbb{R}_+, \nu)} &\leq \nu(\text{supp } \mathcal{F}(f))^{1/q} \|\mathcal{F}(f)\|_\infty \\ &\leq \nu(\text{supp } \mathcal{F}(f))^{1/q} \|f\|_{L^1(\mathbb{R}_+, \mu)} \\ &\leq \nu(\text{supp } \mathcal{F}(f))^{1/q} \mu(\text{supp } f)^{1/q} \|f\|_{L^p(\mathbb{R}_+, \mu)}. \end{aligned} \quad \square$$

Proposition 3.3. *Let $f \in L^2(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then*

$$1 < \nu(\text{supp } \mathcal{F}(f))^{\frac{q-2}{2q}} \mu(\text{supp } f)^{\frac{2-p}{2p}},$$

where $q = \frac{p}{p-1}$.

Proof. Let $f \in L^2(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then by (1.1), Hölder's inequality and Riez's interpolation, we get

$$\begin{aligned} \|\mathcal{F}(f)\|_{L^2(\mathbb{R}_+, \nu)} &\leq \nu(\text{supp } \mathcal{F}(f))^{\frac{q-2}{2q}} \|\mathcal{F}(f)\|_{L^q(\mathbb{R}_+, \nu)} \\ &\leq \nu(\text{supp } \mathcal{F}(f))^{\frac{q-2}{2q}} \|f\|_{L^p(\mathbb{R}_+, \mu)} \\ &\leq \nu(\text{supp } \mathcal{F}(f))^{\frac{q-2}{2q}} \mu(\text{supp } f)^{\frac{2-p}{2p}} \|f\|_{L^2(\mathbb{R}_+, \mu)}, \end{aligned}$$

by Plancherel's formula we get the desired result. \square

Lemma 3.1. *Any nonzero function in $\mathcal{C}_0(\mathbb{R}_+^d)$ has linearly independent dilates.*

Proof. In the case $d = 1$ this Lemma was proved in [8]. The case $d > 1$ we reduce to the case $d = 1$. Let $f \in \mathcal{C}_0(\mathbb{R}_+^d)$ such that $f \neq 0$, if $x = (x_1, \dots, x_{d-1}, 0) = r(x', 0) \in \mathbb{R}_+^d$, $\sum_{i=1}^{d-1} |x'_i|^2 = 1$, $r \in \mathbb{R}_+$, we get $g(r) = f(r(x', 0))$.

If $x_d > 0$, $x = r(\theta x', 1)$, where $\sum_{i=1}^{d-1} |x'_i|^2 = 1$, and $r, \theta \in \mathbb{R}^+$, we get $g(r) = f(r(\theta x', 1))$. In both cases $g(r) \in \mathcal{C}_0(\mathbb{R}_+)$. \square

Lemma 3.2. *Let S_0 and Σ_0 be a pair of measurable subsets of \mathbb{R}_+ with $0 < \mu(S_0), \nu(\Sigma_0) < \infty$, then exist an infinite sequence of distinct numbers $(\rho_j)_{j=0}^\infty \subset (0, \infty)$ such that*

$$\mu(\cup_{j=0}^\infty \rho_j S_0) < 2\mu(S_0) \quad \text{and} \quad \nu\left(\cup_{j=0}^\infty \frac{1}{\rho_j} \Sigma_0\right) < 2\nu(\Sigma_0).$$

Proof. Let S_1 be a measurable subset of \mathbb{R}_+ of finite Lebesgue measure such that $S_0 \subset S_1$. Define $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $h(\rho) = \mu(S_1 \cup \rho S_0)$. Since $\chi_{\rho S_0}$ and χ_{S_1} are in $L^2(\mathbb{R}_+, \mu)$, we may express h in terms of scalar product in $L^2(\mathbb{R}_+, \mu)$

$$h(\rho) = \|\chi_{\rho S_0} - \chi_{S_1}\|_{L^2(\mathbb{R}_+, \mu)}^2 + \langle \chi_{\rho S_0}, \chi_{S_1} \rangle_{L^2(\mathbb{R}_+, \mu)}.$$

The function $\rho \mapsto h(\rho)$ is a continuous function on $(0, \infty)$. We deduce that there exist an infinite sequence of distinct numbers $(\rho_j)_{j=0}^\infty \subset (0, \infty)$, with $\rho_0 = 1$ such that $\mu(\cup_{j=0}^\infty \rho_j S_0) < 2\mu(S_0)$. We can follow the same techniques to prove that

$$\nu\left(\bigcup_{j=0}^\infty \frac{1}{\rho_j} \Sigma_0\right) < 2\nu(\Sigma_0). \quad \square$$

We are now in position to prove Benedicks-type theorem for the Sturm-Liouville transform.

Theorem 3.1. *Let S and Σ be a pair of measurable subsets of \mathbb{R}_+ , with $0 < \mu(S), \nu(\Sigma) < \infty$, then the pair (S, Σ) is weakly annihilating pair.*

Proof. Suppose that there exist $f_0 \neq 0$ such that $S_0 = \text{supp} f_0$ and $\Sigma_0 = \text{supp} \mathcal{F}(f_0)$ have both finite measure $0 < \mu(\text{supp} f_0), \nu(\text{supp} \mathcal{F}(f_0)) < \infty$. From Lemma 3.2 we can find an infinite sequence of distinct numbers $(\rho_j)_{j=0}^\infty \subset (0, \infty)$, with $\rho_0 = 1$, such that, if we denote by $S = \cup_{j=0}^\infty \rho_j \text{supp} f_0$ and $\Sigma = \cup_{j=0}^\infty \frac{1}{\rho_j} \text{supp} \mathcal{F}(f_0)$ we have $\mu(S) < 2\mu(S_0)$, $\nu(\Sigma) < 2\nu(\Sigma_0)$.

Put $f_i = D_{\rho_i} f_0$, so that $\text{supp} f_i = \rho_i \text{supp} f_0$. As $\mathcal{F}(f_i) = D_{\frac{1}{\rho_i}} \mathcal{F}(f_0)$ we have $\text{supp} \mathcal{F}(f_i) = \frac{1}{\rho_i} \text{supp} \mathcal{F}(f_0)$. Since $\text{supp} \mathcal{F}(f_0)$ has finite measure, $f_0 \in \mathcal{C}_0(\mathbb{R}_+)$. It follows from Lemma 3.1 that $(f_i)_{i=0}^\infty$ are linearly independent vectors belonging to $\text{Im} P_S \cap \text{Im} Q_\Sigma$ which contradicts (2.6). Then, (S, Σ) is weakly annihilating. \square

Theorem 3.2 (Benedicks-type theorem). *Let S and Σ be a pair of measurable subsets of \mathbb{R}_+ with $0 < \mu(S), \nu(\Sigma) < \infty$, then the pair (S, Σ) is strong annihilating pair.*

Proof. Assume there is no such constant $C(S, \Sigma)$. We can find a sequence $f_n \in L^2(\mathbb{R}_+, \mu)$ of norm 1 weakly convergent in $L^2(\mathbb{R}_+, \mu)$ with some limit f such that

$$\text{supp} f_n \subset S \quad \text{and} \quad \|\chi_{\Sigma^c} \mathcal{F}(f_n)\|_{L^2(\mathbb{R}_+, \nu)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\mathcal{F}(f_n)$ is the scalar product of f_n and $\chi_S \varphi_\lambda(\cdot)$, it follows that $\mathcal{F}(f_n)$ converge to $\mathcal{F}(f)$. Since $|\mathcal{F}(f_n)|$ as bounded by $\sqrt{\mu(S)}$, it follows from Lebesgue's theorem that $\mathcal{F}(f_n) \chi_\Sigma$ converges to $\mathcal{F}(f)$ in $L^2(\mathbb{R}_+, \nu)$ and the limit f has norm 1. But the function f has support in S and spectrum in Σ , since (S, Σ) is a weak annihilating pair, it follows that $f = 0$, which gives a contradiction. \square

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REFERENCES

- [1] A. Achak, R. Daher and H. Lahlali, *Beurling's theorem for Bessel-Struve transform*, C. R. Math. Acad. Sci. Paris **354**(1) (2016), 81–85.
- [2] M. Benedicks, *On Fourier transforms of function supported on sets of finite Lebesgue measure*, J. Math. Anal. Appl. **106** (1985), 180–183.
- [3] A. Beurling, *The Collect Works of Arne Beurling*, Birkhäuser, Boston, 1989.
- [4] A. Bonami and B. Demange, *A survey on uncertainty principles related to quadratic forms*, Collect. Math. **2** (2006), 1–36.
- [5] M. G. Cowling and J. F. Price, *Generalizations of Heisenberg Inequality*, Lecture Notes in Mathematics **992** Springer, Berlin, 1983, 443–449.
- [6] D. L. Donoho and P. B. Stark, *Uncertainty principles and signal recovery*, SIAM J. Appl. Math. **49** (1989), 906–931.
- [7] G. B. Folland and A. Sitaram, *The uncertainty principle a mathematical survey*, J. Fourier Anal. Appl. **3** (1997), 207–238.
- [8] S. Ghobber and Ph. Jaming, *Uncertainty principles for integral operators*, Studia Math. **220** (2014), 197–220.
- [9] G. H. Hardy, *A theorem concerning Fourier transform*, J. London Math. Soc. **8** (1933), 227–231.
- [10] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, Springer-Verlag, Berlin, 1994.
- [11] T. Kawazoe and H. Mejjaoli, *Uncertainty principles for the Dunkl transform*, Hiroshima Math. J. **40** (2010), 241–268.
- [12] H. J. Landau and W. L. Miranker, *The recovery of distorted band-limited signal*, J. Math. Anal. App. **2** (1961), 97–104.
- [13] H. J. Landau and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty II*, Bell System Technical Journal **40** (1961), 65–84.
- [14] H. Mejjaoli *Generalized Dunkl-Sobolev spaces of exponential type and applications*, Journal of Inequalities in Pure and Applied Mathematics **10**(2) (2009), 24 pages.
- [15] A. Miyachi, *A generalization of theorem of Hardy*, Harmonic Analysis Seminar, Izunagaoka, Shizuoka-Ken, Japan, 1997, 44–51.
- [16] N. Ben Salem. and A. Rashed Nasr, *Heisenberg-type inequalities for the Weinstein operator*, Integral Transforms Spec. Funct. **26**(9) 2015, 700–718.
- [17] A. Papoulis, *A new algorithm in spectral analysis and band-limited extrapolation*, IEEE Trans. Circuits Syst. I. Regul. Pap. **9** (1975), 735–742.
- [18] D. Slepian and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty I*, Bell System Technical Journal **40** (1961), 43–63.
- [19] F. Soltani, *L^p uncertainty principles on Sturm-Liouville Hypergroups*, Acta Math. Hungar. **142**(2) (2014), 433–443.
- [20] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **83** (1956), 482–492.
- [21] K. Tosio, *Perturbation Theory for Linear Operators*, Springer, New York, 1966.

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