Kragujevac Journal of Mathematics
Volume 47(4) (2023), Pages 613–625.

GENERALIZATION OF CERTAIN INEQUALITIES CONCERNING THE POLAR DERIVATIVE OF A POLYNOMIAL

IMTIAZ HUSSAIN[1] AND A. LIMAN[1]

Abstract. In this paper, we prove some more general results concerning the maximum modulus of the polar derivative of a polynomial. A variety of interesting results follow as special cases from our results.

1. Introduction

Let $\mathbb{P}_n$ denote the space of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n and let $P'(z)$ be its derivative then

$$\max_{|z|=1} \left|P'(z)\right| \leq n \max_{|z|=1} \left|P(z)\right|. \tag{1.1}$$

Inequality (1.1) is a famous result due to Bernstein (for reference see [3]) and is best possible with equality holds for $P(z) = \lambda z^n$, where λ is a complex number. Where as concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have (for reference see [15]),

$$\max_{|z|=R} \left|P(z)\right| \leq R^n \max_{|z|=1} \left|P(z)\right|, \quad R \geq 1. \tag{1.2}$$

Inequality (1.2) holds for $P(z) = \lambda z^n$, where λ is a complex number.

If we restrict ourselves to the class of polynomials $P \in \mathbb{P}_n$, with $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{2} \max_{|z|=1} \left|P(z)\right| \tag{1.3}$$

Key words and phrases. Polar derivative, maximum modulus, zeros, inequalities.
2010 *Mathematics Subject Classification.* Primary: 30A10, 30C10, 30C15.
DOI
Received: April 16, 2020.
Accepted: October 10, 2020.

and

$$\max_{|z|=R\geq 1}\Big|P(z)\Big| \leq \frac{R^n+1}{2}\max_{|z|=1}\Big|P(z)\Big|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [10], where as inequality (1.4) was proved by Ankeny and Rivlin [1].

Inequality (1.1) can be seen as a special case of the following inequality which is also due to Bernstein [3].

Theorem 1.1. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \leq |F(z)|$ for $|z| = 1$, then for $|z| \geq 1$, we have*

$$\Big|f'(z)\Big| \leq \Big|F'(z)\Big|. \tag{1.5}$$

Equality holds in (1.5) *for $f(z) = e^{i\eta}F(z)$, $\eta \in \mathbb{R}$.*

Inequality (1.1) can be obtained from inequality (1.5) by taking $F(z) = Mz^n$, where $M = \max_{|z|=1}|f(z)|$. In the same way, inequality (1.2) follows from the following result which is a special case of Bernstein-Walsh lemma [14], Corollary 12.1.3.

Theorem 1.2. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \leq |F(z)|$ for $|z| = 1$, then*

$$\Big|f(z)\Big| < \Big|F(z)\Big|, \quad \text{for } |z| > 1,$$

unless $f(z) = e^{i\eta}F(z)$ for some $\eta \in \mathbb{R}$.

In 2011, Govil et al. [4] proved a more general result which provides a compact generalization of inequalities (1.1), (1.2), (1.3) and (1.4) and includes Theorem 1.1 and Theorem 1.2 as special cases. In fact, they proved that if $f(z)$ and $F(z)$ are as in Theorem 1.1, then for any β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$\Big|f(Rz) - \beta f(rz)\Big| \leq \Big|F(Rz) - \beta F(rz)\Big|, \quad \text{for } |z| \geq 1. \tag{1.6}$$

Further, as a generalization of (1.6), Liman et al. [8] in the same year 2011 and under the same hypothesis as in Theorem 1.1, proved that

$$\begin{aligned}&\left|f(Rz) - \beta f(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}f(rz)\right|\\ \leq&\left|F(Rz) - \beta F(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}F(rz)\right|,\end{aligned} \tag{1.7}$$

for every $\beta, \gamma \in \mathbb{C}$ with $|\beta| \leq 1$, $|\gamma| \leq 1$ and $R > r \geq 1$.

Jain [6] proved a result concerning the minimum modulus of polynomials by showing that if $f \in \mathbb{P}_n$ and $f(z)$ has all its zeros in $|z| \leq 1$, then for every β with $|\beta| \leq 1$ and $R \geq 1$,

$$\min_{|z|=1}\left|f(Rz) + \beta\Big(\frac{R+1}{2}\Big)^n f(z)\right| \geq \left|R^n + \beta\Big(\frac{R+1}{2}\Big)^n\right|\min_{|z|=1}|f(z)|. \tag{1.8}$$

Mezerji et al. [13] besides proving some other results also obtained a generalization of (1.8) by proving that if $f \in \mathbb{P}_n$ and $f(z)$ has all its zeros in $|z| \le k, k \le 1$, then for every $|\beta| \le 1$ and $R \ge 1$

$$\min_{|z|=1}\left|f(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n f(z)\right| \ge \frac{1}{k^n}\left|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\right|\min_{|z|=1}|f(z)|. \tag{1.9}$$

Recently, Kumar [7] found that there is a room for the generalization of the condition $R \ge 1$ in (1.8) and (1.9) to $R \ge r > 0$ and proved that if $f \in \mathbb{P}_n$ and $f(z)$ has all its zeros in $|z| \le k$, $k > 0$, then for every β with $|\beta| \le 1$, $|z| \ge 1$ and $R \ge r$, $Rr \ge k^2$,

$$\min_{|z|=1}\left|f(Rz)+\beta\Big(\frac{R+k}{r+k}\Big)^n f(rz)\right| \ge \frac{1}{k^n}\left|R^n+\beta r^n\Big(\frac{R+k}{r+k}\Big)^n\right|\min_{|z|=k}|f(z)|. \tag{1.10}$$

For $f \in \mathbb{P}_n$, let $D_\alpha f(z)$ denote the polar derivative of $f(z)$ of degree n with respect to α (see [11]) then

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z).$$

The polynomial $D_\alpha f(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty}\frac{D_\alpha f(z)}{\alpha} := f'(z)$$

uniformly with respect to z for $|z| \le R$, $R > 0$.

The latest development and research can be found in the papers by Jiraphorn Somsuwan and Meneeruk Nakprasit [16] and Abdullah Mir [12].

Recently, Liman et al. [9] besides proving some other results also proved the following generalization of (1.6) and (1.7) to the polar derivative $D_\alpha f(z)$ of a polynomial $f(z)$ with respect to α, $|\alpha| \ge 1$.

Theorem 1.3. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \le 1$ and $f(z)$ be a polynomial of degree $m(\le n)$ such that $|f(z)| \le |F(z)|$ for $|z| = 1$. If $\alpha, \beta, \gamma \in \mathbb{C}$ be such that $|\alpha| \ge 1$, $|\beta| \le 1$ and $|\lambda| < 1$, then for $R > r \ge 1$ and $|z| \ge 1$, we have*

$$\begin{aligned}&\left|z\Big[(n-m)\Big\{f(Rz)-\beta f(rz)\Big\}+D_\alpha f(Rz)-\beta D_\alpha f(rz)\Big]\right.\\&\left.\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)-\beta f(rz)\Big\}\right|\\&\le\left|z\Big\{D_\alpha F(Rz)-\beta D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)-\beta F(rz)\Big\}\right|.\end{aligned} \tag{1.11}$$

Equality holds in (1.11) *for $f(z) = e^{i\eta}F(z)$, $\eta \in \mathbb{R}$.*

2. Main Results

The main aim of this paper is to obtain some more general results for the maximum modulus of the polar derivative of a polynomial under certain constraints on the zeros and on the functions considered. We first prove the following generalization of inequalities (1.6) and (1.7) and of Theorem 1.3.

Theorem 2.1. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq k, k > 0$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\Big|f(z)\Big| \leq |F(z)|, \quad \text{for } |z| = k. \tag{2.1}$$

If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1$, $|\beta| \leq 1$, $|\gamma| \leq 1$ and $|\lambda| < 1$, then for $R > r$, $rR \geq k^2$ and $|z| \geq 1$, we have

$$\begin{aligned}&\Bigg|z\Bigg[(n-m)\Bigg\{f(Rz) + \psi f(rz)\Bigg\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Bigg] \\ &\quad + \frac{n\lambda}{2}(|\alpha|-1)\Bigg\{f(Rz) + \psi f(rz)\Bigg\}\Bigg| \\ &\leq \Bigg|z\Bigg\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Bigg\} + \frac{n\lambda}{2}(|\alpha|-1)\Bigg\{F(Rz) + \psi F(rz)\Bigg\}\Bigg|,\end{aligned} \tag{2.2}$$

where

$$\psi = \psi_k(R, r, \beta, \gamma) = \gamma\Bigg\{\Big(\frac{R+k}{r+k}\Big)^n - |\beta|\Bigg\}. \tag{2.3}$$

The result is sharp and equality in (2.2) holds for $f(z) = e^{i\eta}F(z)$, η is real and $F(z)$ has all its zeros in $|z| \leq k$.

We now present and discuss some consequences of Theorem 2.1. Suppose $f \in \mathbb{P}_n$ and $f(z) \neq 0$ in $|z| < k$, the polynomial $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})} \in \mathbb{P}_n$ and $Q(z)$ has all its zeros in $|z| \leq \frac{1}{k}$. Note that

$$|Q(z)| = \frac{1}{k^n}|f(k^2 z)|, \quad \text{for } |z| = \frac{1}{k}.$$

Applying Theorem 2.1 with $F(z)$ replaced by $k^n Q(z)$, we get the following corollary.

Corollary 2.1. *If $f \in \mathbb{P}_n$ and $f(z) \neq 0$ in $|z| < k$, $k > 0$, then for every $|\alpha| \geq 1$, $|\beta| \leq 1$, $|\gamma| \leq 1$ and $|\lambda| < 1$, we have for $R > r$, $rR \geq \frac{1}{k^2}$ and $|z| \geq 1$,*

$$\begin{aligned}&\Bigg|z\Bigg\{D_\alpha f(Rk^2 z) + \phi D_\alpha f(rk^2 z)\Bigg\} + \frac{n\lambda}{2}(|\alpha|-1)\Bigg\{f(Rk^2 z) + \phi f(rk^2 z)\Bigg\}\Bigg| \\ &\leq k^n\Bigg|z\Bigg\{D_\alpha Q(Rz) + \phi D_\alpha Q(rz)\Bigg\} + \frac{n\lambda}{2}(|\alpha|-1)\Bigg\{Q(Rz) + \phi Q(rz)\Bigg\}\Bigg|,\end{aligned} \tag{2.4}$$

$Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$ and

$$\phi = \phi_k(R, r, \beta, \gamma) = \gamma\Bigg\{\Big(\frac{Rk+1}{rk+1}\Big)^n - |\beta|\Bigg\}. \tag{2.5}$$

Equality holds in (2.4) for $f(z) = e^{i\eta}Q(z), \eta \in \mathbb{R}$.

Remark 2.1. For $k=1$ and $\gamma=0$, Corollary 2.1 in particular yields a result of Liman et al. [9, Corollary 1.4]. Taking $\beta=\lambda=0$ in Corollary 2.1 we get the following result.

Corollary 2.2. *If $f\in\mathbb{P}_n$ and $f(z)\neq 0$ in $|z|<k$, $k>0$, then for every $|\alpha|\geq 1$, $|\gamma|\leq 1$, we have for $R>r$, $rR\geq\frac{1}{k^2}$ and $|z|\geq 1$,*

$$\begin{aligned}&\left|D_\alpha f(Rk^2z)+\gamma\Big(\frac{Rk+1}{rk+1}\Big)^n D_\alpha f(rk^2z)\right|\\ &\leq k^n\left|D_\alpha Q(Rz)+\gamma\Big(\frac{Rk+1}{rk+1}\Big)D_\alpha Q(rz)\right|,\end{aligned}\tag{2.6}$$

$Q(z)=z^n\overline{f(\frac{1}{\bar z})}$.

Inequality (2.6) should be compared with a result recently proved by Kumar [7, Lemma 2.2], where $f(z)$ is replaced by $D_\alpha f(z)$, $|\alpha|\geq 1$.

Remark 2.2. For $r=1$, Corollary 2.2 gives the polar derivative analog of a result due to Mezerji et al. ([13], Lemma 4). If we take $\beta=0$ in Theorem 2.1 we get the following.

Corollary 2.3. *Let $F\in\mathbb{P}_n$, having all zeros in $|z|\leq k$, $k>0$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\Big|f(z)\Big|\leq\Big|F(z)\Big|,\quad \text{for } |z|=k.$$

If $\alpha,\gamma,\lambda\in\mathbb{C}$ be such that $|\alpha|\geq 1$, $|\gamma|\leq 1$ and $|\lambda|<1$, then for $R>r$, $rR\geq k^2$ and $|z|\geq 1$, we have

$$\begin{aligned}&\Bigg|z\Bigg[(n-m)\Big\{f(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n f(rz)\Big\}+D_\alpha f(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n D_\alpha f(rz)\Bigg]\\ &\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n f(rz)\Big\}\Bigg|\\ &\leq\Bigg|z\Big\{D_\alpha F(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n D_\alpha F(rz)\Big\}\\ &\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\Big(\frac{R+k}{r+k}\Big)^n F(rz)\Big\}\Bigg|.\end{aligned}\tag{2.7}$$

Equality holds in (2.7) for $f(z)=e^{i\eta}F(z)$, $\eta\in\mathbb{R}$ and $F(z)$ has all its zeros in $|z|\leq k$.

If we apply Theorem 2.1 to polynomials $f(z)$ and $\frac{z^n}{k^n}\min_{|z|=k}|f(z)|$, we get the following result.

Corollary 2.4. *If $f\in\mathbb{P}_n$ and $f(z)$ has all its zeros in $|z|\leq k, k>0$, then for every $\alpha,\beta,\gamma,\lambda\in\mathbb{C}$ such that $|\alpha|\geq 1$, $|\beta|\leq 1$, $|\gamma|\leq 1$ and $|\lambda|<1$, we have for $R>r$, $rR\geq k^2$ and $|z|\geq 1$,*

$$\left|z\Big\{D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\right|$$

$$\geq \frac{n|z|^n}{k^n}\left|\alpha(R^{n-1}+\psi r^{n-1})+\frac{\lambda}{2}(|\alpha|-1)(R^n+\psi r^n)\right|\min_{|z|=k}|f(z)|, \tag{2.8}$$

where ψ is defined by the equation (2.3). Equality holds in (2.8) for $f(z)=az^n$, $a\neq 0$.

Taking $\lambda=0$ in Corollary 2.4 we get the following result.

Corollary 2.5. *If $f\in\mathbb{P}_n$ and $f(z)$ has all its zeros in $|z|\leq k$, $k>0$, then for every $\alpha,\beta,\gamma,\in\mathbb{C}$ such that $|\alpha|\geq 1$, $|\beta|\leq 1$, $|\gamma|\leq 1$ and for $R>r$, $rR\geq k^2$, we have*

$$\min_{|z|=1}\left|D_\alpha f(Rz)+\psi D_\alpha f(rz)\right|\geq \frac{n|\alpha|}{k^n}\left|R^{n-1}+\psi r^{n-1}\right|\min_{|z|=k}|f(z)|, \tag{2.9}$$

ψ is defined by the equation (2.3). Equality holds in (2.8) for $f(z)=az^n$, $a\neq 0$.

Remark 2.3. For $\beta=0$, the above inequality (2.9) gives the polar derivative analog of (1.10).

Theorem 2.2. *Let $F\in\mathbb{P}_n$, having all its zeros in $|z|\leq k$, $k>0$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\left|f(z)\right|\leq|F(z)|, \quad \text{for } |z|=k.$$

If $\alpha,\beta,\gamma,\in\mathbb{C}$ be such that $|\alpha|\geq 1$, $|\beta|\leq 1$ and $|\gamma|\leq 1$, then for $R>r$, $rR\geq k^2$ and $|z|\geq 1$, we have

$$\begin{aligned}&\left|z\left[(n-m)\left\{f(Rz)+\psi f(rz)\right\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\right]\right|\\&\quad+\frac{n}{2}(|\alpha|-1)\left|F(Rz)+\psi F(rz)\right|\\&\leq\left|z\left\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\right\}\right|+\frac{n}{2}(|\alpha|-1)\left|f(Rz)+\psi f(rz)\right|,\end{aligned} \tag{2.10}$$

where ψ is defined by the equation (2.3). Equality holds in (2.10) for $f(z)=e^{i\eta}F(z)$, $\eta\in\mathbb{R}$ and $F(z)$ has all its zeros in $|z|\leq k$.

Remark 2.4. $\gamma=0$ and $k=1$, Theorem 2.2 gives in particular a result of Liman et al. [9, Theorem 2]. From Theorem 2.2 we have the following.

Corollary 2.6. *If $f\in\mathbb{P}_n$, and $f(z)$ does not vanish in $|z|<k$, $k>0$, then for every $\alpha,\beta,\gamma\in\mathbb{C}$ with $|\alpha|\geq 1$, $|\beta|\leq 1$, $|\gamma|\leq 1$, we have for $R>r$, $rR\geq\frac{1}{k^2}$ and $|z|\geq 1$,*

$$\begin{aligned}&\left|z\left\{D_\alpha f(Rk^2z)+\phi D_\alpha f(rk^2z)\right\}\right|+\frac{n}{2}(|\alpha|-1)k^n\left|Q(Rz)+\phi Q(rz)\right|\\&\leq k^n\left|z\left\{D_\alpha Q(Rz)+\phi D_\alpha Q(rz)\right\}\right|+\frac{n}{2}(|\alpha|-1)\left|f(Rk^2z)+\phi f(rk^2z)\right|,\end{aligned} \tag{2.11}$$

where $Q(z)=z^n\overline{f(\frac{1}{\bar z})}$ and ϕ is defined by the equation (2.5).

Remark 2.5. We recover a result of Liman et al. [9, Corollary 2.3] from Corollary 2.5 when we take $\gamma=0$ and $k=1$.

3. Lemmas

We need the following lemmas to prove our theorems. The first lemma is due to Aziz and Zargar [2].

Lemma 3.1. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq k$, $k \geq 0$, then for every $R > r$, $rR \geq k^2$*

$$\Big|f(Rz)\Big| > \left(\frac{R+k}{r+k}\right)^n \Big|f(rz)\Big|, \quad \textit{for } |z| = 1.$$

Lemma 3.2. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$,*

$$2\Big|zD_\alpha f(z)\Big| \geq n(|\alpha| - 1)\Big|f(z)\Big|, \quad \textit{for } |z| = 1.$$

The above lemma is due to Shah [17].

Lemma 3.3. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq k$, then for $|\alpha| \geq k$, the polar derivative*

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z),$$

of $f(z)$ at the point α also has all its zeros in $|z| \leq k$.

The above lemma is due to Laguerre [11, page 49].

4. Proof of the Theorems

Proof of Theorem 2.1. By hypothesis, $F(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ and $f(z)$ is a polynomial of degree at most n such that

$$\Big|f(z)\Big| \leq |F(z)|, \quad \text{for } |z| = k, \tag{4.1}$$

therefore, if $F(z)$ has a zero of multiplicity ν at $z = ke^{i\theta_0}$, then $f(z)$ must also have a zero of multiplicity at least ν at $z = ke^{i\theta_0}$. We assume that $\frac{f(z)}{F(z)}$ is not a constant, otherwise, the inequality (2.2) is obvious. It follows by the maximum modulus principle that

$$\Big|f(z)\Big| < |F(z)|, \quad \text{for } |z| > k.$$

Suppose $F(z)$ has m zeros on $|z| = k$, where $0 \leq m < n$, so that we can write

$$F(z) = F_1(z)F_2(z),$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree $n - m$ whose all zeros lie in $|z| < k$. This gives with the help of (4.1) that

$$f(z) = P_1(z)F_1(z),$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Now, from inequality (4.1), we get

$$|P_1(z)| \leq |F_2(z)|, \quad \text{for } |z| = k,$$

and $F_2(z) \neq 0$ for $|z| = k$. Therefore, for a given complex number δ with $|\delta| > 1$, it follows from Rouche's theorem that the polynomial $P_1(z) - \delta F_2(z)$ of degree $n-m \geq 1$ has all its zeros in $|z| < k$. Hence, the polynomial

$$P(z) = F_1(z)(P_1(z) - \delta F_2(z)) = f(z) - \delta F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$P(z) = (z - \eta e^{i\gamma})H(z),$$

where $\eta < k$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leq k$. Applying Lemma 3.1 to $H(z)$, we obtain for $R > r$, $rR \geq k^2$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}\left|P(Re^{i\theta})\right| =& \left|Re^{i\theta} - \eta e^{i\gamma}\right|\left|H(Re^{i\theta})\right| \\ >& \left|Re^{i\theta} - \eta e^{i\gamma}\right|\left(\frac{R+k}{r+k}\right)^{n-1}\left|H(re^{i\theta})\right| \\ =& \left(\frac{R+k}{r+k}\right)^{n-1}\frac{\left|Re^{i\theta} - \eta e^{i\gamma}\right|}{\left|re^{i\theta} - \eta e^{i\gamma}\right|}\left|re^{i\theta} - \eta e^{i\gamma}\right|\left|H(re^{i\theta})\right|. \end{aligned} \tag{4.2}$$

Now for $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta} - \eta e^{i\gamma}}{re^{i\theta} - \eta e^{i\gamma}}\right|^2 =& \frac{R^2 + \eta^2 - 2R\eta\cos(\theta - \gamma)}{r^2 + \eta^2 - 2r\eta\cos(\theta - \gamma)} \\ \geq& \left(\frac{R+\eta}{r+\eta}\right)^2, \quad \text{for } R > r \text{ and } rR \geq k^2 \\ >& \left(\frac{R+k}{r+k}\right)^2, \quad \text{since } \eta < k. \end{aligned}$$

This implies

$$\left|\frac{Re^{i\theta} - \eta e^{i\gamma}}{re^{i\theta} - \eta e^{i\gamma}}\right| > \frac{R+k}{r+k},$$

which on using in (4.2) gives for $R > r$, $rR \geq k^2$ and $0 \leq \theta < 2\pi$,

$$\left|P(Re^{i\theta})\right| > \left(\frac{R+k}{r+k}\right)^n \left|P(re^{i\theta})\right|.$$

Equivalently,

$$\left|P(Rz)\right| > \left(\frac{R+k}{r+k}\right)^n \left|P(rz)\right|, \tag{4.3}$$

for $R > r$, $rR \geq k^2$ and $|z| = 1$. This implies for every $|\beta| \leq 1$, $R > r$, $rR \geq k^2$ and $|z| = 1$,

$$\left|P(Rz) - \beta P(rz)\right| \geq \left|P(Rz)\right| - |\beta|\left|P(rz)\right| > \left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\}\left|P(rz)\right|. \tag{4.4}$$

Again, since $r < R$, it follows that $\left(\frac{r+k}{R+k}\right)^n < 1$, inequality (4.3) implies that

$$|P(rz)| < |P(Rz)|, \quad \text{for } |z| = 1.$$

Also, all the zeros of $P(Rz)$ lie in $|z| \leq \frac{k}{R}$ and $R^2 > rR \geq k^2$, we have $\frac{k}{R} < 1$. A direct application of Rouche's theorem shows that the polynomial $P(Rz) - \beta f(rz)$ has all its zeros in $|z| < 1$, for every $|\beta| \leq 1$. Applying Rouche's theorem again, it follows from (4.4) that for every $|\gamma| \leq 1$, $|\beta| \leq 1$, $R > r$, $rR \geq k^2$, all the zeros of the polynomial

$$g(z) := P(Rz) - \beta P(rz) + \gamma\left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\}P(rz) = P(Rz) + \psi P(rz) \tag{4.5}$$

lie in $|z| < 1$. Using Lemma 3.2 we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $|z| = 1$

$$2\Big|zD_\alpha g(z)\Big| \geq n(|\alpha| - 1)\Big|g(z)\Big|.$$

Hence, for any complex number λ with $|\lambda| < 1$, we have for $|z| = 1$,

$$2\Big|zD_\alpha g(z)\Big| > n|\lambda|(|\alpha| - 1)\Big|g(z)\Big|.$$

Therefore, it follows by Lemma 3.3 that all the zeros of

$$\begin{aligned} W(z) :=& 2zD_\alpha g(z) + n\lambda(|\alpha| - 1)g(z) \\ =& 2zD_\alpha P(Rz) + 2z\psi D_\alpha P(rz) + n\lambda\Big(|\alpha| - 1\Big)\Big(P(Rz) + \psi P(rz)\Big) \end{aligned} \tag{4.6}$$

lie in $|z| < 1$.

Replacing $P(z)$ by $f(z) - \delta F(z)$ and using definition of polar derivative give

$$\begin{aligned} W(z) =& 2z\Bigg[n\Big\{f(Rz) - \delta F(Rz)\Big\} + (\alpha - Rz)\Big\{f(Rz) - \delta F(Rz)\Big\}'\Bigg] \\ &+ 2z\psi\Bigg[n\Big\{f(rz) - \delta F(rz)\Big\} + (\alpha - rz)\Big\{f(rz) - \delta F(rz)\Big\}'\Bigg] \\ &+ n\lambda(|\alpha| - 1)\Big\{f(Rz) - \delta F(Rz)\Big\} + n\lambda\psi(|\alpha| - 1)\Big\{f(rz) - \delta F(rz)\Big\}, \end{aligned}$$

which on simplification gives

$$\begin{aligned} W(z) =& 2z\Bigg[(n - m)f(Rz) + mf(Rz) + (\alpha - Rz)\Big(f(Rz)\Big)' \\ &- \delta\Big\{nF(rz) + (\alpha - rz)\Big(F(Rz)\Big)'\Big\}\Bigg] \\ &+ 2z\psi\Bigg[(n - m)f(rz) + mf(rz) + (\alpha - rz)\Big(f(rz)\Big)' \\ &- \delta\Big\{nF(rz) + (\alpha - rz)\Big(F(rz)\Big)'\Big\}\Bigg] \\ &+ n\lambda(|\alpha| - 1)\Big\{f(Rz) - \delta F(Rz)\Big\} + n\lambda\psi(|\alpha| - 1)\Big\{f(rz) - \delta F(rz)\Big\} \end{aligned}$$

$$
\begin{aligned}
&=2z\Big\{\Big(n-m)f(Rz)+D_\alpha f(Rz)-\delta D_\alpha F(Rz)\Big\}\\
&\quad+2z\psi\Big\{\Big(n-m)f(rz)+D_\alpha f(rz)-\delta D_\alpha F(rz)\Big\}\\
&\quad+n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\}\\
&=2z\Big\{(n-m)f(Rz)+\psi(n-m)f(rz)+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big\}\\
&\quad+n\lambda\psi(|\alpha|-1)f(Rz)+n\lambda\psi(|\alpha|-1)f(rz)\\
&\quad-\delta\Big\{2zD_\alpha F(Rz)+2z\psi D_\alpha F(rz)\\
&\quad+n\lambda(|\alpha|-1)F(Rz)+n\lambda\psi(|\alpha|-1)F(rz)\Big\}.
\end{aligned}
\tag{4.7}
$$

Since by (4.6), $W(z)$ has all its zeros in $|z|<1$, therefore, by (4.7), we get for $|z|\geq 1$

$$
\begin{aligned}
&\Bigg|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big]\\
&\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\Bigg|\\
&\leq\Bigg|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\Bigg|.
\end{aligned}
\tag{4.8}
$$

To see that the inequality (4.8) holds, note that if the inequality (4.8) is not true, then there is a point $z=z_0$ with $|z_0|\geq 1$, such that

$$
\begin{aligned}
&\Bigg|z_0\Big[(n-m)\Big\{f(Rz_0)+\psi f(rz_0)\Big\}+D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]\\
&\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz_0)+\psi f(rz_0)\Big\}\Bigg|\\
&>\Bigg|z_0\Big\{D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}\Bigg|.
\end{aligned}
\tag{4.9}
$$

Now, because by hypothesis all the zeros of $F(z)$ lie in $|z|\leq k$, the polynomial $F(Rz)$ has all its zeros in $|z|\leq\frac{k}{R}<1$, and therefore, if we use Rouche's theorem and Lemmas 3.1 and 3.3 and argument similar to the above we will get that all the zeros of

$$z\Big(D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}$$

lie in $|z|<1$ for every $|\alpha|\geq 1$, $|\lambda|<1$ and $R>r$, $rR\geq k^2$, that is,

$$z\Big(D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}\neq 0,$$

for every z_0 with $|z_0| \geq 1$. Therefore, if we take

$$\delta = \frac{z_0\Big[(n-m)\Big\{f(Rz_0)+\psi f(rz_0)\Big\} + D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]}{z_0\Big(D_\alpha F(Rz_0)+\psi F(rz_0)D_\alpha\Big) + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}}$$
$$+ \frac{\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz_0)+\psi f(rz_0)\Big\}}{z_0\Big(D_\alpha F(Rz_0)+\psi F(rz_0)D_\alpha\Big) + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}},$$

then δ is a well-defined real or complex number, and in view of (4.9) we also have $|\delta| > 1$. Hence, with the choice of δ, we get from (4.7) that $W(z_0) = 0$ for some z_0, satisfying $|z_0| \geq 1$, which is clearly a contradiction to the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus for every $R > r$, $rR \geq k^2$, $|\alpha| \geq 1$, $|\lambda| < 1$ and $|z| \geq 1$, inequality (4.8) holds and this completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Since all the zeros of $F(z)$ lie in $|z| \leq k$, $k > 0$, for $R > r$, $rR \geq k^2$, $|\beta| \leq 1$, $|\gamma| \leq 1$, it follows as in the proof of Theorem 2.1, that all the zeros of

$$h(z) := F(Rz) - \beta F(rz) + \gamma\Bigg\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\Bigg\}F(rz) = F(Rz) + \psi F(rz)$$

lie in $|z| < 1$. Hence, by Lemma 3.2 we get for $|\alpha| \geq 1$,

$$2\Big|zD_\alpha h(z)\Big| \geq n(|\alpha|-1)\Big|h(z)\Big|, \quad \text{for } |z| \geq 1.$$

This gives for every λ with $|\lambda| < 1$

$$\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}\Big| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz)+\psi F(rz)\Big| \geq 0, \tag{4.10}$$

for $|z| \geq 1$. Therefore, it is possible to choose the argument of λ in the right hand side of (4.8) such that for $|z| \geq 1$

$$\begin{aligned}&\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\Big|\\ &=\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}\Big| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz)+\psi F(rz)\Big|.\end{aligned} \tag{4.11}$$

Hence, from (4.8), we get by using (4.11) for $|z| \geq 1$

$$\begin{aligned}&\Big|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\} + D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big]\Big|\\ &- \frac{n|\lambda|}{2}(|\alpha|-1)\Big|f(Rz)+\psi f(rz)\Big|\\ &\leq\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}\Big| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz)+\psi F(rz)\Big|.\end{aligned} \tag{4.12}$$

Letting $|\lambda| \to 1$ in (4.12), we immediately get (2.10) and this proves Theorem 2.2 completely. $\square$

Acknowledgements. The work is sponsored by National Board of Higher Mathematics (NBHM), Department of Atomic Energy, Government of India under the Post Doctoral Fellowship reference No. 0204/52/2019/R&D-II/315.

The authors are highly thankful to the referees for the valuable suggestions regarding the paper.

References

[1] N. C. Ankeny and T. J. Rivlin, *On a theorem of S. Bernstein*, Pacific J. Math. **5**(2) (1955), 849–852.

[2] A. Aziz and B. A. Zargar, *Inequalities for a polynomial and its derivative*, Math. Ineq. Appl. **4**(1) (1998), 543–550.

[3] S. Bernstein, *Sur la limitation des derivées des polynomes,* Comptes Rendus de l' Academic des Sciences (Paris) **190** (1930), 338–340.

[4] N. K. Govil, A. Liman and W. M. Shah, *Some inequalities concerning derivative and maximum modulus of polynomials*, Aust. J. Math. Anal. Appl. **8** (2011), 1–8.

[5] N. K. Govil, M. A. Qazi and Q. I. Rahman, *Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius*, Math. Ineq. Appl. **6**(3) (2003), 453–467.

[6] V. K. Jain, *Inequalities for a polynomial and its derivative*, Proc. Acad. Sci. (Math. Sci.) **32**(2) (1997), 45–52.

[7] P. N. Kumar, *On the generalization of polynomial inequalities in the complex domain,* J. Contemp. Math. Anal. **50**(1) (2015), 14–21. `https://doi.org/10.3103/S1068362315010021`

[8] A. Liman, R. N. Mohapatra and W. M. Shah, *Inequalities for polynomials not vanishing in a disk*, Appl. Math. Comput. **218**(3) (2011), 949–955. `https://DOI:10.1016/j.amc.2011.01.077`

[9] A. Liman, I. Q. Peer and W. M. Shah, *On some inequalities concerning the polar derivative of a polynomial*, Ramanujan J. **38**(2) (2015), 349–360. `https://doi.org/10.1007/s11139-014-9640-1`

[10] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513. `https://DOI:10.1090/S0002-9904-1944-08177-9`

[11] M. Marden, *The TeXbook*, Math. Surveys **3**, Amer. Math. Soc., Providence, RI, 1966.

[12] A. Mir, *Some sharp upper bound estimates for the maximal modulus of polar derivative of a polynomial*, Annali Dell'Universita'Di Ferrara **65** (2019), 327–336. `https://doi.org/10.1007/s11565-019-00317-2`

[13] H. A. S. Mezerji, M. A. Baseri, M. Bidhkam and A. Zireh, *Generalization of certain inequalities for a polynomial and its derivative,* Lobachevskii J. Math. **33**(1) (2012), 68–74.

[14] Q. I. Rahman and G. Schmeisser, *The TeXbook,* Oxford University Press, New York, 2002.

[15] M. Riesz, *Über einen satz des Herrn Serge Bernstein,* Acta Math. **40** (1916), 337–347. `https://doi.org/10.1007/BF02418550`

[16] J. Somsuwan and M. Nakprasit, *Some bounds for the polar derivative of a polynomial,* Int. J. Math. Math. Sci. (2018), Article ID 5034607. `https://doi.org/10.1155/2018/5034607`

[17] W. M. Shah *A generalization of a theorem of Paul Turán*, J. Ramanujan Math. Soc. **1** (1996), 67–72.

[1]Department of Mathematics,
National Institute of Technology,
Srinagar–190006, J&K, India
Email address: dar.imtiaz5@gmail.com
Email address: abliman@rediffmail.com