

**BASIC INEQUALITIES FOR  $(m, M)$ - $\Psi$ -CONVEX FUNCTIONS  
WHEN  $\Psi = -\ln$**

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ABSTRACT. In this paper we establish some basic inequalities for  $(m, M)$ - $\Psi$ -convex functions when  $\Psi = -\ln$ . Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

1. INTRODUCTION

Assume that the function  $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  ( $I$  is an interval) is convex on  $I$  and  $m \in \mathbb{R}$ . We shall say that the function  $\Phi : I \rightarrow \mathbb{R}$  is  $m$ - $\Psi$ -lower convex if  $\Phi - m\Psi$  is a convex function on  $I$ . We may introduce the class of functions (see [1])

$$(1.1) \quad \mathcal{L}(I, m, \Psi) := \{\Phi : I \rightarrow \mathbb{R} \mid \Phi - m\Psi \text{ is convex on } I\}.$$

Similarly, for  $M \in \mathbb{R}$  and  $\Psi$  as above, we can introduce the class of  $M$ - $\Psi$ -upper convex functions by (see [1])

$$(1.2) \quad \mathcal{U}(I, M, \Psi) := \{\Phi : I \rightarrow \mathbb{R} \mid M\Psi - \Phi \text{ is convex on } I\}.$$

The intersection of these two classes will be called the class of  $(m, M)$ - $\Psi$ -convex functions and will be denoted by (see [1])

$$(1.3) \quad \mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi).$$

*Remark 1.1.* If  $\Phi \in \mathcal{B}(I, m, M, \Psi)$ , then  $\Phi - m\Psi$  and  $M\Psi - \Phi$  are convex and then  $(\Phi - m\Psi) + (M\Psi - \Phi)$  is also convex which shows that  $(M - m)\Psi$  is convex, implying that  $M \geq m$  (as  $\Psi$  is assumed not to be the trivial convex function  $\Psi(t) = 0$ ,  $t \in I$ ).

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The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [7], S. S. Dragomir and N. M. Ionescu introduced the concept of *g-convex dominated functions*, for a function  $f : I \rightarrow \mathbb{R}$ . We recall this, by saying, for a given convex function  $g : I \rightarrow \mathbb{R}$ , the function  $f : I \rightarrow \mathbb{R}$  is *g-convex dominated* iff  $g + f$  and  $g - f$  are convex functions on  $I$ . In [7], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of *g-convex dominated functions* can be obtained as a particular case from  $(m, M)$ - $\Psi$ -convex functions by choosing  $m = -1$ ,  $M = 1$  and  $\Psi = g$ .

The following lemma holds (see [1]).

**Lemma 1.1.** *Let  $\Psi, \Phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be differentiable functions on  $\overset{\circ}{I}$ , the interior of  $I$  and  $\Psi$  is a convex function on  $\overset{\circ}{I}$ .*

(i) *For  $m \in \mathbb{R}$ , the function  $\Phi \in \mathcal{L}(\overset{\circ}{I}, m, \Psi)$  if and only if*

$$(1.4) \quad m(\Psi(t) - \Psi(s) - \Psi'(s)(t-s)) \leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s),$$

*for all  $t, s \in \overset{\circ}{I}$ .*

(ii) *For  $M \in \mathbb{R}$ , the function  $\Phi \in \mathcal{U}(\overset{\circ}{I}, M, \Psi)$  if and only if*

$$(1.5) \quad \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \leq M(\Psi(t) - \Psi(s) - \Psi'(s)(t-s)),$$

*for all  $t, s \in \overset{\circ}{I}$ .*

(iii) *For  $M, m \in \mathbb{R}$  with  $M \geq m$ , the function  $\Phi \in \mathcal{B}(\overset{\circ}{I}, m, M, \Psi)$  if and only if both (1.4) and (1.5) hold.*

Another elementary fact for twice differentiable functions also holds (see [1]).

**Lemma 1.2.** *Let  $\Psi, \Phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable on  $\overset{\circ}{I}$  and  $\Psi$  is convex on  $\overset{\circ}{I}$ .*

(i) *For  $m \in \mathbb{R}$ , the function  $\Phi \in \mathcal{L}(\overset{\circ}{I}, m, \Psi)$  if and only if*

$$(1.6) \quad m\Psi''(t) \leq \Phi''(t), \quad \text{for all } t \in \overset{\circ}{I}.$$

(ii) *For  $M \in \mathbb{R}$ , the function  $\Phi \in \mathcal{U}(\overset{\circ}{I}, M, \Psi)$  if and only if*

$$(1.7) \quad \Phi''(t) \leq M\Psi''(t), \quad \text{for all } t \in \overset{\circ}{I}.$$

(iii) *For  $M, m \in \mathbb{R}$  with  $M \geq m$ , the function  $\Phi \in \mathcal{B}(\overset{\circ}{I}, m, M, \Psi)$  if and only if both (1.6) and (1.7) hold.*

For various inequalities concerning these classes of function, see the survey paper [3].

In what follows, we consider the class of functions  $\mathcal{B}(I, m, M, -\ln)$  for  $M, m \in \mathbb{R}$ , with  $M \geq m$  that is obtained for  $\Psi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ ,  $\Psi(t) = -\ln t$ .

If  $\Phi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  is a differentiable function on  $\overset{\circ}{I}$  then by Lemma 1.1 we have  $\Phi \in \mathcal{B}(I, m, M, -\ln)$  if and only if

$$(1.8) \quad m \left( \ln s - \ln t - \frac{1}{s}(s-t) \right) \leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \\ \leq M \left( \ln s - \ln t - \frac{1}{s}(s-t) \right),$$

for any  $s, t \in \overset{\circ}{I}$ .

If  $\Phi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  is a twice differentiable function on  $\overset{\circ}{I}$  then by Lemma 1.2 we have  $\Phi \in \mathcal{B}(I, m, M, -\ln)$  if and only if

$$(1.9) \quad m \leq t^2 \Phi''(t) \leq M,$$

which is a convenient condition to verify in applications.

In this paper we establish some basic inequalities for  $(m, M)$ - $\Psi$ -convex functions when  $\Psi = -\ln$ . Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

For recent results concerning inequalities for weighted arithmetic mean and geometric mean (see [4, 5] and [8–15]).

## 2. SOME INEQUALITIES FROM DEFINITION OF CONVEXITY

We define the weighted arithmetic and geometric means

$$A_\nu(a, b) := (1 - \nu)a + \nu b \text{ and } G_\nu(a, b) := a^{1-\nu}b^\nu,$$

where  $\nu \in [0, 1]$  and  $a, b > 0$ . If  $\nu = \frac{1}{2}$ , then we write for brevity  $A(a, b)$  and  $G(a, b)$ , respectively.

The following double inequality holds, see also [6].

**Theorem 2.1.** *Let  $M, m \in \mathbb{R}$  with  $M > m$  and  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ . Then for any  $a, b > 0$  and  $\nu \in [0, 1]$  we have*

$$(2.1) \quad \ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^m \leq (1 - \nu)\Phi(a) + \nu\Phi(b) - \Phi((1 - \nu)a + \nu b) \\ \leq \ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^M.$$

*Proof.* Since  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ , then  $\Phi_m := \Phi + m \ln$  is convex and by the definition of convexity, we have

$$\Phi((1 - \nu)a + \nu b) + m \ln A_\nu(a, b) \\ \leq (1 - \nu)(\Phi(a) + m \ln a) + \nu(\Phi(b) + m \ln b) \\ = (1 - \nu)\Phi(a) + \nu\Phi(b) + (1 - \nu)m \ln a + \nu m \ln b \\ = (1 - \nu)\Phi(a) + \nu\Phi(b) + m \ln G_\nu(a, b),$$

that is equivalent to

$$m \ln \frac{A_\nu(a, b)}{G_\nu(a, b)} \leq (1 - \nu) \Phi(a) + \nu \Phi(b) - \Phi((1 - \nu)a + \nu b)$$

and the first inequality in (2.1) is proved.

Similarly, by the convexity of  $\Phi_M := -M \ln -\Phi$  we get the second part of (2.1).  $\square$

For  $m, M$  with  $M > m > 0$  we define

$$(2.2) \quad M_p := \begin{cases} M^p, & \text{if } p > 1, \\ m^p, & \text{if } p < 0, \end{cases} \quad \text{and } m_p := \begin{cases} m^p, & \text{if } p > 1, \\ M^p, & \text{if } p < 0. \end{cases}$$

Consider the function  $\Phi(t) = t^p, p \in (-\infty, 0) \cup (1, \infty)$ . This is a convex function and  $\Phi''(t) = p(p - 1)t^{p-2}, t > 0$ . Consider  $\kappa(t) := t^2\Phi''(t) = p(p - 1)t^p$ . We observe that

$$\max_{t \in [m, M]} \kappa(t) = p(p - 1)M_p \text{ and } \min_{t \in [m, M]} \kappa(t) = p(p - 1)m_p.$$

**Corollary 2.1.** *Let  $m, M$  with  $M > m > 0$  and  $p \in (-\infty, 0) \cup (1, \infty)$ . Then for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$  we have*

$$(2.3) \quad \ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^{p(p-1)m_p} \leq (1 - \nu) a^p + \nu b^p - ((1 - \nu)a + \nu b)^p \\ \leq \ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^{p(p-1)M_p},$$

where  $M_p$  and  $m_p$  are defined by (2.2).

By taking the exponential in (2.3) we get the equivalent inequality

$$(2.4) \quad \exp \left( \frac{(1 - \nu) a^p + \nu b^p - ((1 - \nu)a + \nu b)^p}{p(p - 1)M_p} \right) \\ \leq \frac{A_\nu(a, b)}{G_\nu(a, b)} \\ \leq \exp \left( \frac{(1 - \nu) a^p + \nu b^p - ((1 - \nu)a + \nu b)^p}{p(p - 1)m_p} \right),$$

for any  $p \in (-\infty, 0) \cup (1, \infty), \nu \in [0, 1]$  and any  $a, b \in [m, M]$ .

If we take  $p = 2$  in (2.4) and perform the calculations, then we get

$$(2.5) \quad \exp \left( \frac{1}{2} (1 - \nu) \nu \frac{(b - a)^2}{M^2} \right) \leq \frac{A_\nu(a, b)}{G_\nu(a, b)} \leq \exp \left( \frac{1}{2} (1 - \nu) \nu \frac{(b - a)^2}{m^2} \right),$$

for any  $a, b \in [m, M]$ .

If  $a, b > 0$  then by taking  $M = \max\{a, b\}$  and  $m = \min\{a, b\}$  in (2.5) we have

$$(2.6) \quad \exp \left( \frac{1}{2} (1 - \nu) \nu \frac{(b - a)^2}{\max^2\{a, b\}} \right) \leq \frac{A_\nu(a, b)}{G_\nu(a, b)} \leq \exp \left( \frac{1}{2} (1 - \nu) \nu \frac{(b - a)^2}{\min^2\{a, b\}} \right).$$

Since

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{b-a}{\max\{a,b\}}\right)^2 = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2$$

and

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{b-a}{\min\{a,b\}}\right)^2 = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2$$

for any  $a, b > 0$ , then (2.6) can be written as

$$\begin{aligned} (2.7) \quad & \exp\left(\frac{1}{2}(1-\nu)\nu\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right) \\ & \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ & \leq \exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right). \end{aligned}$$

This inequality was obtained in a different way in [5].

If we take  $p = -1$  in (2.4) and perform the calculations, then we get

$$(2.8) \quad \exp\left(\frac{1}{2}(1-\nu)\nu\frac{m(b-a)^2}{abA_\nu(a,b)}\right) \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{M(b-a)^2}{abA_\nu(a,b)}\right),$$

for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$ .

If  $a, b > 0$  then by taking  $M = \max\{a, b\}$  and  $m = \min\{a, b\}$  in (2.8) and since  $ab = \max\{a, b\} \min\{a, b\}$  we have

$$\begin{aligned} (2.9) \quad & \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right) \\ & \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ & \leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right), \end{aligned}$$

for any  $\nu \in [0, 1]$ .

Since

$$\frac{1}{\max\{a,b\}} \leq \frac{1}{A_\nu(a,b)} \leq \frac{1}{\min\{a,b\}},$$

hence,

$$\exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2\right) \leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right)$$

and

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right) \leq \exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right),$$

showing that the double inequality (2.9) is better than (2.7).

### 3. SOME PERTURBED INEQUALITIES

Recall the following result obtained by Dragomir in 2006 [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}
 (3.1) \quad & n \min_{j \in \{1, 2, \dots, n\}} \{p_j\} \left( \frac{1}{n} \sum_{j=1}^n f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^n x_j\right) \right) \\
 & \leq \frac{1}{P_n} \sum_{j=1}^n p_j f(x_j) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \\
 & \leq n \max_{j \in \{1, 2, \dots, n\}} \{p_j\} \left( \frac{1}{n} \sum_{j=1}^n f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^n x_j\right) \right),
 \end{aligned}$$

where  $f : C \rightarrow \mathbb{R}$  is a convex function defined on convex subset  $C$  of the linear space  $X$ ,  $\{x_j\}_{j \in \{1, 2, \dots, n\}}$  are vectors in  $C$  and  $\{p_j\}_{j \in \{1, 2, \dots, n\}}$  are nonnegative numbers with  $P_n = \sum_{j=1}^n p_j > 0$ .

For  $n = 2$ , we deduce from (3.1) that

$$\begin{aligned}
 (3.2) \quad & 2r \left( \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right) \\
 & \leq \nu f(x) + (1-\nu) f(y) - f(\nu x + (1-\nu)y) \\
 & \leq 2R \left( \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right),
 \end{aligned}$$

for any  $x, y \in C$  and  $\nu \in [0, 1]$ , where  $r := \min\{\nu, 1-\nu\}$  and  $R := \max\{\nu, 1-\nu\}$ .

**Theorem 3.1.** *Let  $M, m \in \mathbb{R}$  with  $M > m$  and  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ . Then for any  $a, b > 0$  and  $\nu \in [0, 1]$  we have*

$$\begin{aligned}
 (3.3) \quad & \ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \right)^m \\
 & \leq (1-\nu) \Phi(a) + \nu \Phi(b) - \Phi((1-\nu)a + \nu b) - 2r \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right) \right) \\
 & \leq \ln \left( \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^M
 \end{aligned}$$

and

$$(3.4) \quad \ln \left( \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \frac{G_\nu(a, b)}{A_\nu(a, b)} \right)^m$$

$$\begin{aligned} &\leq 2R \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi \left( \frac{a+b}{2} \right) \right) - (\nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b)) \\ &\leq \ln \left( \frac{G_\nu(a, b)}{A_\nu(a, b)} \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \right)^M, \end{aligned}$$

where  $r := \min \{ \nu, 1 - \nu \}$  and  $R := \max \{ \nu, 1 - \nu \}$ .

*Proof.* Since  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ , then  $f_m := \Phi + m \ln$  is convex and by (3.2) we have

$$\begin{aligned} (3.5) \quad &2r \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi \left( \frac{a+b}{2} \right) \right) - 2rm \ln \frac{A(a, b)}{G(a, b)} \\ &\leq \nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b) - m \ln \frac{A_\nu(a, b)}{G_\nu(a, b)} \\ &\leq 2R \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi \left( \frac{a+b}{2} \right) \right) - 2Rm \ln \frac{A(a, b)}{G(a, b)}, \end{aligned}$$

for any  $a, b > 0$  and  $\nu \in [0, 1]$ .

Since  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ , then also  $f_M := -\Phi - M \ln$  is convex and by (3.2) we have

$$\begin{aligned} (3.6) \quad &2r \left( \Phi \left( \frac{a+b}{2} \right) - \frac{\Phi(a) + \Phi(b)}{2} \right) + 2rM \ln \frac{A(a, b)}{G(a, b)} \\ &\leq \Phi(\nu a + (1-\nu)b) - \nu\Phi(a) - (1-\nu)\Phi(b) + M \ln \frac{A_\nu(a, b)}{G_\nu(a, b)} \\ &\leq 2R \left( \Phi \left( \frac{a+b}{2} \right) - \frac{\Phi(a) + \Phi(b)}{2} \right) + 2RM \ln \frac{A(a, b)}{G(a, b)}, \end{aligned}$$

for any  $a, b > 0$  and  $\nu \in [0, 1]$ .

From the first inequality in (3.5) we have

$$\begin{aligned} &\ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \right)^m \\ &\leq \nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b) - 2r \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi \left( \frac{a+b}{2} \right) \right), \end{aligned}$$

while from the first inequality in (3.6) we also have

$$\begin{aligned} &\nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b) - 2r \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi \left( \frac{a+b}{2} \right) \right) \\ &\leq \ln \left( \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^M, \end{aligned}$$

for any  $a, b > 0$  and  $\nu \in [0, 1]$ .

These prove the desired result (3.3).

From the second inequality in (3.5) we have

$$\begin{aligned} & \ln \left( \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \frac{G_\nu(a, b)}{A_\nu(a, b)} \right)^m \\ & \leq 2R \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi \left( \frac{a+b}{2} \right) \right) - (\nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b)), \end{aligned}$$

while from the second inequality in (3.6) we also have

$$\begin{aligned} & 2R \left( \frac{\Phi(a) + \Phi(b)}{2} - \Phi \left( \frac{a+b}{2} \right) \right) - (\nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b)) \\ & \leq \ln \left( \frac{G_\nu(a, b)}{A_\nu(a, b)} \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \right)^M, \end{aligned}$$

for any  $a, b > 0$  and  $\nu \in [0, 1]$ .

These prove the desired result (3.4).  $\square$

**Corollary 3.1.** *Let  $m, M$  with  $M > m > 0$  and  $p \in (-\infty, 0) \cup (1, \infty)$ . Then for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$  we have*

$$\begin{aligned} (3.7) \quad & \ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \right)^{p(p-1)m_p} \\ & \leq (1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p - 2r \left( \frac{a^p + b^p}{2} - \left( \frac{a+b}{2} \right)^p \right) \\ & \leq \ln \left( \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^{p(p-1)M_p} \end{aligned}$$

and

$$\begin{aligned} (3.8) \quad & \ln \left( \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \frac{G_\nu(a, b)}{A_\nu(a, b)} \right)^{p(p-1)m_p} \\ & \leq 2R \left( \frac{a^p + b^p}{2} - \left( \frac{a+b}{2} \right)^p \right) - ((1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p) \\ & \leq \ln \left( \frac{G_\nu(a, b)}{A_\nu(a, b)} \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \right)^{p(p-1)M_p}, \end{aligned}$$

where  $r := \min\{\nu, 1-\nu\}$  and  $R := \max\{\nu, 1-\nu\}$  and  $M_p$  and  $m_p$  are defined by (2.2).



Observe, by simple calculation, we have that

$$\begin{aligned}
 (3.9) \quad & (1 - \nu) a^2 + \nu b^2 - ((1 - \nu) a + \nu b)^2 - 2r \left( \frac{a^2 + b^2}{2} - \left( \frac{a + b}{2} \right)^2 \right) \\
 & = (1 - \nu) \nu (b - a)^2 - \frac{r}{2} (b - a)^2 = r \left( R - \frac{1}{2} \right) (b - a)^2
 \end{aligned}$$

and

$$\begin{aligned}
 (3.10) \quad & 2R \left( \frac{a^2 + b^2}{2} - \left( \frac{a + b}{2} \right)^2 \right) - ((1 - \nu) a^2 + \nu b^2 - ((1 - \nu) a + \nu b)^2) \\
 & = \frac{R}{2} (b - a)^2 - (1 - \nu) \nu (b - a)^2 = R \left( \frac{1}{2} - r \right) (b - a)^2,
 \end{aligned}$$

for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$ .

If we write the inequalities (3.7) and (3.8) for  $p = 2$ , then we get

$$\begin{aligned}
 (3.11) \quad & \ln \left( \frac{A_\nu(a, b)}{G_\nu(a, b)} \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \right)^{2m^2} \leq r \left( R - \frac{1}{2} \right) (b - a)^2 \\
 & \leq \ln \left( \left( \frac{G(a, b)}{A(a, b)} \right)^{2r} \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^{2M^2}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.12) \quad & \ln \left( \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \frac{G_\nu(a, b)}{A_\nu(a, b)} \right)^{2m^2} \leq R \left( \frac{1}{2} - r \right) (b - a)^2 \\
 & \leq \ln \left( \frac{G_\nu(a, b)}{A_\nu(a, b)} \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \right)^{2M^2},
 \end{aligned}$$

for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$ .

From the first inequality in (3.11) we have

$$(3.13) \quad \frac{A_\nu(a, b)}{G_\nu(a, b)} \leq \left( \frac{A(a, b)}{G(a, b)} \right)^{2r} \exp \left( \frac{1}{2m^2} r \left( R - \frac{1}{2} \right) (b - a)^2 \right),$$

while from the second inequality in (3.11) we have

$$(3.14) \quad \left( \frac{A(a, b)}{G(a, b)} \right)^{2r} \exp \left( \frac{1}{2M^2} r \left( R - \frac{1}{2} \right) (b - a)^2 \right) \leq \frac{A_\nu(a, b)}{G_\nu(a, b)}.$$

From the first inequality in (3.12) we have

$$(3.15) \quad \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \exp \left( -\frac{1}{2m^2} R \left( \frac{1}{2} - r \right) (b - a)^2 \right) \leq \frac{A_\nu(a, b)}{G_\nu(a, b)},$$

while from the second inequality in (3.12) we have

$$(3.16) \quad \frac{A_\nu(a, b)}{G_\nu(a, b)} \leq \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \exp \left( -\frac{1}{2M^2} R \left( \frac{1}{2} - r \right) (b - a)^2 \right).$$

In conclusion, from (3.13)-(3.16) we have the following result:

$$(3.17) \quad \begin{aligned} & \max \left\{ \left( \frac{A(a, b)}{G(a, b)} \right)^{2r} \exp \left( \frac{1}{2M^2} r \left( R - \frac{1}{2} \right) (b - a)^2 \right), \right. \\ & \left. \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \exp \left( -\frac{1}{2m^2} R \left( \frac{1}{2} - r \right) (b - a)^2 \right) \right\} \\ & \leq \frac{A_\nu(a, b)}{G_\nu(a, b)} \\ & \leq \min \left\{ \left( \frac{A(a, b)}{G(a, b)} \right)^{2r} \exp \left( \frac{1}{2m^2} r \left( R - \frac{1}{2} \right) (b - a)^2 \right), \right. \\ & \left. \left( \frac{A(a, b)}{G(a, b)} \right)^{2R} \exp \left( -\frac{1}{2M^2} R \left( \frac{1}{2} - r \right) (b - a)^2 \right) \right\}, \end{aligned}$$

for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$ .

We need the following lemma (see [4]).

**Lemma 3.1.** *If the function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable convex function on  $\overset{\circ}{I}$ , then for any  $a, b \in \overset{\circ}{I}$  and  $\nu \in [0, 1]$  we have*

$$(3.18) \quad \begin{aligned} 0 & \leq (1 - \nu) f(a) + \nu f(b) - f((1 - \nu)a + \nu b) \\ & \leq \nu(1 - \nu)(b - a)(f'(b) - f'(a)). \end{aligned}$$

We have the following theorem.

**Theorem 3.2.** *Let  $M, m \in \mathbb{R}$  with  $M > m$  and  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ . Then for any  $a, b > 0$  and  $\nu \in [0, 1]$  we have*

$$(3.19) \quad \begin{aligned} & m \left( \nu(1 - \nu) \frac{(b - a)^2}{ab} - \ln \frac{A_\nu(a, b)}{G_\nu(a, b)} \right) \\ & \leq \nu(1 - \nu)(b - a)(\Phi'(b) - \Phi'(a)) \\ & \quad - ((1 - \nu)\Phi(a) + \nu\Phi(b) - \Phi((1 - \nu)a + \nu b)) \\ & \leq M \left( \nu(1 - \nu) \frac{(b - a)^2}{ab} - \ln \frac{A_\nu(a, b)}{G_\nu(a, b)} \right). \end{aligned}$$

*Proof.* Since  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ , then  $f_m := \Phi + m \ln$  is convex and by (3.18) we have

$$0 \leq (1 - \nu)\Phi(a) + \nu\Phi(b) - \Phi((1 - \nu)a + \nu b) - m \ln \frac{A_\nu(a, b)}{G_\nu(a, b)}$$

$$\begin{aligned} &\leq \nu(1-\nu)(b-a) \left( \Phi'(b) - \Phi'(a) + \frac{m}{b} - \frac{m}{a} \right) \\ &= \nu(1-\nu)(b-a) (\Phi'(b) - \Phi'(a)) - \frac{m}{ab} \nu(1-\nu)(b-a)^2, \end{aligned}$$

that is equivalent to

$$\begin{aligned} &m \left( \nu(1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \right) \\ &\leq \nu(1-\nu)(b-a) (\Phi'(b) - \Phi'(a)) - ((1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b)), \end{aligned}$$

for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$  and the first inequality in (3.19) is proved.

Since  $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$ , then also  $f_M := -\Phi - M \ln$  is convex and by (3.18) we have

$$\begin{aligned} 0 &\leq -(1-\nu)\Phi(a) - \nu\Phi(b) + f((1-\nu)a + \nu b) + M \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ &\leq -\nu(1-\nu)(b-a) (\Phi'(b) - \Phi'(a)) + M\nu(1-\nu) \frac{(b-a)^2}{ab}, \end{aligned}$$

that is equivalent to

$$\begin{aligned} &\nu(1-\nu)(b-a) (\Phi'(b) - \Phi'(a)) - (1-\nu)\Phi(a) - \nu\Phi(b) + f((1-\nu)a + \nu b) \\ &\leq M \left( \nu(1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \right), \end{aligned}$$

for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$  and the second inequality in (3.19) is proved.  $\square$

**Corollary 3.2.** *Let  $m, M$  with  $M > m > 0$  and  $p \in (-\infty, 0) \cup (1, \infty)$ . Then for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$  we have*

$$\begin{aligned} (3.20) \quad &p(p-1)m_p \left( \nu(1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \right) \\ &\leq p\nu(1-\nu)(b-a) (b^{p-1} - a^{p-1}) - ((1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p) \\ &\leq p(p-1)M_p \left( \nu(1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \right), \end{aligned}$$

where  $M_p$  and  $m_p$  are defined by (2.2).

The case  $p = 2$  is of interest. Observe that

$$\begin{aligned} &2\nu(1-\nu)(b-a)^2 - ((1-\nu)a^2 + \nu b^2 - ((1-\nu)a + \nu b)^2) \\ &= 2\nu(1-\nu)(b-a)^2 - \nu(1-\nu)(b-a)^2 = \nu(1-\nu)(b-a)^2 \end{aligned}$$

and by (3.20) we have

$$2m^2 \left( \nu(1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \right) \leq \nu(1-\nu)(b-a)^2$$

$$\leq 2M^2 \left( \nu(1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \right),$$

which is equivalent to

$$(3.21) \quad \exp \left( \nu(1-\nu)(b-a)^2 \left( \frac{1}{ab} - \frac{1}{2m^2} \right) \right) \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ \leq \exp \left( \nu(1-\nu)(b-a)^2 \left( \frac{1}{ab} - \frac{1}{2M^2} \right) \right),$$

for any  $a, b \in [m, M]$  and  $\nu \in [0, 1]$ .

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