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CRITICAL EXPONENTS CURVE FOR SEMILINEAR SYSTEM OF WEAKLY COUPLED EFFECTIVELY DAMPED WAVES WITH DIFFERENT POWER NONLINEARITIES

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Abstract. In this paper we prove a blow-up result for the semi linear system of weakly coupled effectively damped waves with different power nonlinearities

$$u_{tt} - \Delta u + b(t)u_t = |v|^p, \quad v_{tt} - \Delta v + b(t)v_t = |u|^q,$$
$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x),$$

where $b(t)$ will be explained in detail in the next sections. We apply the so called "test function method" to determine the range for the exponents $p, q > 0$ in the nonlinear terms in which local in time existence may not globally prolonged with respect to the t variable under suitable integral sign assumptions for the Cauchy data u_0, u_1, v_0, v_1. Since we prove the blow-up in a complementary range for powers of the nonlinear terms to that for the global existence of small data solutions (see [7]), the main blow-up of this paper is optimal.

1. Introduction

The sharpness of the results for the global (in time) existence of small data solutions or the notion of "blow-up of local (in time) solutions" means that if the pivotal condition for the global (in time) existence is not satisfied, then the solution does, in general, not exist globally (in time) regardless of the size of the data. Among several methods to prove blow-up results, the test function method is an important method which was introduced in the paper [19] and applied by Zhang for damped waves in [28].

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A fundamental step to use this method consists in the modification of the choice of a suitable scaling for the test function with respect to the time and space variables. In particular, the scaling with respect to t is given by the function $F(R)$, introduced in [3, Definition 2.2] which is strongly related to the coefficient $b(t)$.

Let us consider the Cauchy problem for the classical damped wave equation with power nonlinearity

$$u_{tt} - \Delta u + u_t = |u|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.1}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

The nonexistence result for $p = p_{Fuj}(n)$ has been established in [28]. Todorova and Yordanov proved in [26] that $p_{Fuj}(n) = 1 + \frac{2}{n}$ is critical.

In the following we recall an important result which the reader can find in the book of Ebert and Reissig [8]. The proof of Theorem 1.1 explains the basics and the philosophy of the test function method.

Theorem 1.1. *Let* $(u_0, u_1) \in \mathcal{A}_{1,1} = (H^1 \cap L^1) \times (L^2 \cap L^1)$ *satisfy the assumption*

$$\int_{\mathbb{R}^n} \Big(u_0(x) + u_1(x)\Big) dx > 0, \tag{1.2}$$

with $n \geq 1$ *and* $p \in (1, 1 + \frac{2}{n}]$. *Then there exists a unique locally* (*in time*) *defined energy solution* u *to* (1.1) *in* $\mathcal{C}([0,T), H^1) \cap \mathcal{C}^1([0,T), L^2)$ *for some* $T > 0$. *This solution cannot be continued to the interval* $[0,\infty)$ *in time.*

The Cauchy problem (1.1) has also been investigated by many authors [9–17, 20–23, 28, 29].

Let us now consider the weakly coupled system of semilinear classical damped waves

$$\begin{aligned} &u_{tt} - \Delta u + u_t = |v|^p, \quad v_{tt} - \Delta v + v_t = |u|^q, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \end{aligned} \tag{1.3}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $p, q \geq 1$ and $pq > 1$. Motivated by some previous papers concerned with the case of the Cauchy problem for a semilinear single equation, the authors in [24] and [25] studied the blow-up behavior of solutions of the system (1.3). In the following theorem we will recall the result of F. Sun and M. Wang published in [25].

Theorem 1.2. *Let* $n \geq 1$. *Assume that* $q \geq p \geq 1$ *and* $\frac{n}{2} \leq \frac{q+1}{pq-1}$. *If the data satisfy*

$$(u_i, v_i) \in [W^{1-i,1}(\mathbb{R}^n) \cap W^{1-i,\infty}(\mathbb{R}^n)]^2, \quad \text{for } i = 0, 1,$$

and

$$\int_{\mathbb{R}^n} u_i(x) dx > 0, \quad \int_{\mathbb{R}^n} v_i(x) dx > 0, \quad \text{for } i = 0, 1,$$

then the Sobolev solution (u, v) *of the Cauchy problem* (1.3) *does not exist globally* (*in time*).

2. Blow-up Result for Weakly Coupled Systems of Semilinear Damped Waves with Different Coefficients in the Dissipation Terms

Firstly, let us consider the Cauchy problem for a semilinear classical damped wave equation, namely

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{2.1}$$

where the dissipation term $b(t)u_t$ is supposed to be effective in the sense of Wirth [27]. The damping term $b(t)u_t$ is called effective in the model (2.1) if $b = b(t)$ satisfies the following properties:

- b is a positive and monotonic function with $tb(t) \to \infty$ as $t \to \infty$;
- $((1+t)^2 b(t))^{-1} \in L^1(0,\infty)$;
- $b \in \mathcal{C}^3[0,\infty)$ and $|b^{(k)}(t)| \lesssim \frac{b(t)}{(1+t)^k}$ for $k = 1,2,3$;
- $\frac{1}{b} \notin L^1(0,\infty)$ and there exists a constant $a \in [0,1)$ such that $tb'(t) \le ab(t)$.

Typical examples are

$$b(t) = \frac{\mu}{(1+t)^r}, \quad b(t) = \frac{\mu}{(1+t)^r}(\log(e+t))^\gamma, \quad b(t) = \frac{\mu}{(1+t)^r(\log(e+t))^\gamma},$$

for some $\mu > 0$, $\gamma > 0$ and $r \in (-1,1)$.

We introduce for $m \in [1,2)$ the function space

$$\mathcal{A}_{m,1} := (H^1 \cap L^m) \times (L^2 \cap L^m),$$

with the norm

$$\|(u,v)\|_{\mathcal{A}_{m,1}} := \|u\|_{H^1} + \|u\|_{L^m} + \|v\|_{L^2} + \|v\|_{L^m}.$$

We denote by $B(t,0)$ the primitive of $1/b(t)$ which vanishes at $t = 0$, that is,

$$B(t,0) := \int_0^t \frac{1}{b(r)}dr.$$

In [2] the authors determined the critical exponent $p = p_{Fuj}(n) := 1 + \frac{2}{n}$. That means after proving the global existence for some admissible range $p > p_{Fuj}(n)$, the authors proved also that, in general, the solution cannot be globally defined for $1 < p \le p_{Fuj}(n)$ under suitable sign assumptions for the Cauchy data. In other words, we have, in general, only local solutions (in time). The case $b(t) = \frac{\mu}{(1+t)^r}$ with $\mu > 0$ and $r > 0$ was studied in [18].

Let us consider now the Cauchy problem for the following system:

$$\begin{aligned} &u_{tt} - \Delta u + b(t)u_t = |v|^p, \quad v_{tt} - \Delta v + b(t)v_t = |u|^q, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \end{aligned} \tag{2.2}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$. As we already remarked during the treatment of the models (1.3) and (1.1) the test function method is not influenced by higher regularity of the data. We restrict ourselves to prove the sharpness of our results for the Cauchy problem (2.2), where the data are supposed to belong to the energy space $\mathcal{A}_{1,1} := (H^1 \cap L^1) \times (L^2 \cap L^1)$.

In [7] the authors proved the global (in time) existence of small data solution to (2.2), which means that the solution exists globally for

$$\frac{n}{2} > \frac{\max\{p;q\}+1}{pq-1}.$$

Theorem 2.1 ([7])**.** *Let $n \leq \frac{2m^2}{2-m}$ and $n < \frac{2m}{m-1}$. The data (u_0,u_1), (v_0,v_1) are supposed to belong to $\mathcal{A}_{m,1} \times \mathcal{A}_{m,1}$ with $m \in [1,2)$. Finally, the exponents p and q satisfy the assumptions*

$$\frac{2}{m} \leq \min\{p;q\} < p_{Fuj,m}(n) < \max\{p;q\}, \quad \text{if } n \leq 2, \tag{2.3}$$

$$\frac{2}{m} \leq \min\{p;q\} < p_{Fuj,m}(n) < \max\{p;q\} \leq p_{GN}(n), \quad \text{if } n > 2, \tag{2.4}$$

and

$$m\left(\frac{\max\{p;q\}+1}{pq-1}\right) < \frac{n}{2}.$$

Then there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (*in time*) *energy solution to* (2.2) *in*

$$\Big(\mathcal{C}([0,\infty),H^1) \cap \mathcal{C}^1([0,\infty),L^2)\Big)^2.$$

In the following we will prove the optimality of our results from Theorem 2.1. That means, if

$$\frac{n}{2} \leq \frac{\max\{p;q\}+1}{pq-1},$$

then, under suitable integral sign assumptions on the initial data, the local (in time) energy solution cannot be extended globally. The ideas of the proof of the following theorem are based on the paper [3] which is devoted to study a general case of model (2.1).

Theorem 2.2. *Let $b = b(t)$ such that $b(t)u_t$, $b(t)v_t$ are effective dissipation terms. Moreover, let*

$$\liminf_{t\longrightarrow\infty} \frac{b'(t)}{b(t)^2} > -1, \quad \limsup_{t\longrightarrow\infty} \frac{tb'(t)}{b(t)} < 1,$$

and let p,q such that

$$\frac{n}{2} \leq \frac{\max\{p,q\}+1}{pq-1},$$

where $pq > 1$. Then there exists no global classical solution $(u,v) \in \big(\mathcal{C}^2([0,\infty)\times\mathbb{R}^n)\big)^2$ to (2.2) *with initial data $((u_0,u_1),(v_0,v_1)) \in \mathcal{A}_{1,1} \times \mathcal{A}_{1,1}$ such that*

$$\begin{aligned} \int_{\mathbb{R}^n} u_0(x) + \hat{b}_1^{-1} u_1(x)dx &> 0, \\ \int_{\mathbb{R}^n} v_0(x) + \hat{b}_1^{-1} v_1(x)dx &> 0, \end{aligned} \tag{2.5}$$

where $\hat{b}_1$ *is defined in* (2.6).

Before proving this theorem we show the following lemma which will be used later in the proof.

Lemma 2.1. *Let* $g = g(t) \in \mathcal{C}([0,\infty))$ *be a solution of the following initial value problem for an ordinary differential equation*

$$-g'(t) + g(t)b(t) = 1, \quad g(0) = \frac{1}{\hat{b}_1}. \tag{2.6}$$

If $b = b(t)$ *satisfies the assumptions of Theorem* 2.2, *then it holds* $g(t) \approx \frac{1}{b(t)}$ *and*

$$|g'(t) - 1| \le C. \tag{2.7}$$

The proof of Lemma 2.1 can be concluded from [3] and [18].

Proof. For the sake of brevity we assume that $q > p$. We multiply (2.2) by the positive function $g = g(t)$ which is defined in Lemma 2.1. In this way we obtain

$$\begin{aligned}(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + (-g'(t) + g(t)b(t))u_t =& g(t)|v|^p,\\ (g(t)v)_{tt} - \Delta(g(t)v) - (g'(t)v)_t + (-g'(t) + g(t)b(t))v_t =& g(t)|u|^q.\end{aligned}$$

From the definition of $g = g(t)$ we may conclude

$$\begin{aligned}(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + u_t &= g(t)|v|^p,\\ (g(t)v)_{tt} - \Delta(g(t)v) - (g'(t)v)_t + v_t &= g(t)|u|^q.\end{aligned}$$

We introduce the test functions $\eta \in \mathcal{C}_0^\infty[0,\infty)$ with $0 \le \eta(t) \le 1$, where

$$\eta(t) = \begin{cases} 1, & \text{for } 0 \le t \le \frac{1}{2},\\ 0, & \text{for } t \ge 1,\end{cases}$$

$\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $0 \le \phi(x) \le 1$, where

$$\phi(x) = \begin{cases} 1, & \text{for } 0 \le |x| \le \frac{1}{2},\\ 0, & \text{for } |x| \ge 1.\end{cases}$$

Moreover, one can choose test functions η, ϕ and $1 < \alpha, \beta, \alpha', \beta' < p$ such that

$$\max\left\{\frac{|\eta'(t)|^\beta}{\eta(t)}; \frac{|\eta''(t)|^\alpha}{\eta(t)}\right\} \le C, \quad \text{for } \frac{1}{2} \le t \le 1,$$

and

$$\max\left\{\frac{|\nabla\phi(x)|^{\beta'}}{\phi(x)}; \frac{|\Delta\phi(x)|^{\alpha'}}{\phi(x)}\right\} \le C, \quad \text{for } \frac{1}{2} < |x| < 1,$$

where we choose $1 < \alpha, \beta, \alpha', \beta' < \min\{p; q\}$. Let R be a large parameter in $[0,\infty)$ and

$$Q_R := [0, F(R)] \times B_R, \quad B_R := \{x \in \mathbb{R}^n : |x| \le R\}.$$

We define the test function

$$\psi_R(t,x) := \eta_R(t)\phi_R(x) = \eta\Big(\frac{t}{F(R)}\Big)\phi\Big(\frac{x}{R}\Big),$$

where $F(R) = B^{-1}(R^2,0)$ and $B^{-1}(t,0)$ is the inverse function of $B(t,0)$. It follows that $F : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $F(0) = 0$ and $\lim_{R\to\infty} F(R) = \infty$. Moreover, we have $R \lesssim F(R)$ as a result of $b(t) \gtrsim (1+t)^{-1}$.

We have after integrating by parts

$$\begin{aligned}\int_{Q_R} g(t)|v|^p\psi_R d(t,x) = &-\int_{B_R}(u_0+\hat{b}_1^{-1}u_1)\psi_R dx\\ &+\int_{Q_R}\Big(g(t)u\partial_t^2\psi_R+(g'(t)-1)u\partial_t\psi_R+g(t)u\Delta\psi_R\Big)d(t,x)\end{aligned}$$

and

$$\begin{aligned}\int_{Q_R} g(t)|u|^q\psi_R d(t,x) = &-\int_{B_R}(v_0+\hat{b}_1^{-1}v_1)\psi_R dx\\ &+\int_{Q_R}\Big(g(t)v\partial_t^2\psi_R+(g'(t)-1)v\partial_t\psi_R+g(t)v\Delta\psi_R\Big)d(t,x).\end{aligned}$$

For sufficiently large R, thanks to (2.5), this implies

$$\int_{Q_R} g(t)|v|^p\psi_R d(t,x) \lesssim \int_{Q_R}\Big|g(t)u\partial_t^2\psi_R+(g'(t)-1)u\partial_t\psi_R+g(t)u\Delta\psi_R\Big|\,d(t,x)$$

and

$$\int_{Q_R} g(t)|u|^q\psi_R d(t,x) \lesssim \int_{Q_R}\Big|g(t)v\partial_t^2\psi_R+(g'(t)-1)v\partial_t\psi_R+g(t)v\Delta\psi_R\Big|\,d(t,x).$$

Using Lemma 2.1, Hölder's inequality with $\frac{1}{q}+\frac{1}{q'}=1$ and (2.7) we get

$$\begin{aligned}&\int_{Q_R}|ug(t)\partial_t^2\psi_R|d(t,x)\\ &\le\Big(\int_{Q_R}|u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}}\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\partial_t^2\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}},\end{aligned}\tag{2.8}$$

$$\begin{aligned}&\int_{Q_R}|u(g'(t)-1)\partial_t\psi_R|d(t,x)\\ &\le\Big(\int_{Q_R}|u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}}\Big(\int_{Q_R}g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}},\end{aligned}\tag{2.9}$$

$$\begin{aligned}&\int_{Q_R}|ug(t)\Delta\psi_R|d(t,x)\\ &\le\Big(\int_{Q_R}|u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}}\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\Delta\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}.\end{aligned}\tag{2.10}$$

We apply a change of variables $t = F(R)\tau$ and $x = Ry$. Then we have

$$d(t,x) = F(R)R^n d(\tau,y),\quad \partial_t\psi_R = F(R)^{-1}\partial_\tau\psi_R,\quad \partial_t^2\psi_R = F(R)^{-2}\partial_\tau^2\psi_R,$$

and

$$\Delta_x\psi_R = R^{-2}\Delta_y\psi_R, \frac{F(R)}{2}\le t\le F(R), \frac{R}{2}\le|x|\le R \Leftrightarrow \frac{1}{2}\le\tau,|y|\le 1.$$

With this change of variables we get for (2.8) the chain of inequalities

$$
\begin{aligned}
&\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\partial_t^2\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\\
=&\Big(\int_{\frac12}^1\int_{\frac12}^1\psi_R^{-\frac{q'}{q}}g(t)(F(R)\tau)F(R)^{-2q'}|\psi_R|^{\frac{q'}{\alpha}}F(R)R^n d\tau dy\Big)^{\frac{1}{q'}}\\
\lesssim&\Big(F(R)^{-2q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}g(t)dt\Big)^{\frac{1}{q'}}\\
\lesssim&\Big(F(R)^{-2q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}\frac{1}{b(t)}dt\Big)^{\frac{1}{q'}}\\
\lesssim&\Big(F(R)^{-2q'}R^nB(F(R),0)\Big)^{\frac{1}{q'}}\\
\lesssim&F(R)^{\frac{n+2-2q'}{q'}}.
\end{aligned}
$$

Consequently, we arrive at

$$
\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\partial_t^2\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{2.11}
$$

In the same way we can prove for (2.10) the estimate

$$
\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\Delta\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{2.12}
$$

Finally, let us turn to (2.9). We have

$$
\begin{aligned}
\Big(\int_{Q_R}g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}&\lesssim\Big(F(R)^{-q'}\int_{Q_R}b(t)^{q'-1}\psi_R^{-\frac{q'}{q}}|\psi_R|^{\frac{q'}{\beta}}d(t,x)\Big)^{\frac{1}{q'}}\\
&\lesssim\Big(F(R)^{-q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}b(t)^{q'-1}dt\Big)^{\frac{1}{q'}}.
\end{aligned}
$$

Since $F(0)=0$ and

$$
F'(R)=(B^{-1}(R^2,0))'=\frac{2R}{B'(F(R))}=2Rb(F(R)),
$$

using $b(t)\approx b(\frac{t}{2})$ and $B(t,0)-B(\frac{t}{2},0)\approx B(t,0)$ from [2, Remark 4.1], we get

$$
\int_{\frac{F(R)}{2}}^{F(R)}b(t)^{q'-1}dt\approx(b(F(R)))^{q'}\int_{\frac{F(R)}{2}}^{F(R)}b(t)^{-1}dt\approx(b(F(R)))^{q'}R^2.
$$

Moreover, we have

$$
\frac{b(F(R))}{F(R)}\approx\frac{1}{B(F(R),0)}=R^{-2}.
$$

Finally, we obtain

$$
\Big(\int_{Q_R}g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{2.13}
$$

Consequently, from (2.11) to (2.13) we get

$$\int_{Q_R} g(t)|v|^p\psi_R d(t,x) \lesssim F(R)^{\frac{n+2-2q'}{q'}}\Big(\int_{Q_R}|u|^q g\psi_R d(t,x)\Big)^{\frac{1}{q}}. \tag{2.14}$$

Analogously, one can get also

$$\int_{Q_R} g(t)|u|^q\psi_R d(t,x) \lesssim F(R)^{\frac{n+2-2p'}{p'}}\Big(\int_{Q_R}|v|^p g\psi_R d(t,x)\Big)^{\frac{1}{p}}, \quad \text{where } \frac{1}{p}+\frac{1}{p'}=1. \tag{2.15}$$

From (2.14) and (2.15) we obtain

$$\Big(\int_{Q_R} g(t)|v|^p\psi_R d(t,x)\Big)^{\frac{pq-1}{pq}} \le F(R)^{s_1},$$

$$\Big(\int_{Q_R} g(t)|u|^q\psi_R d(t,x)\Big)^{\frac{pq-1}{pq}} \le F(R)^{s_2}, \tag{2.16}$$

where

$$s_1=\frac{n+2}{q'}-2+\Big(\frac{n+2}{p'}-2\Big)\frac{1}{q} \quad \text{and} \quad s_2=\frac{n+2}{p'}-2+\Big(\frac{n+2}{q'}-2\Big)\frac{1}{p}.$$

The assumption $\frac{n}{2}\le\frac{q+1}{pq-1}$ implies that $s_2\le 0$. We consider two cases.

- If $s_2<0$, then letting $R\to\infty$ in the inequality (2.16) we obtain

  $$\int_0^\infty\int_{\mathbb{R}^n} g(t)|u|^q d(t,x)=0.$$

  This implies $u\equiv 0$. This is a contradiction to the assumptions.
- If $s_2=0$, then there exists a positive number R_0 such that

  $$\int_\Omega g(t)|u|^q\psi_R d(t,x)\le R_0,$$

  where $\Omega=\{(t,x)\in[0,\infty)\times\mathbb{R}^n : \frac{F(R)}{2}\le t\le F(R), \frac{R}{2}\le|x|\le R\}$. From $\partial_t\psi_R=\partial_{tt}\psi_R=\Delta\psi_R=0$ for $(t,x)\in Q_R\diagdown\Omega$, one can prove similarly to (2.14) and (2.15) the following estimates:

$$\int_0^\infty\int_{\mathbb{R}^n} g(t)|v|^p\psi_R d(t,x)+\int_{B_R}(u_0+\hat{b}_1^{-1}u_1)\psi_R dx \lesssim F(R)^{\frac{n+2-2q'}{q'}}\Big(\int_\Omega|u|^q g\psi_R d(t,x)\Big)^{\frac{1}{q}},$$

$$\int_0^\infty\int_{\mathbb{R}^n} g(t)|u|^q\psi_R d(t,x)+\int_{B_R}(v_0+\hat{b}_1^{-1}v_1)\psi_R dx \lesssim F(R)^{\frac{n+2-2p'}{p'}}\Big(\int_\Omega|v|^p g\psi_R d(t,x)\Big)^{\frac{1}{p}}.$$

  Last estimates for $s_2=0$ leads to

  $$\int_0^\infty\int_{\mathbb{R}^n} g(t)|u|^q d(t,x)+\int_{B_R}(v_0+\hat{b}_1^{-1}v_1)\psi_R dx\lesssim 0,$$

  for $R\to\infty$. This is also a contradiction. The proof is completed. □

3. Concluding Remarks

Recently, in [1] the author proved the blow-up of solutions for a model with constant coefficients considering the additional regularity L^m by taking a lower bound for the initial data $u_0(x) \in L^1_{loc}$ and $u_0(x) \geq \epsilon |x|^{-\frac{n}{m}} \log |x|$. Assuming a similar condition in our case by mixing additional regularities, we get from $\int_{B_R}(u_0 + \hat{b}_1^{-1} u_1)\psi_R(0,x)dx$ and $\int_{B_R}(v_0 + \hat{b}_1^{-1} v_1)\psi_R(0,x)dx$ a lower bound with respect to $R \lesssim F(R)$ after using $\psi_R(0,x) = \phi_R(x)$. This generated R cannot leads to the requested contraction. Finally, this means that the mentioned approach is not suitable for our model.

Assuming the weakly coupled system of semilinear damped waves (2.2) with different coefficients in the dissipation terms $b_1(t)u_t$ and $b_2(t)u_t$.

$$
\begin{aligned}
&u_{tt} - \Delta u + b_1(t)u_t = |v|^p, \quad v_{tt} - \Delta v + b_2(t)v_t = |u|^q, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x),
\end{aligned} \tag{3.1}
$$

The global existence (in time) of solutions of this Cauchy problem was treated in [4–7], where the data are defined in different classes of regularity which are the followings: low regular data, data from energy space, data from Sobolev spaces with suitable regularity and, finally, large regular data. The blow-up of (3.1) where $b_1(t) = \frac{\mu}{(1+t)^{r_1}}$, $b_2(t) = \frac{\mu}{(1+t)^{r_2}}$, $r_1, r_2 \in (-1,1)$, with data from energy space can be treated in a separated forthcoming project.

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