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SEQUENTIAL FRACTIONAL KORTEWEG-DE VRIES EQUATIONS WITH REGULAR PERTURBATIONS: EXISTENCE AND STABILITY ANALYSIS

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Abstract. This paper investigates a sequential (α, β)-fractional Korteweg-de Vries (KdV) equation that involves a regular nonlinear perturbation term. The problem is formulated using sequential Caputo fractional derivatives, which extend the classical KdV equation by incorporating nonlocal dispersion effects of different fractional orders. We establish existence and uniqueness results for the corresponding solutions under suitable assumptions on the nonlinear perturbation term. Furthermore, we investigate the behavior of the solutions and the effect of perturbations on their stability. A quantitative estimate describing the relationship between the solutions of the perturbed equation (when $\epsilon \neq 0$) and those of the associated unperturbed system (when $\epsilon = 0$) is derived. This estimate provides valuable information on the stability and qualitative behavior of solutions. Our results contribute to the theoretical analysis of nonlinear sequential fractional evolution equations and provide a method for the study of fractional dispersive wave problems.

1. Introduction

Fractional calculus has become an important mathematical tool for modeling phenomena that exhibit memory and hereditary effects. Unlike classical derivatives, fractional operators allow the description of nonlocal dynamics that frequently arise in complex systems. Consequently, fractional differential equations have found numerous applications in engineering, and applied sciences [8, 10, 20, 21].

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Among nonlinear evolution equations, the Korteweg-de Vries (KdV) equation plays a central role in the study of dispersive wave phenomena. Originally introduced to describe shallow water waves, the KdV equation has been widely used in various areas such as fluid mechanics, plasma physics, and nonlinear optics [2, 3, 11, 16, 22].

In recent years, considerable attention has been devoted to generalizations of the KdV equation. These models incorporate fractional derivatives in order to capture nonlocal dispersion effects and anomalous transport processes that cannot be described by classical models [4, 12, 13, 27]. More specifically, in [23], the authors derived exact traveling wave solutions of the KdV-Burgers equation: $u_t + auu_x + bu_{xxx} = cu_{xx}$, with arbitrary real coefficients, applied to beam-permeated multi-ion plasma fluids. Furthermore, in [1], A. Alalyani et al. presented the VHPM method as a numerical technique for approximating solutions to the KDVB equation given by:

$$u_t + a_1 uu_x + a_2 u_{xxx} + a_3 u_{xx} = 0.$$

Recently, the authors of [6] studied the following problem

$$\begin{cases} \delta \frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + D_x^\alpha u = 0, & (x,t) \in [ct, X] \times [0,T], \\ \frac{\partial^k u}{\partial x^k}(ct,t) = 0, & k = 1, 2, \ldots, m-1, \\ u(x,0) = g(x), & x \in [0, X], \end{cases}$$

where δ, β, γ are positive constants, $D_x^\alpha u$ is the derivative of Caputo, $\alpha \in (m-1, m)$.

Sequential fractional derivatives provide a natural extension of classical fractional operators by allowing the combination of derivatives of different orders. This approach offers additional flexibility in modeling complex dispersive interactions and opens new perspectives for the analysis of nonlinear fractional evolution equations [5, 17, 24–26].

Motivated by the above-cited papers, this work investigates the following KdV-type equation with sequential fractional effects and a general nonlinear regular perturbation:

$$u_t + duu_{xxx} + bu_x u_{xx} + cu^2 u_x + D_x^\alpha D_x^\beta u = \varepsilon g(u(x,t), u_x(x,t))), \tag{1.1}$$

where $u = u(x,t)$, $d, b, c \in \mathbb{R}$, and ε is a small perturbation parameter, the operators D_x^α, D_x^β are Caputo fractional derivatives with respect to x, of orders $\alpha \in (1,2)$, $\beta \in (2,3)$ and $\alpha + \beta > 4$, the function: $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity.

The goal is to study how the parameter ε affects the existence, uniqueness, and structure of traveling wave solutions. We propose the following conditions at $x_0(t) = \omega t$:

$$\begin{cases} u(\omega t, t) = a, \\ u_x(\omega t, t) = 0, \\ u_{xx}(\omega t, t) = 0, \\ {}^C D_x^\beta u(\omega t, t) = 0, \\ {}^C D_x^{\beta+1} u(\omega t, t) = 0. \end{cases} \tag{1.2}$$

Thus, we can distinguish between the unperturbed equation corresponding to $\varepsilon = 0$ and the perturbed problem when $\varepsilon \neq 0$ is small. We investigate both cases taking into account a.

The problem (1.1)–(1.2) is important to be investigated because it extends the classical results of perturbation analysis to sequential fractional nonlinear wave problems, addressing a critical gap in modern applied mathematics. While regular perturbation theory is well-established for integer-order PDEs like the KdV equation [7, 14, 19], its application to equations with sequential fractional derivatives remains largely unexplored. The problem investigates how a small and general nonlinear perturbation affects the existence of traveling waves in such fractional setting. Understanding this is vital because many physical systems exhibit a fractional dynamics and are subject to small influences.

The paper is organized as follows. In Section 2, we recall the Caputo fractional derivative and discuss a domain analysis, establishing some auxiliary results that will be used throughout the paper. Section 3 is devoted to the study of the unperturbed problem, where the existence and uniqueness of traveling wave solutions are obtained via the Banach fixed point theorem, and the existence of at least one traveling wave solution is further proved using Schaefer's fixed point theorem. In Section 4, we investigate the perturbed problem, where analogous existence and uniqueness results are derived under suitable Lipschitz conditions on the perturbation term, and the existence of at least one traveling wave solution is also established. Finally, Section 5 addresses the stability and asymptotic analysis, [15, 17, 21] showing the relationship between the solutions of both the perturbed and unperturbed problems.

2. Caputo Derivatives and Domain Analysis

In this section, we recall the Caputo derivative and then present three auxiliary results that will be used later.

Definition 2.1. The Caputo derivative of order $\mu \in (n-1, n)$ is defined as:

$$D_x^{\mu} f(x) = \frac{1}{\Gamma(n-\mu)} \int_{x_0}^{x} \frac{f^{(n)}(s)}{(x-s)^{\mu-n+1}} ds,$$

where f is a function of a class C^n over any interval $[x_0, x_1]$.

To study traveling wave solutions, which are particular solutions that maintain their shape while moving at constant speed, we put $u(x,t) = h(x - \omega t)$, then using $\xi = s - \omega t$, we obtain:

$$D_x^{\beta} u(x,t) = D_v^{\beta} h(v) \quad \text{and} \quad D_x^{\alpha} D_x^{\beta} u(x,t) = D_v^{\alpha} D_v^{\beta} h(v),$$

provided that $v = x - \omega t$.

We consider the domain: $(x,t) \in [\omega t, X] \times [0, T]$, with its interpretation:

-temporal domain: $t \in [0, T]$,

-spatial domain for each t: $x \in [\omega t, X]$, and parameters: $\omega \in \mathbb{R}$, $X > 0$, $T > 0$. For each $t \in [0, T]$, we have $x \in [\omega t, X]$.

Substituting $v = x - \omega t$ gives the following. If $x = \omega t$ implies $v = \omega t - \omega t = 0$, and when $x = X$ implies $v = X - \omega t$. Thus, for each fixed $t \in [0, T]$, we have:

$$v \in [0, X - \omega t]. \tag{2.1}$$

We now determine the set of values for which v is achieved while t varies in $[0, T]$. We have

$$\min v = \min_{\substack{t\in[0,T]\\ x\in[\omega t,X]}} (x - \omega t) = \min_{t\in[0,T]} 0 = 0.$$

This minimum is obtained for all $t \in [0, T]$, when $x = \omega t$.

On the other hand, we have

$$\max v = \max_{\substack{t\in[0,T]\\ x\in[\omega t,X]}} (x - \omega t) = \max_{t\in[0,T]} (X - \omega t).$$

Also, we can state that.

Case A: $\omega > 0$. The function $f(t) = X - \omega t$ is strictly decreasing and maximum occurs at $t = 0$: $\max v = X - \omega \cdot 0 = X$.

Case B: $\omega = 0$. The function $f(t) = X$ is constant and the maximum is: $\max v = X$.

Case C: $\omega < 0$. The function $f(t) = X - \omega t$ is strictly increasing (since $-\omega > 0$) and maximum occurs at $t = T$: $\max v = X - \omega T = X + |\omega| T$.

Hence, we have the following auxiliary result for the characterization of the domain analysis.

Theorem 2.1. *The traveling wave variable $v = x - \omega t$ with $(x, t) \in [\omega t, X] \times [0, T]$ satisfies $v \in [0, \max(X, X - \omega T)]$. More precisely*

$$v \in \begin{cases} [0, X], & \text{if } \omega \geq 0, \\ [0, X - \omega T] = [0, X + |\omega| T], & \text{if } \omega < 0. \end{cases} \tag{2.2}$$

For fixed domain, we have the following secondition auxiliary result.

Corollary 2.1. *The domain of v is independent of the wave direction and equals $[0, X]$ for all $t \in [0, T]$ if and only if $X - \omega T \leq X$ if and only if $-\omega T \leq 0$ if and only if $\omega T \geq 0$. Since $T > 0$, this simplifies to $\omega \geq 0$.*

For our applications in fixed point theory, we require S, such that

$$v \in [0, S], \quad \text{for all } (x, t) \in [\omega t, X] \times [0, T].$$

So, we can prove the following auxiliary result.

Theorem 2.2. *For all $\omega \in \mathbb{R}$, the value*

$$S = X + |\omega| T, \tag{2.3}$$

guarantees that $v \in [0, S]$.

Proof. For $\omega \geq 0$: $\max v = X \leq X + |\omega| T = S$. But, for $\omega < 0$: $\max v = X + |\omega| T = S$. Thus, in all cases, $v \in [0, S]$. □

3. Unperturbed Problem Formulation

According to the definitions [9,18], the unperturbed problem corresponds to the case where the small perturbation parameter ε is exactly zero. We consider the following unperturbed generalized fifth-order KdV-type equation with fractional sequential derivatives, which corresponds to the case $\varepsilon = 0$:

$$u_t + duu_{xxx} + bu_x u_{xx} + cu^2 u_x + D_x^\alpha D_x^\beta u = 0,$$

where $u = u(x,t)$, $d, b, c, \omega \in \mathbb{R}$, $1 < \alpha < 2$, $2 < \beta < 3$. The operators D_x^α and D_x^β are Caputo fractional derivatives.

This equation contains no perturbation term because $\varepsilon = 0$.

3.1. **Traveling wave solutions and integral representation.** Using the traveling wave transformation $u(x,t) = h(v)$ with $v = x - \omega t$, we obtain

$$-\omega h' + dhh''' + bh'h'' + ch^2 h' + D_v^\alpha D_v^\beta h = 0. \tag{3.1}$$

Applying $J_v^{\alpha+\beta}$ to both sides and using the imposed conditions:

$$h(0) = a, \quad h'(0) = 0, \quad h''(0) = 0,$$
$$D_v^\beta h(0) = 0, \quad D_v^{\beta+1} h(0) = 0,$$

we obtain the equivalent integral problem:

$$h(v) = a + J_v^{\alpha+\beta} \left[\omega h'(v) - dh(v)h'''(v) - bh'(v)h''(v) - ch^2(v)h'(v) \right]. \tag{3.2}$$

3.2. **Required Functional space.** Let $S > 0$ be fixed. We consider the Banach space:

$$Y := C^3([0,S]), \quad \|h\|_{C^3} = \|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty + \|h^{(3)}\|_\infty.$$

(Instead of $Y := C^3([0,S])$, we can simply take $Y := C^3$.)

For $r > 0$, define the closed ball: $B_r := \{h \in Y : \|h\|_{C^3} \leq r\}$, then, define the nonlinear operator $T_0 : Y \to Y$ by:

$$(T_0 h)(v) = a + J_v^{\alpha+\beta} F_0(h)(v),$$

where $F_0(h)(v) = \omega h'(v) - dh(v)h'''(v) - bh'(v)h''(v) - ch^2(v)h'(v)$. Its derivatives are given by:

$$(T_0 h)^{(k)}(v) = J_v^{\alpha+\beta-k} F_0(h)(v), \quad k = 1, 2, 3.$$

Define the quantity:

$$C_1 = \frac{S^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{S^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{S^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} + \frac{S^{\alpha+\beta-3}}{\Gamma(\alpha+\beta-2)}.$$

We propose to the reader the following three main results.

3.3. **Unperturbed main results.**

Theorem 3.1 (Stability)**.** *Assume $h \in B_r$ and*

$$|a| + C_1 \left[|\omega| r + (|d| + |b|) r^2 + |c| r^3\right] \leq r. \tag{3.3}$$

Then, $T_0 h \in B_r$.

Proof. For $h \in B_r$, we have by definition:

$$\begin{aligned} |\omega h'(v)| &\leq |\omega| r, \\ |dh(v)h'''(v)| &\leq |d| r^2, \\ |bh'(v)h''(v)| &\leq |b| r^2, \\ |ch^2(v)h'(v)| &\leq |c| r^3. \end{aligned}$$

Therefore, $\|F_0(h)\|_\infty \leq |\omega| r + (|d| + |b|) r^2 + |c| r^3 = K_0(r)$. Now, we estimate $\|T_0 h\|_{C^3}$. Using the results in Section 3.2, we can write

$$\begin{aligned} \|T_0 h\|_\infty &\leq |a| + \frac{S^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} K_0(r), \\ \|(T_0 h)'\|_\infty &\leq \frac{S^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} K_0(r), \\ \|(T_0 h)''\|_\infty &\leq \frac{S^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} K_0(r), \\ \|(T_0 h)^{(3)}\|_\infty &\leq \frac{S^{\alpha+\beta-3}}{\Gamma(\alpha+\beta-2)} K_0(r). \end{aligned}$$

Hence, $\|T_0 h\|_{C^3} \leq |a| + C_1 K_0(r) = |a| + C_1 \left[|\omega| r + (|d| + |b|) r^2 + |c| r^3\right]$. By (3.3), we conclude that $T_0 h \in B_r$. □

Theorem 3.2 (Contraction)**.** *Let $L_0(r) = |\omega| + 2(|d| + |b|) r + 3|c| r^2$. If $C_1 L_0(r) < 1$, then T_0 is a contraction on B_r.*

Proof. Let $h_1, h_2 \in B_r$ and consider

$$F_0(h_1)(v) - F_0(h_2)(v) := \omega(h_1' - h_2') - d(h_1 h_1''' - h_2 h_2''') - b(h_1' h_1'' - h_2' h_2'') - c(h_1^2 h_1' - h_2^2 h_2').$$

It is clear that the following inequalities are valid using the above norm definition and the fact that we work on the ball or radius r :

$$\begin{aligned} |\omega(h_1' - h_2')| &\leq |\omega| \|h_1 - h_2\|_{C^3}, \\ |d(h_1 h_1''' - h_2 h_2''')| &\leq |d| \Big(r \|h_1''' - h_2'''\|_\infty + r \|h_1 - h_2\|_\infty\Big) \leq 2|d| r \|h_1 - h_2\|_{C^3}, \\ |b(h_1' h_1'' - h_2' h_2'')| &\leq |b| \Big(r \|h_1'' - h_2''\|_\infty + r \|h_1' - h_2'\|_\infty\Big) \leq 2|b| r \|h_1 - h_2\|_{C^3}, \\ |c(h_1^2 h_1' - h_2^2 h_2')| &\leq |c| \Big(r^2 \|h_1' - h_2'\|_\infty + 2r^2 \|h_1 - h_2\|_\infty\Big) \leq 3|c| r^2 \|h_1 - h_2\|_{C^3}. \end{aligned}$$

Hence, $\|F_0(h_1) - F_0(h_2)\|_\infty \leq L_0(r)\|h_1 - h_2\|_{C^3}$. Therefore,

$$\begin{aligned}
\|T_0h_1 - T_0h_2\|_{C^3} &= \sum_{k=0}^{3} \|(T_0h_1)^{(k)} - (T_0h_2)^{(k)}\|_\infty \\
&= \sum_{k=0}^{3} \|J_v^{\alpha+\beta-k}[F_0(h_1) - F_0(h_2)]\|_\infty \\
&\leq \sum_{k=0}^{3} \frac{S^{\alpha+\beta-k}}{\Gamma(\alpha+\beta-k+1)}\|F_0(h_1) - F_0(h_2)\|_\infty \\
&= C_1\|F_0(h_1) - F_0(h_2)\|_\infty \\
&\leq C_1L_0(r)\|h_1 - h_2\|_{C^3},
\end{aligned}$$

where $L_0(r)$ is given in the proof of the above stability theorem.

Since by hypothesis the quantity $C_1L_0(r) < 1$, then T_0 is a contraction. □

Theorem 3.3 (Existence and Uniqueness)**.** *Let*

$$P_0(r) = |\omega|r + (|d| + |b|)r^2 + |c|r^3, \quad Q_0(r) = |\omega| + 2(|d| + |b|)r + 3|c|r^2.$$

If there is an $r > 0$, such that

$$\max\left\{\frac{|a| + C_1P_0(r)}{r}, C_1Q_0(r)\right\} < 1, \tag{3.4}$$

then there is a unique traveling wave solution $h_0 \in B_r$ of the unperturbed problem.

Proof. The above max-condition guarantees both stability: $|a| + C_1P_0(r) < r$ (so $T_0(B_r) \subset B_r$) and contraction: $C_1Q_0(r) < 1$ (so T_0 is contractive on B_r). Then, thanks to Banach's fixed point theorem, T_0 has a unique fixed point in B_r. □

For cases where the contraction condition is not satisfied, we can still establish the existence by proving the following result.

Theorem 3.4 (Existence)**.** *Suppose there is $\bar{r} > 0$, with*

$$C_1\left[|\omega|r + (|d| + |b|)r^2 + |c|r^3\right] < r - |a|, \quad \textit{for all } r \geq \bar{r}. \tag{3.5}$$

Then, $h = T_0h$ has at least one solution h.

Proof. We employ Schaefer's fixed point theorem. The proof proceeds through the following steps.

(a) **Continuity**. T_0 is continuous; this step is trivial and hence its proof is omitted.

(b) **Compactness**. It is clear that T_0 maps bounded sets into bounded sets (see Theorem 3.1). Moreover, T_0 maps bounded sets into equicontinuous sets; to show this, we use the linearity property of the Riemann-Liouville fractional integral $J_v^{\alpha+\beta}F_0(h)(v)$. Therefore, by the Arzeláá -Ascoli theorem, T_0 is compact.

(c) **A priori bound**. One can easily check, using the condition imposed above, that the Schaefer set

$$\mathcal{S} = \{h \in C^3([0,1]) : h = \lambda T_0 h \text{ for some } \lambda \in [0,1]\}$$

is bounded.

By Schaefer's theorem, T_0 has at least one fixed point which is a solution of the unperturbed wave problem. □

4. Perturbed Problem

To derive the integral equivalent equation of the perturbed problem introduced by (1.1)–(1.2), we start from the following sequential problem:

$$D_v^\alpha D_v^\beta h(v) = \omega h'(v) - dh(v)h^{(3)}(v) - bh'(v)h''(v) - ch^2(v)h'(v) + \varepsilon g(h(v), h'(v)), \tag{4.1}$$

where $1 < \alpha < 2$, $2 < \beta < 3$, $\alpha + \beta > 4$.

According to the definitions in [9,18], the regularly perturbed problem corresponds to the case $0 < \varepsilon \ll 1$, i.e., (4.1) that includes $\varepsilon g(h(v), h'(v))$.

So, applying J_v^α, we can write

$$\begin{aligned} D_v^\beta h(v) =& C_0 + C_2 v + J_v^\alpha \Big[\omega h'(v) - dh(v)h^{(3)}(v) - bh'(v)h''(v) \\ & - ch^2(v)h'(v) + \varepsilon g(h(v), h'(v))\Big]. \end{aligned} \tag{4.2}$$

As before, by J_v^β, we get

$$\begin{aligned} h(v) =& C_3 + C_4 v + C_5 v^2 + J_v^\beta [C_0 + C_2 v] \\ & + J_v^{\alpha+\beta}\Big[\omega h'(v) - dh(v)h^{(3)}(v) - bh'(v)h''(v) \\ & - ch^2(v)h'(v) + \varepsilon g(h(v), h'(v))\Big]. \end{aligned} \tag{4.3}$$

Since,

$$J_v^\beta[1] = \frac{v^\beta}{\Gamma(\beta+1)}, \quad J_v^\beta[v] = \frac{v^{\beta+1}}{\Gamma(\beta+2)},$$

then

$$\begin{aligned} h(v) =& C_3 + C_4 v + C_5 v^2 + \frac{C_0}{\Gamma(\beta+1)} v^\beta + \frac{C_2}{\Gamma(\beta+2)} v^{\beta+1} \\ & + J_v^{\alpha+\beta}\Big[\omega h'(v) - dh(v)h^{(3)}(v) - bh'(v)h''(v) \\ & - ch^2(v)h'(v) + \varepsilon g(h(v), h'(v))\Big]. \end{aligned} \tag{4.4}$$

Equation (4.4) is the integral form of (1.1), including $C_0, C_2 \ldots, C_5$.

Using the classical initial conditions (1.2), we get

$$h(0) = C_3 = a, \quad h'(0) = C_4 = 0, \quad h''(0) = 2C_5 = 0,$$

which implies $C_5 = 0$.

Recalling the quantities:

$$D_v^\beta\left[v^\beta\right]=\Gamma(\beta+1),\qquad D_v^\beta\left[v^{\beta+1}\right]=\Gamma(\beta+2)v,$$

we get

$$D_v^{\beta+1}\left[v^\beta\right]=0,\qquad D_v^{\beta+1}\left[v^{\beta+1}\right]=\Gamma(\beta+2).$$

Thus, we have

$$D_v^\beta h(0)=\frac{C_0}{\Gamma(\beta+1)}\Gamma(\beta+1)+0=C_0,$$
$$D_v^{\beta+1}h(0)=+0+\frac{C_2}{\Gamma(\beta+2)}\Gamma(\beta+2)=C_2.$$

Since $D_v^\beta h(0)=0$ and $D_v^{\beta+1}h(0)=0$, then

$$C_0=0,\quad C_2=0.$$

Consequently,

(4.5)

$$h(v)=a+J_v^{\alpha+\beta}\Big[\omega h'(v)-dh(v)h^{(3)}(v)-bh'(v)h''(v)-ch^2(v)h'(v)+\varepsilon g(h(v),h'(v))\Big].$$

We have thus obtained the equivalent fractional integral equation that is required to study the sequential problem (1.1)–(1.2) of Section 1.

4.1. **Fixed point theory and hypotheses.** As before, we can consider the Banach space: $Y:=C^3([0,S])$ with the norm

$$\|h\|_{C^3}:=\|h\|_\infty+\|h'\|_\infty+\|h''\|_\infty+\|h^{(3)}\|_\infty.$$

(Instead of $Y:=C^3([0,S])$, we can simply take $Y:=C^3$.)

For $r>0$, we put $B_r:=\{h\in C^3([0,S])\ :\ \|h\|_{C^3}\le r\}$.

Let $\alpha\in(1,2)$, $\beta\in(2,3)$ and consider

$$Rh(v):=a+J^{\alpha+\beta}\Big(\omega h'(v)-dh(v)h^{(3)}(v)-bh'(v)h''(v)-ch^2(v)h'(v)+\varepsilon g(h(v),h'(v))\Big).$$

For $h\in C^3$, put

$$F(h)(v):=\omega h'(v)-dh(v)h^{(3)}(v)-bh'(v)h''(v)-ch^2(v)h'(v)+\varepsilon g(h(v),h'(v)).$$

Then, for $k=1,2,3$, we can write

$$(Rh)^{(k)}(v)=J^{\alpha+\beta-k}F(h)(v),$$

using $\frac{d^k}{dv^k}J^{\alpha+\beta}=J^{\alpha+\beta-k}$ whenever $\alpha+\beta-k>0$.

We assume the following.

(H1) g is Lipschitz on $\{(u,v):|u|\le r,\ |v|\le r\}$, i.e., there exists $L_g>0$ such that for all $(u_1,v_1),(u_2,v_2)$ with $|u_i|\le r,|v_i|\le r$,

$$|g(u_1,v_1)-g(u_2,v_2)|\le L_g\left(|u_1-u_2|+|v_1-v_2|\right).$$

(H2) Set $G_0:=|g(0,0)|$. Then, for $h\in B_r$, $|g(h(v),h'(v))|\le G_0+2L_gr$.

4.2. **Perturbed main results.** We prove the following theorem.

Theorem 4.1. *Assume* **(H1)** *and* **(H2)** *are valid,* $h \in B_r$ *and*

$$|a| + C_1 \left(|\omega| r + (|d| + |b|) r^2 + |c| r^3 + \varepsilon (G_0 + 2L_g r)\right) \leq r. \tag{4.6}$$

Then, $Rh \in B_r$.

Proof. Since $h \in B_r$, hence we can use the following evident inequalities:

$$\begin{aligned}
\|Rh\|_\infty \leq & |a| + \frac{S^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \|F(h)\|_\infty, \\
\|(Rh)'\|_\infty \leq & \frac{S^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \|F(h)\|_\infty, \\
\|(Rh)''\|_\infty \leq & \frac{S^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \|F(h)\|_\infty, \\
\|(Rh)^{(3)}\|_\infty \leq & \frac{S^{\alpha+\beta-3}}{\Gamma(\alpha+\beta-2)} \|F(h)\|_\infty.
\end{aligned}$$

So, by the norm definition, we have

$$\|Rh\|_{C^3} \leq |a| + C_1 \|F(h)\|_\infty.$$

If $\|h\|_{C^3} \leq r$, then thanks to **(H1)** and **(H2)**, we obtain

$$\|F(h)\|_\infty \leq |\omega| r + (|d| + |b|) r^2 + |c| r^3 + \varepsilon (G_0 + 2L_g r).$$

Thus, $\|Rh\|_{C^3} \leq |a| + C_1 \left(|\omega| r + (|d| + |b|) r^2 + |c| r^3 + \varepsilon (G_0 + 2L_g r)\right)$. The condition (4.6) guarantees $\|Rh\|_{C^3} \leq r$, so $Rh \in B_r$. □

We also prove the following theorem.

Theorem 4.2. *Assume* **(H1)** *and* **(H2)** *are valid, and define*

$$K_r := |\omega| + 2|d| r + 2|b| r + 3|c| r^2 + \varepsilon L_g.$$

If $C_1 K_r < 1$, *then* R *is a contraction on* B_r.

Proof. We have

$$\begin{aligned}
F(h_1)(v) - F(h_2)(v) = & \omega (h_1' - h_2') - d \left(h_1 h_1^{(3)} - h_2 h_2^{(3)}\right) \\
& - b \left(h_1' h_1'' - h_2' h_2''\right) - c \left(h_1^2 h_1' - h_2^2 h_2'\right) \\
& + \varepsilon \left(g(h_1, h_1') - g(h_2, h_2')\right).
\end{aligned}$$

Using the fact that $\|h_i^{(k)}\|_\infty \leq r$, we can write

$$\begin{aligned}
|h_1 h_1^{(3)} - h_2 h_2^{(3)}| \leq & r \|h_1^{(3)} - h_2^{(3)}\|_\infty + r \|h_1 - h_2\|_\infty, \\
|h_1' h_1'' - h_2' h_2''| \leq & r \|h_1'' - h_2''\|_\infty + r \|h_1' - h_2'\|_\infty, \\
|h_1^2 h_1' - h_2^2 h_2'| \leq & 3r^2 \|h_1 - h_2\|_{C^3},
\end{aligned}$$

and by **(H1)** and **(H2)**, we have

$$|g(h_1, h_1') - g(h_2, h_2')| \leq L_g\Big(\|h_1 - h_2\|_\infty + \|h_1' - h_2'\|_\infty\Big) \leq L_g\|h_1 - h_2\|_{C^3}.$$

Hence,

$$\|F(h_1)-F(h_2)\|_\infty \leq \Big(|\omega| + 2|d|r + 2|b|r + 3|c|r^2 + \varepsilon L_g\Big)\,\|h_1-h_2\|_{C^3} = K_r\|h_1-h_2\|_{C^3}.$$

Consequently,

$$\|Rh_1 - Rh_2\|_{C^3} \leq C_1\|F(h_1) - F(h_2)\|_\infty \leq C_1K_r\|h_1 - h_2\|_{C^3}.$$

If $C_1K_r < 1$, then R is a contraction. This ends the proof. □

Remark 4.1. Thanks to the Banach fixed point theorem, and under the two combined conditions:

(a) stability: $|a| + C_1\left(|\omega|r + (|d| + |b|)r^2 + |c|r^3 + \varepsilon(G_0 + 2L_gr)\right) \leq r$,
(b) contraction: $C_1\left(|\omega| + 2|d|r + 2|b|r + 3|c|r^2 + \varepsilon L_g\right) < 1$,

we state that R has a unique fixed point in B_r.

Theorem 4.3 (A unified version)**.** *Let*

$$\begin{aligned} P(r) :=& |\omega|r + (|d| + |b|)r^2 + |c|r^3 + \varepsilon(G_0 + 2L_gr), \\ Q(r) :=& |\omega| + 2(|d| + |b|)r + 3|c|r^2 + \varepsilon L_g \end{aligned}$$

and

$$C_1 = \frac{S^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{S^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{S^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} + \frac{S^{\alpha+\beta-3}}{\Gamma(\alpha+\beta-2)}.$$

Consider the conditions:

$$|a| + C_1P(r) < r, \tag{4.7}$$

$$C_1Q(r) < 1. \tag{4.8}$$

Then, the conjunction of (4.7) *and* (4.8) *is equivalent to:*

$$\max\left\{\frac{|a| + C_1P(r)}{r}, C_1Q(r)\right\} < 1. \tag{4.9}$$

Proof. Assume (4.7) and (4.8) hold. From (4.7) we have

$$\frac{|a| + C_1P(r)}{r} \leq 1,$$

and from (4.8) we have $C_1Q(r) < 1$. Therefore, both quantities inside the maximum in (4.9) are strictly less than 1. Thus, (4.9) holds.

Now, suppose (4.9) holds. Then, $\frac{|a|+C_1P(r)}{r} < 1$ implies $|a| + C_1P(r) < r$, which is the strict form of (4.7). Also, $C_1Q(r) < 1$ follows directly from the secondition term in the maximum. Hence, (4.7) and (4.8) are satisfied. □

Theorem 4.4. *The condition*

$$\max\left\{\frac{|a|+C_1P(r)}{r}, C_1Q(r)\right\}<1$$

yields an explicit upper bound $\varepsilon_0>0$, such that for $0<\varepsilon<\varepsilon_0$, both stability and contraction hold.

Proof. To see this, set

$$\begin{aligned} A :=&|\omega|r+(|d|+|b|)r^2+|c|r^3, \quad B := G_0+2L_gr,\\ A' :=&|\omega|+2(|d|+|b|)r+3|c|r^2, \quad B' := L_g, \end{aligned}$$

so that $P(r)=A+\varepsilon B$, $Q(r)=A'+\varepsilon B'$. Then,

$$\frac{|a|+C_1P(r)}{r}<1 \quad \text{and} \quad C_1Q(r)<1$$

are equivalent to

$$|a|+C_1A+C_1\varepsilon B<r \quad \text{and} \quad C_1(A'+\varepsilon B')<1.$$

Solving each for ε gives

$$\varepsilon<\frac{r-|a|-C_1A}{C_1B}, \qquad \varepsilon<\frac{1/C_1-A'}{B'}.$$

Therefore, one can define

$$\varepsilon_0:=\min\left\{\frac{r-|a|-C_1A}{C_1B}, \frac{1/C_1-A'}{B'}\right\},$$

and the above theorem applies for $0<\varepsilon<\varepsilon_0$. □

We consider the following hypothesis.

(H3) $g:\mathbb{R}^2\to\mathbb{R}$ is continuous and $|g(u,p)|\le M_g$.

We prove the following lemmas that will be used later.

Lemma 4.1 (Continuity)**.** *Suppose that* **(H1)** *and* **(H3)** *hold. Then, the operator $R:Y\to Y$ is continuous.*

Proof. Let $h_n\to h$ in Y, i.e., $\|h_n-h\|_{C^3}\to 0$. This means

$$\|h_n-h\|_\infty+\|h_n'-h'\|_\infty+\|h_n''-h''\|_\infty+\|h_n^{(3)}-h^{(3)}\|_\infty\to 0.$$

We have the following inequalities:

$$\begin{aligned} \|\omega(h_n'-h')\|_\infty \le&|\omega|\|h_n'-h'\|_\infty,\\ \|d(h_nh_n^{(3)}-hh^{(3)})\|_\infty \le&|d|\left(\|h_n\|_\infty\|h_n^{(3)}-h^{(3)}\|_\infty+\|h_n^{(3)}\|_\infty\|h_n-h\|_\infty\right),\\ \|b(h_n'h_n''-h'h'')\|_\infty \le&|b|\Big(\|h_n'\|_\infty\|h_n''-h''\|_\infty+\|h_n''\|_\infty\|h_n'-h'\|_\infty\Big),\\ \|c(h_n^2h_n'-h^2h')\|_\infty \le&|c|\left(\|h_n^2\|_\infty\|h_n'-h'\|_\infty+\|h_n'\|_\infty\|h_n^2-h^2\|_\infty\right),\\ \|\varepsilon(g(h_n,h_n')-g(h,h'))\|_\infty \le&\varepsilon L_g\Big(\|h_n-h\|_\infty+\|h_n'-h'\|_\infty\Big). \end{aligned}$$

Since $h_n \to h$ in $\|\cdot\|_{C^3}$, all terms $\to 0$. Thus, $\|F(h_n) - F(h)\|_\infty \to 0$. Now consider the quantity $\|Rh_n - Rh\|_{C^3}$. By definition, we have

$$\begin{aligned}\|Rh_n - Rh\|_{C^3} = \|Rh_n - Rh\|_\infty + \|(Rh_n)' - (Rh)'\|_\infty \\ + \|(Rh_n)'' - (Rh)''\|_\infty + \|(Rh_n)^{(3)} - (Rh)^{(3)}\|_\infty.\end{aligned}$$

For each $k = 0, 1, 2, 3$, we can write

$$\begin{aligned}\|(Rh_n)^{(k)} - (Rh)^{(k)}\|_\infty &= \|J_v^{\alpha+\beta-k}[F(h_n) - F(h)]\|_\infty \\ &\le \frac{S^{\alpha+\beta-k}}{\Gamma(\alpha+\beta-k+1)}\|F(h_n) - F(h)\|_\infty.\end{aligned}$$

Therefore, $\|Rh_n - Rh\|_{C^3} \le C_1\|F(h_n) - F(h)\|_\infty \to 0$. Hence, R is continuous. □

Lemma 4.2 (Boundedness)**.** *Suppose that* **(H3)** *is valid. Then, for any set* $B_r = \{h \in Y : \|h\|_{C^3} \le r\}$*, the image* $R(B_r)$ *is bounded in* Y*.*

Proof. For $h \in B_r$, we have:

$$\|h\|_\infty \le r, \quad \|h'\|_\infty \le r, \quad \|h''\|_\infty \le r, \quad \|h^{(3)}\|_\infty \le r.$$

We need to estimate $\|F(h)\|_\infty$. Using **(H3)**, we can write

$$\begin{aligned}|\omega h'(v)| &\le |\omega| r, \\ |dh(v)h^{(3)}(v)| &\le |d| r^2, \\ |bh'(v)h''(v)| &\le |b| r^2, \\ |ch^2(v)h'(v)| &\le |c| r^3, \\ |\varepsilon g(h(v), h'(v))| &\le \varepsilon M_g.\end{aligned}$$

Thus, we have $\|F(h)\|_\infty \le |\omega| r + (|d| + |b|)r^2 + |c| r^3 + \varepsilon M_g =: K(r)$. Now, since

$$\begin{aligned}\|Rh\|_\infty &\le |a| + \frac{S^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}K(r), \\ \|(Rh)'\|_\infty &\le \frac{S^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}K(r), \\ \|(Rh)''\|_\infty &\le \frac{S^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}K(r), \\ \|(Rh)^{(3)}\|_\infty &\le \frac{S^{\alpha+\beta-3}}{\Gamma(\alpha+\beta-2)}K(r),\end{aligned}$$

we have

$$\begin{aligned}\|Rh\|_{C^3} \le& |a| + \left(\frac{S^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{S^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{S^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} + \frac{S^{\alpha+\beta-3}}{\Gamma(\alpha+\beta-2)}\right) K(r) \\ =& |a| + C_1 K(r).\end{aligned}$$

So, $R(B_r) \subset B_{|a|+C_1K(r)}$. □

Lemma 4.3 (Equicontinuity)**.** *If* $(H3)$ *is valid, then for any bounded set* $B_r \subset Y$*, the family* $R(B_r)$ *is equicontinuous in* C^3*.*

Proof. Let $h \in B_r$, $v_1, v_2 \in [0, S]$. For $k = 0, 1, 2, 3$, the reader can observe that

$$|(Rh)^{(k)}(v_2) - (Rh)^{(k)}(v_1)| = |J_v^{\alpha+\beta-k} F(h)(v_2) - J_v^{\alpha+\beta-k} F(h)(v_1)|.$$

Using the linearity for Riemann-Liouville integrals and by **H3**, we get

$$|J_v^{\gamma}\phi(v_2) - J_v^{\gamma}\phi(v_1)| \leq \frac{|v_2^{\gamma} - v_1^{\gamma}|}{\Gamma(\gamma+1)}\|\phi\|_{\infty},$$

where $\gamma = \alpha + \beta - k > 1$. Thus,

$$|(Rh)^{(k)}(v_2) - (Rh)^{(k)}(v_1)| \leq \frac{|v_2^{\alpha+\beta-k} - v_1^{\alpha+\beta-k}|}{\Gamma(\alpha+\beta-k+1)}K(r).$$

Since v^{γ} is uniformly continuous on $[0, S]$, given $\epsilon > 0$, there exists $\delta > 0$ such that: $|v_2 - v_1| < \delta$ implies $|v_2^{\gamma} - v_1^{\gamma}| < \epsilon\Gamma(\gamma+1)/K(r)$. This bound is uniform for all $h \in B_r$, hence $R(B_r)$ is equicontinuous in C^3. □

Lemma 4.4 (Compactness)**.** *Suppose that* **(H3)** *is valid. Then, the operator* $R : Y \to Y$ *is compact.*

Proof. We have already proved that:

(i) R maps bounded sets to bounded sets (Lemma 4.2),

(ii) R maps bounded sets to equicontinuous sets (Lemma 4.3).

Hence, by Arzeláá-Ascoli theorem, we conclude that we have relatively compactness in C^3. This ends the proof of this lemma. □

Lemma 4.5. *Suppose that* ***(H3)*** *is valid. Then, the set* $\{h \in Y : h = \lambda Rh$ *for some* $\lambda \in [0, 1]\}$ *is bounded provided that there exists* $\bar{r} > 0$*, such that*

$$C_1\left[|\omega|r + (|d| + |b|)r^2 + |c|r^3 + \varepsilon M_g\right] < r,$$

for $r \geq \bar{r}$*.*

Proof. Let $h = \lambda Rh$ with $\lambda \in [0, 1]$. Then,

$$h^{(k)}(v) = \lambda J_v^{\alpha+\beta-k} F(h)(v), \quad k = 0, 1, 2, 3.$$

Let $r := \|h\|_{C^3}$. Then, thanks to **(H3)**, we can write

$$\|F(h)\|_{\infty} \leq |\omega|r + (|d| + |b|)r^2 + |c|r^3 + \varepsilon M_g = K(r).$$

Now estimate $\|h\|_{C^3}$ as follows:

$$\|h\|_{\infty} \leq \frac{S^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}K(r),$$

$$\|h'\|_{\infty} \leq \frac{S^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}K(r),$$

$$\|h''\|_\infty \leq \frac{S^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}K(r),$$
$$\|h^{(3)}\|_\infty \leq \frac{S^{\alpha+\beta-3}}{\Gamma(\alpha+\beta-2)}K(r).$$

Therefore, we have

$$r = \|h\|_{C^3} \leq C_1 K(r) = C_1\left[|\omega|r + (|d|+|b|)r^2 + |c|r^3 + \varepsilon M_g\right].$$

If there exists $\bar{r} > 0$, such that

$$C_1\left[|\omega|r + (|d|+|b|)r^2 + |c|r^3 + \varepsilon M_g\right] < r,$$

for $r \geq \bar{r}$, this gives a contradiction. Hence, $r < \bar{r}$. □

Remark 4.2. This lemma has been proved when $a = 0$. On the other hand, using the lemmas established above, we have the following existence result.

Theorem 4.5 (Existence)**.** *Suppose that* **(H1)** *and* **(H3)** *are valid and there exists a* $\bar{r} > 0$*, such that*

$$C_1\left[|\omega|r + (|d|+|b|)r^2 + |c|r^3 + \varepsilon M_g\right] < r,$$

for $r \geq \bar{r}$*. Then, the equation* $h = Rh$ *has at least one solution* $h \in C^3([0,S])$*.*

Proof. The proof is a direct application of the above lemmas. □

5. Stability and Asymptotic Analysis

Having established complete existence and uniqueness results for both the unperturbed ($\varepsilon = 0$) and perturbed ($\varepsilon \neq 0$) cases, we now investigate their relationship. We prove the following theorem.

Theorem 5.1. *Suppose that the following conditions are valid for some* $r > 0$ *and* $S > 0$*.*

(a) *There exists* $M_g > 0$*, such that* $|g(u,v)| \leq M_g$*, for all* $(u,v) \in \mathbb{R}^2$*.*

(b) *There exists* $L_g > 0$ *such that for all* $(u_1,v_1),(u_2,v_2)$*, with* $|u_i| \leq r$*,* $|v_i| \leq r$*:*

$$|g(u_1,v_1) - g(u_2,v_2)| \leq L_g(|u_1-u_2| + |v_1-v_2|).$$

(c) *The unique explicit value*

$$\varepsilon_0 = \min\left\{\frac{r - |a| - C_1 A}{C_1 B}, \frac{1/C_1 - A_0}{B_0}\right\}$$

satisfies $0 < \varepsilon_0$*, where*

$$A = |\omega|r + (|d|+|b|)r^2 + |c|r^3,$$
$$B = M_g + 2L_g r,$$
$$A_0 = |\omega| + 2(|d|+|b|)r + 3|c|r^2,$$
$$B_0 = L_g.$$

Then, for each $\varepsilon \in [0, \varepsilon_0]$, there are unique solutions $h_\varepsilon \in B_r$ and $h_0 \in B_r$ satisfying the inequality:

$$\|h_\varepsilon - h_0\|_{C^3} \leq \frac{C_1 M_g}{1-\kappa}\varepsilon,$$

where

$$C_1 = \sum_{k=0}^{3} \frac{S^{\alpha+\beta-k}}{\Gamma(\alpha+\beta-k+1)}, \quad \kappa = C_1\left[|\omega| + 2(|d|+|b|)r + 3|c|r^2 + \varepsilon_0 L_g\right] < 1.$$

Proof. We proceed using the following steps. **Step 1.** The definition of ε_0 ensures that for all $\varepsilon \in [0, \varepsilon_0]$, we have:

- stability condition via $|a| + C_1[|\omega|r + (|d|+|b|)r^2 + |c|r^3 + \varepsilon M_g] \leq r$, and
- contraction condition via $C_1[|\omega| + 2(|d|+|b|)r + 3|c|r^2 + \varepsilon L_g] \leq \kappa < 1$.

Step 2. For each $\varepsilon \in [0, \varepsilon_0]$, the operator $T_\varepsilon : B_r \to B_r$ defined by

$$T_\varepsilon(h) = a + J_v^{\alpha+\beta}[F_0(h) + \varepsilon g(h, h')]$$

satisfies both stability and contraction conditions. By Banach's fixed point theorem, there exists a unique $h_\varepsilon \in B_r$, with $T_\varepsilon(h_\varepsilon) = h_\varepsilon$. **Step 3.** Consider the difference:

$$h_\varepsilon - h_0 = T_\varepsilon(h_\varepsilon) - T_0(h_0).$$

We have

$$h_\varepsilon - h_0 = [T_\varepsilon(h_\varepsilon) - T_\varepsilon(h_0)] + [T_\varepsilon(h_0) - T_0(h_0)].$$

Step 4. By the contraction property, we have

$$\|T_\varepsilon(h_\varepsilon) - T_\varepsilon(h_0)\|_{C^3} \leq \kappa \|h_\varepsilon - h_0\|_{C^3}.$$

Step 5. Note that

$$T_\varepsilon(h_0) - T_0(h_0) = J_v^{\alpha+\beta}[\varepsilon g(h_0, h_0')].$$

Therefore, we get

$$\|T_\varepsilon(h_0) - T_0(h_0)\|_{C^3} \leq C_1 \varepsilon \|g(h_0, h_0')\|_\infty \leq C_1 M_g \varepsilon.$$

Step 6. Combining Step 4 and 5, we can write

$$\|h_\varepsilon - h_0\|_{C^3} \leq \kappa \|h_\varepsilon - h_0\|_{C^3} + C_1 M_g \varepsilon.$$

Hence,

$$(1-\kappa)\|h_\varepsilon - h_0\|_{C^3} \leq C_1 M_g \varepsilon.$$

Since $\kappa < 1$ (by hypothesis), we obtain

$$\|h_\varepsilon - h_0\|_{C^3} \leq \frac{C_1 M_g}{1-\kappa}\varepsilon.$$

This ends the proof. □

6. Conclusion

In this work, we analyzed a sequential fractional KdV equation of two different orders, incorporating a nonlinear perturbation term. The fractional derivatives are taken in the Caputo sense, and the problem generalizes the classical KdV equation by introducing nonlocal dispersion effects.

Using the traveling wave transformation, we reduced the fractional partial differential equation to a fractional ordinary differential equation. For the unperturbed problem (when $\epsilon = 0$), we applied Banach's fixed point theorem to prove the existence and uniqueness of traveling wave solutions under some conditions on the parameters of the sequential problem. Additionally, using Schaefer's fixed point theorem, we established other existence results under weaker assumptions.

For the perturbed problem (when $\epsilon \neq 0$), we showed that the Lipschitz conditions of the perturbation term guarantees uniqueness of the solution. We also proved an existence result by imposing suitable sufficient conditions on the parameter a and the nonlinear term.

We also presented a stability and asymptotic analysis. Specifically, we proved the continuous dependence of solutions on the perturbation parameter and showed that as the perturbation tends to zero, the solution of the perturbed problem converges uniformly to the solution of the unperturbed problem. A key contribution is the quantitative estimate that links the solution of the perturbed problem to that of the unperturbed one.

Collectively, our main results extend classical KdV theory to the sequential fractional nonlocal considerations. They provide analytical results for studying nonlinear fractional dispersive wave problems under small regular perturbations, with explicit control over the effect of the perturbation.

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