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CONVERGENCE OF SOLUTIONS OF CERTAIN NON-HOMOGENEOUS THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with differential equations of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

where a is a positive constant and g,h and p are continuous in their respective arguments, with functions g and h not necessarily differentiable. By introducing a complete Lyapunov function, as well as restricting the incrementary ratio $\eta^{-1}\{h(\xi+\eta) - h(\xi)\}, (\eta \neq 0)$, of h to a closed sub-interval of the Routh-Hurwitz interval, we prove the convergence of solutions for this equation. This generalizes earlier results.

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1. INTRODUCTION

This paper considers the convergence of solutions of the third-order nonlinear differential equations of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}) \tag{1}$$

in which $a > 0$ is a constant, functions g, h and p are continuous in their respective arguments.

Any two solutions $x_1(t)$,$x_2(t)$ of (1) are said to *converge* if

$$x_2(t) - x_1(t) \longrightarrow 0, \quad \dot{x}_2(t) - \dot{x}_1(t) \longrightarrow 0, \ddot{x}_2(t) - \ddot{x}_1(t) \longrightarrow 0 \tag{2}$$

as $t \to \infty$. If the relations (2) are true of each (arbitrary) pair of solutions of (1) we shall describe this by saying that all solutions of (1) *converge*.

Many results have been obtained on the convergence of solutions of third-order differential equations ([2, 5, 7, 9, 10]). In [10], Tejumola established the convergence of solutions of (1) assuming that h (of class C') and g satisfies the Routh-Hurwitz condition

$$h'(x) \leq c \leq ab, \quad \frac{h(x_2) - h(x_1)}{x_2 - x_1} \geq \delta > 0, \quad (x_1 \neq x_2), \tag{3}$$

for some constants b, c and δ, and

$$0 < b \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq b_0 < \infty, \quad (y_1 \neq y_1) \tag{4}$$

respectively, with additional conditions that

$$\{h(x_2 - x_1) - h(x_2) + h(x_1)\}^2 \leq B(x_2 - x_1)^2, \tag{5}$$

and

$$\{g(y_2 - y_1) - g(y_2) + g(y_1)\}^2 \leq C(y_2 - y_1)^2 \tag{6}$$

with constants $B,\ C$ sufficiently small.

In this work, we consider a somewhat different approach to [5] in that we assume $g(\dot{x})$, $h(x)$ are not necessarily differentiable but satisfy (4) and

$$0 < \delta \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq kab, \quad (x_1 \neq x_2), \tag{7}$$

respectively, where $k < 1$ is a positive constant whose estimate is given later. Moreover, we do not need additional conditions (5) and (6).

2. MAIN RESULTS

The main results in this paper, which are in some respects generalizations of [4], are the following :

Theorem 1. *Suppose that* $g(0) = h(0)$ *and that*

(i) there are constants $b_0 > 0,\ b > 0$ *such that* $g(y)$ *satisfies inequalities (4) ;*

(ii) there are constants $\delta > 0,\ k < 1$ *such that for any* $\xi, \eta, (\eta \neq 0)$*,the incrementary ratio for* h *satisfies*

$$\eta^{-1}\{h(\xi + \eta) - h(\xi)\} \quad \textit{lies} \quad \textit{in} \quad I_0 \tag{8}$$

with $I_0 = [\delta, kab]$*;*

(iii) there is a continuous function $\phi(t)$ *such that*

$$\begin{aligned} &|p(t, x_2, y_2, z_2) - p(t, x_1, y_1, z_1)| \\ &\qquad \leq \phi(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|\} \end{aligned} \tag{9}$$

holds for arbitrary $t,\ x_1,\ y_1,\ z_1,\ x_2,\ y_2$ *and* z_2*.*

Then, there exists a constant D_1 *such that if*

$$\int_0^t \phi^\nu(\tau)d\tau \leq D_1 t \tag{10}$$

for some ν*, in the range* $1 \leq \nu \leq 2$*, then all solutions of (1) converge.*

A very important step in the proof of Theorem 1 will be to give estimate for any two solutions of (1). This in itself , being of independent interest, is giving as:

Theorem 2. *Let $x_1(t), x_2(t)$ be any two solutions of (1). Suppose that all the conditions of Theorem 1 are satisfied , then for each fixed ν, in the range $1 \leq \nu \leq 2$, there exists constants D_2, D_3 and D_4 such that for $t_2 \geq t_1$,*

$$S(t_2) \leq D_2 S(t_1) exp\{-D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \phi^{\nu}(\tau) d\tau\} \tag{11}$$

where

$$S(t) = \{[x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2\}. \tag{12}$$

If we put $x_1(t) = 0$ and $t_1 = 0$, we immediately obtain:

Corollary 1. *If $p = 0$ and hypotheses (i) and (ii) of Theorem 1 hold, then the trivial solution of (1) is exponentially stable in the large.*

Further, if we put $\xi = 0$ in (2.1) with $\eta(\eta \neq 0)$ arbitrary, we obtain :

Corollary 2. *If $p \neq 0$ and hypotheses (i) and (ii) of Theorem 1 hold for arbitrary $\eta(\eta \neq 0)$, and $\xi = 0$, then there exists a constant $D_5 > 0$ such that every solution $x(t)$ of (1) satisfies*

$$|x(t)| \leq D_5 \quad ; \quad |\dot{x}(t)| \leq D_5 \quad ; \quad |\ddot{x}(t)| \leq D_5. \tag{13}$$

3. PRELIMINARY RESULTS

By setting $\dot{x} = y$, $\dot{y} = z$, the equation (1) may be replaced with the system

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -az - g(y) - h(x) + p(t, x, y, z) \tag{14}$$

Let $(x_i(t), y_i(t), z_i(t))$, $(i = 1, 2)$, be any two solutions of (3.1) such that

$$b \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq b_0 \quad (y_2 \neq y_1); \tag{15}$$

and

$$\delta \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq kab, \quad (x_2 \neq x_1), \tag{16}$$

where b_0, b, δ and k are finite constants, k will be determined later.

Our main tool in the proof of the convergence theorems will be the following function, $W = W(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ defined by

$$\begin{aligned} 2W &= \beta(1-\beta)b^2(x_2-x_1)^2 + \beta b(y_2-y_1)^2 + \alpha b a^{-1}(y_2-y_1)^2 \\ &\quad + \alpha a^{-1}(z_2-z_1)^2 + \{(z_2-z_1) + a(y_2-y_1) + (1-\beta)b(x_2-x_1)\}^2, \end{aligned} \tag{17}$$

where $0 < \beta < 1$ and $\alpha > 0$ are constants.
This is an adaptation of the function V used in [6].
Following the argument used in [5], we can easily verify the following for W.

Lemma 1.

(i) $W(0,0,0) = 0$; *and*

(ii) there exist finite constants $D_6 > 0$, $D_7 > 0$ *such that*

$$\begin{aligned} & D_6\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\} \\ & \quad \le W \le D_7\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\}. \end{aligned} \tag{18}$$

If we define the function $W(t)$ by

$$W(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t))$$

and using the fact that the solutions (x_i, y_i, z_i), $(i = 1, 2)$, satisfy (14), then $S(t)$ as defined in (12) becomes

$$S(t) = \{[x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2 + [z_2(t) - z_1(t)]^2\}. \tag{19}$$

We can then prove the following result on the derivative of $W(t)$ with respect to t.

Lemma 2. *Let the hypotheses (i) and (ii) of Theorem 1 hold. Then, there exist positive finite constants* D_8 *and* D_9 *such that*

$$\frac{dW}{dt} \le -2D_8 S + D_9 S^{\frac{1}{2}}|\theta| \tag{20}$$

where $\theta = p(t, x_2, y_2, z_2) - p(t, x_1, y_1, z_1)$.

Proof of Lemma 2. On using (3.1), a direct computation of $\frac{dW}{dt}$ gives after simplification

$$\frac{dW}{dt} = -W_1 - W_2 - W_3 - W_4 - W_5 + W_6 \tag{21}$$

where

$$\begin{aligned} W_1 &= \frac{1}{4}b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 + \frac{1}{4}\alpha(z_2-z_1)^2 \\ &\quad + \frac{1}{4}a[G(y_2,y_1) - b(1-\beta)](y_2-y_1)^2; \end{aligned}$$

$$\begin{aligned} W_2 &= \frac{1}{4}b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 + \frac{1}{4}\alpha(z_2-z_1)^2 \\ &\quad + (1+\alpha a^{-1})H(x_2,x_1)(x_2-x_1)(z_2-z_1); \end{aligned}$$

$$\begin{aligned} W_3 &= \frac{1}{4}b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 \\ &\quad + \frac{1}{4}a[G(y_2,y_1) - b(1-\beta)](y_2-y_1)^2 \\ &\quad + aH(x_2,x_1)(x_2-x_1)(y_2-y_1); \end{aligned}$$

$$\begin{aligned} W_4 &= \frac{1}{4}b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 \\ &\quad + \frac{1}{4}a[G(y_2,y_1) - b(1-\beta)](y_2-y_1)^2 \\ &\quad + b(1-\beta)[G(y_2,y_1) - b](x_2-x_1)(y_2-y_1); \end{aligned}$$

$$\begin{aligned} W_5 &= \frac{1}{4}a[G(y_2,y_1) - b(1-\beta)](y_2-y_1)^2 + \frac{1}{4}\alpha(z_2-z_1)^2 \\ &\quad + (\alpha a^{-1}+1)[G(y_2,y_1) - b](y_2-y_1)(z_2-z_1); \end{aligned}$$

$$W_6 = \{b(1-\beta)(x_2-x_1) + a(y_2-y_1) + (1+\alpha a^{-1})(z_2-z_1)\}\theta(t);$$

with

$$G(y_2,y_1) = \frac{g(y_2)-g(y_1)}{y_2-y_1}, \quad (y_2 \neq y_1); \tag{22}$$

and

$$H(x_2, x_1) \quad = \quad \frac{h(x_2) - h(x_1)}{x_2 - x_1}, \quad (x_2 \neq x_1). \tag{23}$$

Further, let us denote $H(x_2, x_1)$ and $G(y_2, y_1)$ simply by H and G, respectively.

For strictly positive constants $k_1, \quad k_2, \quad k_3, \quad and \quad k_4$ conveniently chosen later, we have

$$\begin{aligned}(\alpha a^{-1} \quad &+ \quad 1)H(x_2 - x_1)(z_2 - z_1) \\ &= \quad \{k_1(\alpha a^{-1} + 1)^{\frac{1}{2}} H^{\frac{1}{2}}(x_2 - x_1) + \frac{1}{2}k_1^{-1}(\alpha a^{-1} + 1)^{\frac{1}{2}} H^{\frac{1}{2}}(z_2 - z_1)\}^2 \\ &\quad -k_1^2(\alpha a^{-1} + 1)H(x_2 - x_1)^2 - \frac{1}{4}k_1^{-2}(\alpha a^{-1} + 1)H(z_2 - z_1)^2;\end{aligned}$$

$$\begin{aligned}aH(x_2 - x_1)(y_2 - y_1) \quad &= \quad \{k_2 a^{\frac{1}{2}} H^{\frac{1}{2}}(x_2 - x_1) + \frac{1}{2}k_2^{-1} a^{\frac{1}{2}} H^{\frac{1}{2}}(y_2 - y_1)\}^2 \\ &\quad -k_2^2 aH(x_2 - x_1)^2 - \frac{1}{4}k_2^{-2} aH(y_2 - y_1)^2;\end{aligned}$$

$$b(1 - \beta)(G - b)(x_2 - x_1)(y_2 - y_1)$$

$$= \{k_3 b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(G - b)^{\frac{1}{2}}(x_2 - x_1) + \tfrac{1}{2}k_3^{-1} b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(G - b)^{\frac{1}{2}}(y_2 - y_1)\}^2$$

$$-k_3^2 b(1 - \beta)(G - b)(x_2 - x_1)^2 - \tfrac{1}{4}k_3^{-2} b(1 - \beta)(G - b)(y_2 - y_1)^2;$$

$$(\alpha a^{-1} + 1)(G - b)(y_2 - y_1)(z_2 - z_1)$$

$$= \{k_4(\alpha a^{-1} + 1)^{\frac{1}{2}}(G - b)^{\frac{1}{2}}(y_2 - y_1)$$

$$+\tfrac{1}{2}k_4^{-1}(\alpha a^{-1} + 1)^{\frac{1}{2}}(G - b)^{\frac{1}{2}}(z_2 - z_1)\}^2$$

$$-k_4^2(\alpha a^{-1} + 1)(G - b)(y_2 - y_1)^2 - \tfrac{1}{4}k_1^{-2}(\alpha a^{-1} + 1)(G - b)(z_2 - z_1)^2;$$

Thus,

$$W_2 \quad = \{k_1(\alpha a^{-1} + 1)^{1/2} H^{1/2}(x_2 - x_1) + \frac{1}{2}k_1^{-1}(\alpha a_1^{-1} + 1)^{1/2} H^{1/2}(z_2 - z_1)\}^2$$

$$+\{\tfrac{1}{4}b(1 - \beta)H - k_1^2(\alpha a^{-1} + 1)H\}(x_2 - x_1)^2$$

$$+\{\tfrac{1}{4}\alpha - \tfrac{1}{4}k_1^{-2}(\alpha a^{-1} + 1)H\}(z_2 - z_1)^2;$$

$$W_3 = \{k_2 a^{1/2} H^{1/2}(x_2 - x_1) + \frac{1}{2} k_2^{-1} a^{1/2} H^{1/2}(y_2 - y_1)\}^2$$
$$+\{\tfrac{1}{4} a[G - b(1-\beta)] - \tfrac{1}{4} k_2^{-2} aH\}(y_2 - y_1)^2$$
$$+\{\tfrac{1}{4} b(1-\beta)H - k_2^2 aH\}(x_2 - x_1)^2;$$

$$W_4 = \{k_3 b^{1/2}(1-\beta)^{1/2}(G-b)^{1/2}(x_2 - x_1)$$
$$+\tfrac{1}{2} k_3^{-1} b^{1/2}(1-\beta)^{1/2}(G-b)^{1/2}(y_2 - y_1)\}^2$$
$$+\{\tfrac{1}{4} b(1-\beta)H - k_3^2 b(1-\beta)(G-b)\}(x_2 - x_1)^2$$
$$+\{\tfrac{1}{4} a[G - b(1-\beta)] - \tfrac{1}{4} k_3^{-2} b(1-\beta)(G-b)\}(y_2 - y_1)^2;$$

$$W_5 = \{k_1(\alpha a^{-1} + 1)^{1/2}(G-b)^{1/2}(y_2 - y_1)$$
$$+\tfrac{1}{2} k_4^{-1}(\alpha a_1^{-1} + 1)^{1/2}(G-b)^{1/2}(z_2 - z_1)\}^2$$
$$+\{\tfrac{1}{4} a[G - b(1-\beta)] - k_4^2(\alpha a^{-1} + 1)(G-b)\}(y_2 - y_1)^2$$
$$+\{\tfrac{1}{4}\alpha - \tfrac{1}{4} k_4^{-2}(\alpha a^{-1} + 1)(G-b)\}(z_2 - z_1)^2.$$

Furthermore, by using (15), we obtain for all $x_i,\ z_i$ (i=1,2) in $I\!R$,

$$W_2 \geq 0, \tag{24}$$

if

$$k_1^2 \leq \frac{(1-\beta)ab}{4(a+\alpha)} \qquad \text{with} \qquad H \leq \frac{\alpha(1-\beta)a^2 b}{16(a+\alpha)^2},$$

and for all x_i, y_i (i=1,2) in $I\!R$,

$$W_3 \geq 0 \tag{25}$$

if

$$k_2^2 \leq \frac{b(1-\beta)}{4a} \qquad \text{with} \qquad H \leq \frac{\beta(1-\beta)b^2}{4a}.$$

Combining all the inequalities in (24) and (25), we have for all x_i, y_i, z_i (i=1,2) in $I\!R$,

$$W_2 \geq 0 \qquad \text{and} \qquad W_3 \geq 0$$

if

$$H \;\leq\; kab \qquad \text{with} \qquad k = \min\{\frac{\alpha(1-\beta)ab}{16(a+\alpha)^2}; \frac{\beta(1-\beta)b}{4a^2}\} < 1. \tag{26}$$

Also, for all x_i, y_i (i=1,2) in $I\!R$,

$$W_4 \geq 0 \tag{27}$$

if

$$\frac{(1-\beta)(b_0-b)}{a\beta} \leq k_3^2 \leq \frac{\delta}{b_0-b},$$

and for all y_i, z_i (i=1,2) in $I\!R$,

$$W_5 \geq 0, \tag{28}$$

if

$$\frac{(a+\alpha)(b_0-b)}{a\alpha} \leq k_4^2 \leq \frac{a^2 b\beta}{4(a+\alpha)(b_0-b)}.$$

We are now left with estimates W_1 and W_6. There exists a positive constant D_{10} such that

$$W_1 \geq D_{10}\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\} \tag{29}$$

where $\quad D_{10} = \frac{1}{4}\min\{b\delta(1-\beta);\; ab\beta;\; \alpha\}$
while

$$W_6 \leq D_{11}\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\}^{1/2}|\theta(t)| \tag{30}$$

where

$$D_{11} = 2\max\{b(1-\beta);\; a;\; (1+\alpha a^{-1})\}.$$

Thus, combining (24), (25), (27) in (21) and using (19) we have that

$$\frac{dW}{dt} \;\leq\; -D_{10}S(t) + D_{11}S^{\frac{1}{2}}(t)|\theta(t)|. \tag{31}$$

This completes the proof of Lemma 2. □

4. PROOF OF THEOREM 2

This follows directly from [5], on using inequality (31) . Let ν be any constant in the range $1 \le \nu \le 2$. Set $2\mu = 2 - \nu$, so that $0 \le 2\mu \le 1$. We re-write (31) in the form

$$\frac{dW}{dt} + D_{10}S \le D_{11}S^{\mu}W^{*} \tag{32}$$

where

$$W^{*} = (|\theta| - D_{10}D_{11}^{-1}S^{\frac{1}{2}})S^{\frac{1}{2}-\mu}.$$

Considering the two cases

(i) $|\theta| \le D_{10}D_{11}^{-1}S^{\frac{1}{2}}$ and

(ii) $|\theta| > D_{10}D_{11}^{-1}S^{\frac{1}{2}}$

separately, we find that in either case, there exists some constant D_{12} such that $W^{*} \le D_{12}|\theta|^{2(1-\mu)}$. Thus, using (9), inequality (32) becomes

$$\frac{dW}{dt} + D_{10}S \le D_{13}S^{\mu}\phi^{2(1-\mu)}S^{1-\mu} \tag{33}$$

where $D_{13} \ge 2D_{11}D_{12}$. This immediately gives

$$\frac{dW}{dt} + (D_{14} - D_{15}\phi^{\nu}(t))W \le 0 \tag{34}$$

after using Lemma 1 on W, with D_{14} and D_{15} as some positive constants. On integrating (34) from t_1 to t_2 , $(t_2 \ge t_1)$, we obtain

$$W(t_2) \le W(t_1)\exp\{-D_{14}(t_2 - t_1) + D_{15}\int_{t_1}^{t_2} \phi^{\nu}(\tau)d\tau\}. \tag{35}$$

Again, using Lemma 1, we obtain (11), with $D_2 = D_7D_6^{-1}$,$D_3 = D_{14}$ and $D_4 = D_{15}$. This completes the proof of Theorem 2. $\square$

5. PROOF OF THEOREM 1

This follows from the estimate (11) and the condition (10) on $\phi(t)$. Choose $D_1 = D_3 D_4^{-1}$ in (10). Then, as $t = (t_2 - t_1) \longrightarrow \infty$, $S(t) \longrightarrow 0$, which proves that as $t \longrightarrow \infty$,

$$x_2(t) - x_1(t) \longrightarrow 0, \quad \dot{x}_2(t) - \dot{x}_1(t) \longrightarrow 0, \quad \ddot{x}_2(t) - \ddot{x}_1(t) \longrightarrow 0.$$

This completes the proof of Theorem 1. □

Remark: As remarked in [5], the results remain valid if we replace $\phi(t)$ in (10) by a constant $D_{16} > 0$.

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