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BOUNDS FOR THE DISTANCE ENERGY OF A GRAPH

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Abstract. The distance energy of a graph G is defined as the sum of the absolute values of the eigenvalues of the distance matrix of G. Recently bounds for the distance energy of a graph of diameter 2 were determined. In this paper we obtain bounds for the distance energy of any connected graph G, thus generalizing the earlier results.

1. INTRODUCTION

In this paper we are concerned with simple graphs, that is graphs without loops, multiple or directed edges. Let G be such a graph, possessing n vertices and m edges. We say that G is an (n,m)-graph.

Let the graph G be connected and let its vertices be labelled as $v_1, v_2, \ldots, v_n$. The distance matrix of a graph G is defined as a square matrix $D = D(G) = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G [3,5]. The eigenvalues of the distance matrix $D(G)$ are denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and are said to be the D-eigenvalues of G. Since the distance matrix is symmetric, its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

The characteristic polynomial and eigenvalues of the distance matrix of a graph are considered in [6–8,14,15,18,36].

The distance energy $E_D = E_D(G)$ of a graph G is defined as [18]

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i| \ .$$

The distance energy is defined in analogy to the graph energy [9]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix $A(G)$ of a graph G [5]. For more results on $E(G)$ see [1,2,4,10–13,16,17,19–25,27–35,37].

If $G = K_n$, the complete graph on n vertices, then $A(K_n) = D(K_n)$ and hence $E_D(G) = E(G) = 2(n-1)$.

In a recent paper [18] Indulal, Gutman, and Vijaykumar reported lower and upper bounds for the distance energy of graphs whose diameter (= maximal distance between vertices) does not exceed two. In this paper we obtain bounds for the distance energy of arbitrary connected (n,m)-graphs, which generalize the results obtained in [18].

We first need the following Lemma.

Lemma 1. *Let G be a connected (n,m)-graph, and let $\mu_1, \mu_2, \dots, \mu_n$ be its D-eigenvalues. Then*

$$\sum_{i=1}^{n} \mu_i = 0$$

and

$$\sum_{i=1}^{n} \mu_i^2 = 2 \sum_{1 \le i < j \le n} (d_{ij})^2 \ . \tag{1}$$

Proof.

$$\sum_{i=1}^{n} \mu_i = \text{trace}[D(G)] = \sum_{i=1}^{n} d_{ii} = 0 \ .$$

For $i = 1, 2, \dots, n$, the (i,i)-entry of $[D(G)]^2$ is equal to $\sum_{j=1}^{n} d_{ij}\, d_{ji} = \sum_{j=1}^{n} (d_{ij})^2$. Hence

$$\sum_{i=1}^{n} \mu_i^2 = \text{trace}[D(G)]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij})^2 = 2 \sum_{1 \le i < j \le n} (d_{ij})^2 \ .$$

□

Corollary 1.1 [18]. *Let G be a connected (n,m)-graph, and let $diam(G) \le 2$, where $diam(G)$ denotes the diameter of a graph G. Then*

$$\sum_{i=1}^{n} \mu_i^2 = 2[2n^2 - 2n - 3m] \ .$$

2. BOUNDS FOR THE DISTANCE ENERGY

Theorem 2. *If G is a connected (n,m)-graph, then*

$$\sqrt{2 \sum_{1 \le i < j \le n} (d_{ij})^2} \le E_D(G) \le \sqrt{2n \sum_{1 \le i < j \le n} (d_{ij})^2} \ .$$

Proof. Consider the Cauchy–Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) .$$

Choosing $a_i = 1$ and $b_i = |\mu_i|$, we get

$$\left(\sum_{i=1}^{n} |\mu_i|\right)^2 \leq n \sum_{i=1}^{n} \mu_i^2$$

from which

$$E_D(G)^2 \leq 2n \sum_{1 \leq i < j \leq n} (d_{ij})^2 .$$

This leads to the upper bound for $E_D(G)$.

Now

$$E_D(G)^2 = \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \geq \sum_{i=1}^{n} |\mu_i|^2 = 2 \sum_{1 \leq i < j \leq n} (d_{ij})^2$$

which straightforwardly leads to the lower bound for $E_D(G)$. □

Corollary 2.1. *If G is a connected (n,m)-graph, then $E_D(G) \geq \sqrt{n(n-1)}$.*

Proof. Since $d_{ij} \geq 1$ for $i \neq j$ and there are $n(n-1)/2$ pairs of vertices in G, from the lower bound of Theorem 2,

$$E_D(G) \geq \sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2} \geq \sqrt{2 \frac{n(n-1)}{2}} = \sqrt{n(n-1)} .\square$$

□

Theorem 3. *Let G be a connected (n,m)-graph and let Δ be the absolute value of the determinant of the distance matrix $D(G)$. Then*

$$\sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + n(n-1)\Delta^{2/n}} \leq E_D(G) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} (d_{ij})^2} .$$

Proof. In view of Theorem 2, we only need to demonstrate the validity of the lower bound. This is done analogously to the way in which a lower bound for graph energy is deduced in [26].

By definition of distance energy and Eq. (1)

$$
\begin{aligned}
E_D(G)^2 &= \left(\sum_{i=1}^{n} |\mu_i|\right)^2 = \sum_{i=1}^{n} \mu_i^2 + 2 \sum_{i \le i < j \le n} |\mu_i||\mu_j| \\
&= 2 \sum_{1 \le i < j \le n} (d_{ij})^2 + 2 \sum_{i \le i < j \le n} |\mu_i||\mu_j| \\
&= 2 \sum_{1 \le i < j \le n} (d_{ij})^2 + \sum_{i \ne j} |\mu_i||\mu_j| \ . \qquad (2)
\end{aligned}
$$

Since for nonnegative numbers the arithmetic mean is not smaller than the geometric mean,

$$
\begin{aligned}
\frac{1}{n(n-1)} \sum_{i \ne j} |\mu_i||\mu_j| &\ge \left(\prod_{i \ne j} |\mu_i||\mu_j|\right)^{1/n(n-1)} = \left(\prod_{i=1}^{n} |\mu_i|^{2(n-1)}\right)^{1/n(n-1)} \\
&= \prod_{i=1}^{n} |\mu_i|^{2/n} = \Delta^{2/n} \ . \qquad (3)
\end{aligned}
$$

Combining Eqs. (2) and (3) we arrive at the lower bound. □

Using Eq. (1), Corollary 1.1 and Theorem 3 we have following result.

Corollary 3.1 [18]. *Let G be a connected (n,m)-graph with $diam(G) \le 2$. Then*

$$
\sqrt{4n(n-1) - 6m + n(n-1)\Delta^{n/2}} \le E_D(G) \le \sqrt{2n(2\,n^2 - 2n - 3m)} \ .
$$

For an n-vertex tree T [3,6],

$$
\det D(T) = (-1)^{n-1}(n-1)2^{n-2}
$$

from which we obtain the following:

Corollary 3.2. *For an n-vertex tree T ,*

$$
\sqrt{2 \sum_{1 \le i < j \le n} (d_{ij})^2 + n\left[(n-1)^{n+2} 4^{n-2}\right]^{1/n}} \le E_D(T) \le \sqrt{2n \sum_{1 \le i < j \le n} (d_{ij})^2} \ .
$$

□

Theorem 4. *If G is a connected (n,m)-graph, then*

$$E_D(G) \leq \frac{2}{n}\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sqrt{(n-1)\left[2\sum_{1\leq i<j\leq n}(d_{ij})^2 - \left(\frac{2}{n}\sum_{1\leq i<j\leq n}(d_{ij})^2\right)^2\right]}\ . \qquad (4)$$

Proof. Our proof follows the ideas of Koolen and Moulton [22,23], who obtained an analogous upper bound for ordinary graph energy $E(G)$.

By applying the Cauchy–Schwartz inequality to the two $(n-1)$ vectors $(1,1,\ldots,1)$ and $(|\mu_2|,|\mu_3|,\ldots,|\mu_n|)$, we get

$$\begin{aligned}
\left(\sum_{i=2}^{n}|\mu_i|\right)^2 &\leq (n-1)\left(\sum_{i=2}^{n}\mu_i^2\right) \\
(E_D(G)-\mu_1)^2 &\leq (n-1)\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2-\mu_1^2\right) \\
E_D(G) &\leq \mu_1+\sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2-\mu_1^2\right)}\ .
\end{aligned}$$

Define the function

$$f(x) = x + \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2 - x^2\right)}\ .$$

We set $\mu_1 = x$ and bear in mind that $\mu_1 \geq 1$. From

$$\sum_{i=1}^{n}\mu_i^2 = 2\sum_{1\leq i<j\leq n}(d_{ij})^2$$

we get

$$x^2 = \mu_1^2 \leq 2\sum_{1\leq i<j\leq n}(d_{ij})^2 \qquad \text{i. e.} \qquad x \leq \sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2}\ .$$

Now, $f'(x) = 0$ implies

$$x = \sqrt{\frac{2}{n}\sum_{1\leq i<j\leq n}(d_{ij})^2}\ .$$

Therefore $f(x)$ is a decreasing function in the interval

$$\sqrt{\frac{2}{n}\sum_{1\le i<j\le n}(d_{ij})^2} \le x \le 2\sqrt{\sum_{1\le i<j\le n}(d_{ij})^2}$$

and

$$\sqrt{\frac{2}{n}\sum_{1\le i<j\le n}(d_{ij})^2} \le \frac{2}{n}\sum_{1\le i<j\le n}(d_{ij})^2 \le \mu_1 \ .$$

Hence

$$f(\mu_1) \le f\left(\frac{2}{n}\sum_{1\le i<j\le n}(d_{ij})^2\right)$$

and inequality (4) follows. □

From Eq. (1), Corollary 1.1, and Theorem 4 we obtain:

Corollary 4.1. *Let G be a connected (n,m)-graph with $diam(G) \le 2$. Then*

$$E_D(G) \le \frac{4n^2-4n-6m}{n} + \sqrt{(n-1)\left[4n(n-1)-6m-\left(\frac{4n(n-1)-6m}{n}\right)^2\right]} \ .$$

□

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