BOUNDS FOR THE DISTANCE ENERGY
OF A GRAPH

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Abstract. The distance energy of a graph G is defined as the sum of the absolute values
of the eigenvalues of the distance matrix of G. Recently bounds for the distance energy of
a graph of diameter 2 were determined. In this paper we obtain bounds for the distance energy of any connected graph $G$, thus generalizing the earlier results.

1. INTRODUCTION

In this paper we are concerned with simple graphs, that is graphs without loops, multiple or directed edges. Let $G$ be such a graph, possessing $n$ vertices and $m$ edges. We say that $G$ is an $(n, m)$-graph.

Let the graph $G$ be connected and let its vertices be labelled as $v_1, v_2, \ldots, v_n$. The distance matrix of a graph $G$ is defined as a square matrix $D = D(G) = [d_{ij}]$, where $d_{ij}$ is the distance between the vertices $v_i$ and $v_j$ in $G$ [3,5]. The eigenvalues of the distance matrix $D(G)$ are denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and are said to be the $D$-eigenvalues of $G$. Since the distance matrix is symmetric, its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

The characteristic polynomial and eigenvalues of the distance matrix of a graph are considered in [6–8,14,15,18,36].

The distance energy $E_D = E_D(G)$ of a graph $G$ is defined as [18]

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i| .$$

The distance energy is defined in analogy to the graph energy [9]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix $A(G)$ of a graph $G$ [5]. For more results on $E(G)$ see [1,2,4,10–13,16,17,19–25,27–35,37].

If $G = K_n$, the complete graph on $n$ vertices, then $A(K_n) = D(K_n)$ and hence $E_D(G) = E(G) = 2(n - 1)$.

In a recent paper [18] Indulal, Gutman, and Vijaykumar reported lower and upper bounds for the distance energy of graphs whose diameter (= maximal distance between vertices) does not exceed two. In this paper we obtain bounds for the distance
energy of arbitrary connected \((n, m)\)-graphs, which generalize the results obtained in [18].

We first need the following Lemma.

**Lemma 1.** Let \(G\) be a connected \((n, m)\)-graph, and let \(\mu_1, \mu_2, \ldots, \mu_n\) be its \(D\)-eigenvalues. Then

\[
\sum_{i=1}^{n} \mu_i = 0
\]

and

\[
\sum_{i=1}^{n} \mu_i^2 = 2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 .
\]

**Proof.**

\[
\sum_{i=1}^{n} \mu_i = \text{trace}[D(G)] = \sum_{i=1}^{n} d_{ii} = 0 .
\]

For \(i = 1, 2, \ldots, n\), the \((i, i)\)-entry of \([D(G)]^2\) is equal to \(\sum_{j=1}^{n} d_{ij} d_{ji} = \sum_{j=1}^{n} (d_{ij})^2\). Hence

\[
\sum_{i=1}^{n} \mu_i^2 = \text{trace}[D(G)]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij})^2 = 2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 .
\]

\[\square\]

**Corollary 1.1** [18]. Let \(G\) be a connected \((n, m)\)-graph, and let \(\text{diam}(G) \leq 2\), where \(\text{diam}(G)\) denotes the diameter of a graph \(G\). Then

\[
\sum_{i=1}^{n} \mu_i^2 = 2[2n^2 - 2n - 3m] .
\]

2. BOUNDS FOR THE DISTANCE ENERGY

**Theorem 2.** If \(G\) is a connected \((n, m)\)-graph, then

\[
\sqrt{2} \sum_{1 \leq i < j \leq n} (d_{ij})^2 \leq E_D(G) \leq \sqrt{2n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 .
\]
Proof. Consider the Cauchy–Schwartz inequality
\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right).
\]
Choosing \( a_i = 1 \) and \( b_i = |\mu_i| \), we get
\[
\left( \sum_{i=1}^{n} |\mu_i| \right)^2 \leq n \sum_{i=1}^{n} \mu_i^2
\]
from which
\[
E_D(G)^2 \leq 2n \sum_{1 \leq i < j \leq n} (d_{ij})^2.
\]
This leads to the upper bound for \( E_D(G) \).

Now
\[
E_D(G)^2 = \left( \sum_{i=1}^{n} |\mu_i| \right)^2 \geq \sum_{i=1}^{n} |\mu_i|^2 = 2 \sum_{1 \leq i < j \leq n} (d_{ij})^2
\]
which straightforwardly leads to the lower bound for \( E_D(G) \).

Corollary 2.1. If \( G \) is a connected \((n,m)\)-graph, then \( E_D(G) \geq \sqrt{n(n-1)} \).

Proof. Since \( d_{ij} \geq 1 \) for \( i \neq j \) and there are \( n(n-1)/2 \) pairs of vertices in \( G \), from the lower bound of Theorem 2,
\[
E_D(G) \geq \sqrt{\frac{2}{1 \leq i < j \leq n} (d_{ij})^2} \geq \sqrt{\frac{2n(n-1)}{2}} = \sqrt{n(n-1)} \, .
\]

Theorem 3. Let \( G \) be a connected \((n,m)\)-graph and let \( \Delta \) be the absolute value of the determinant of the distance matrix \( D(G) \). Then
\[
\sqrt{\frac{2}{1 \leq i < j \leq n} (d_{ij})^2 + n(n-1)\Delta^{2/n}} \leq E_D(G) \leq \sqrt{\frac{2n}{1 \leq i < j \leq n} (d_{ij})^2}.
\]

Proof. In view of Theorem 2, we only need to demonstrate the validity of the lower bound. This is done analogously to the way in which a lower bound for graph energy is deduced in [26].
By definition of distance energy and Eq. (1)

\[ E_D(G)^2 = \left( \sum_{i=1}^{n} |\mu_i| \right)^2 = \sum_{i=1}^{n} \mu_i^2 + 2 \sum_{i \leq i < j \leq n} |\mu_i||\mu_j| \]

\[ = 2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + 2 \sum_{i \leq i < j \leq n} |\mu_i||\mu_j| \]

\[ = 2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \sum_{i \neq j} |\mu_i||\mu_j| . \]  

\[ (2) \]

Since for nonnegative numbers the arithmetic mean is not smaller than the geometric mean,

\[ \frac{1}{n(n-1)} \sum_{i \neq j} |\mu_i||\mu_j| \geq \left( \prod_{i \neq j} |\mu_i||\mu_j| \right)^{1/n(n-1)} = \left( \prod_{i=1}^{n} |\mu_i|^{2(n-1)} \right)^{1/n(n-1)} \]

\[ = \prod_{i=1}^{n} |\mu_i|^{2/n} = \Delta^{2/n} . \]  

\[ (3) \]

Combining Eqs. (2) and (3) we arrive at the lower bound. \( \square \)

Using Eq. (1), Corollary 1.1 and Theorem 3 we have following result.

**Corollary 3.1** [18]. Let \( G \) be a connected \((n, m)\)-graph with \( \text{diam}(G) \leq 2 \). Then

\[ \sqrt{4n(n-1) - 6m + n(n-1)\Delta^{n/2}} \leq E_D(G) \leq \sqrt{2n(2n^2 - 2n - 3m)} . \]

For an \( n \)-vertex tree \( T \) [3,6],

\[ \det D(T) = (-1)^{n-1}(n-1)2^{n-2} \]

from which we obtain the following:

**Corollary 3.2.** For an \( n \)-vertex tree \( T \),

\[ \sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + n [(n-1)^{n+2}4^{n-2}]^{1/n}} \leq E_D(T) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} (d_{ij})^2} . \]

\( \square \)
Theorem 4. If $G$ is a connected $(n,m)$-graph, then

$$E_D(G) \leq \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \left( n - 1 \right) \left[ \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 - \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 \right)^2 \right]. \quad (4)$$

Proof. Our proof follows the ideas of Koolen and Moulton [22,23], who obtained an analogous upper bound for ordinary graph energy $E(G)$.

By applying the Cauchy–Schwartz inequality to the two $(n-1)$ vectors $(1, 1, \ldots, 1)$ and $(|\mu_2|, |\mu_3|, \ldots, |\mu_n|)$, we get

$$\left( \sum_{i=2}^{n} |\mu_i| \right)^2 \leq (n - 1) \left( \sum_{i=2}^{n} \mu_i^2 \right)$$

$$(E_D(G) - \mu_1)^2 \leq (n - 1) \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 - \mu_1^2 \right)$$

$$E_D(G) \leq \mu_1 + \sqrt{(n - 1) \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 - \mu_1^2 \right)}.$$ 

Define the function

$$f(x) = x + \sqrt{(n - 1) \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 - x^2 \right)}.$$ 

We set $\mu_1 = x$ and bear in mind that $\mu_1 \geq 1$. From

$$\sum_{i=1}^{n} \mu_i^2 = \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2$$

we get

$$x^2 = \mu_1^2 \leq 2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 \quad \text{i. e.} \quad x \leq \sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2}.$$ 

Now, $f'(x) = 0$ implies

$$x = \sqrt{\frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2}.$$
Therefore \( f(x) \) is a decreasing function in the interval

\[ \sqrt{\frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2} \leq x \leq 2 \sqrt{\frac{1}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2} \]

and

\[ \sqrt{\frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2} \leq \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 \leq \mu_1 . \]

Hence

\[ f(\mu_1) \leq f \left( \frac{2}{n} \sum_{1 \leq i < j \leq n} (d_{ij})^2 \right) \]

and inequality (4) follows. \( \square \)

From Eq. (1), Corollary 1.1, and Theorem 4 we obtain:

**Corollary 4.1.** Let \( G \) be a connected \((n, m)\)-graph with \( \text{diam}(G) \leq 2 \). Then

\[ E_D(G) \leq \frac{4n^2 - 4n - 6m}{n} + \sqrt{(n - 1) \left[ 4n(n - 1) - 6m - \left( \frac{4n(n - 1) - 6m}{n} \right)^2 \right]} . \]

\( \square \)

**References**


