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# THE SPECIAL FUNCTION ش , III.

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**Abstract.** We prove the following exact symbolic formula of the special function ش, in the entire s-complex plane with the negative real axis (including the origin) removed, with a double Laplace transform:

$$\text{ش}\,(s) = L\,\{\ 2\cdot\delta\,(t) + L\,\{\ \tfrac{1}{2\pi i}\cdot[\ \text{ش}\,(t\cdot e^{\,-i\pi}) - \text{ش}\,(t\cdot e^{\,i\pi})]\ \}\ \}$$

where $\delta\,(t)$ stands for the distribution of Dirac and $e$ represents the Euler's number.

## 1. THE EXACT SYMBOLIC FORMULA

In [4] and [5] the author introduces and characterizes the discrete and special function ش in the complex field.

We recall that the explicit formula of the special function ش , in the discrete field, is:

$$\text{ش}\,[k, \Omega\ (I_\ell)] \quad = \left(1 + \frac{1}{3k - \Omega\ (I_\ell)}\ \right)^{2k+1}$$

where it is always valid *the following boundary*[1]:

$$\text{ش}[k,\Omega\ (I_\ell - 1)] \prec 2 \prec \text{ش}[k,\Omega\ (I_\ell)] \quad \text{with}\ \Omega\ (I_\ell - 1) = \Omega\ (I_\ell) - 1\ ;\ \forall\ I_\ell\ ,\ k,\ell \in \mathbf{N}$$

The *auxiliary* integer function $\Omega(I_\ell)$, that really represents a growing "step function", is defined, for the intervals of 8 or 9 following values of $k$, in the following way:

- $\Omega\ (I_1) = 0$ for $k$=1,...,8
- $\Omega\ (I_2) = 1$ for $k$=9,...,16 ; $\Omega\ (I_3)$=2 for $k$=17,...,25 ; $\Omega\ (I_4)$=3 for $k$=26,...,34
- $\Omega\ (I_5) = 4$ for $k$=35,...,43 ; $\Omega\ (I_6)$ =5 for $k$=44,...,51 ; $\Omega\ (I_7)$=6 for $k$=52,...,60
- $\Omega\ (I_8) = 7$ for $k$=61,...,69; $\Omega\ (I_9)$ =8 for $k$=70,...,78 ; $\Omega\ (I_{10})$=9 for $k$=79,...,86
- $\Omega\ (I_{11}) = 10$ for $k$=87,...,95 ; $\Omega\ (I_{12})$=11 for $k$=96,...,104. ; etc.

In particular the author proves in ([4], §§5-6) that the special function ش in the real field is completely monotonic for all the closed intervals $I_\ell$ and therefore the following first approximation "≈" in the complex field:

$$\text{ش}(s) \approx L_s\,[F\,(t)] = L_s\,[2\cdot\Theta\,(t)] = \int_0^\infty e^{-st} d\,[2\cdot\ \Theta\,(t)]\ = 2 \tag{1}$$

where $L_s$ denotes the Laplace-Stieltjes transform and $\Theta\,[t]$ represents the unit step function or Heaviside's function ( $\Theta\,[\ t < 0\ ] = 0\ ;\ \Theta\,[\ t > 0\ ] = 1$ )

Now, we observe that the Laplace-Stieltjes transform is closely to related other integral transforms, including the Fourier transform and the Laplace transform.

In particulary if g has derivative g', then the Laplace-Stieltjes transform of g is the Laplace transform of g'.

Consequently, considering the derivative of Heaviside step function, from (1) we have:

[1] The boundary can include the sign "=" if the integer variable $k$ goes towards zero or the infinity.

$$ش(s) \approx L\left[2 \cdot \delta(t)\right] = \int_0^\infty e^{-st} 2\delta(t)\, \mathrm{dt} = 2 \tag{2}$$

where $L$ denotes the ordinary Laplace transform and $\delta(t)$ stands for distribution of Dirac.

In ([5], §2) the author find practically the following nearly exact integral representation:

$$ش(x) \approx 2 + \frac{1}{2}\int_0^\infty \left\{ \frac{1}{\Gamma(2\cdot t)\cdot\Gamma(1-2\cdot t)} \frac{\left((3t-1)^2\right)^t}{\left((3t-2)^2\right)^t} \left[\left|\frac{3t-2}{3t-1}\right| - \frac{3t-2}{3t-1}\right]\right\} \frac{dt}{(x+t)} \tag{3}$$

where $\Gamma$ denotes the *Eulerian gamma function* (or the *Eulerian integral of second kind*).

The approximation is essentially origined by neglecting the point of discontinuities of the first kind of the special function ش , in the real field, between an interval $I_\ell$ and the following $I_{\ell+1}$ as far as the interval $I\ [0,\infty)$.

We recall that extending the field of definition of the integer variable $k$ of the discrete and special function ش to the real positive number, it's possible to notice that such function is actually assimilable to a piecewise continuous function.

On the contrary of equation (2), in the equation (3) there is an Stieltjes transform, that arises naturally as an iteration of the ordinary Laplace transform.

In fact, if

$$f(x) = \int_0^\infty e^{-xt}\varphi(t)\ dt$$

where

$$\varphi(x) = \int_0^\infty e^{-xt}\psi(t)\ dt$$

then, changing the order of integrations in the double integral by appealing to Fubini's Theorem, we have formally ([1], p. 127 and [8], p. 335):

$$f(x) = L\{L\ [\psi\, t]\} = \int_0^\infty e^{-xu}\, du \int_0^\infty e^{-ut}\psi(t)\ dt = \int_0^\infty \psi(t)\ dt \int_0^\infty e^{-u(x+t)}\, du \tag{4}$$

Hence

$$f(x) = \int_0^\infty \frac{\psi(t)}{x+t}\, dt \tag{5}$$

This last equation we refer to as Stieltjes transform, or from another point view as the Stieltjes integral equation.

However, we shall usually be concerned with the more general case in which the integral equation (5) is replaced by a Stieltjes integral:

$$f(x) = \int_0^\infty \frac{d\alpha\ (t)}{x+t}$$

and in this form the equation was considered by T.J. Stieltjes in connection with his work on continued fractions [6].

In (3) the Stieltjes integral, that is analogous to (5), converges and then we have (see [8],Theorem 7b, p. 340):

$$\lim_{\eta\to 0+} \frac{f(-\xi - i\,\eta) - f(-\xi + i\,\eta)}{2\pi\ i} = \frac{\psi(\xi+) + \psi(\xi-)}{2} \tag{6}$$

for any positive $\xi$ at which $\psi(\xi+)$ and $\psi(\xi-)$ exist.

For, simple computation gives :

$$\frac{f(-\xi - i\,\eta) - f(-\xi + i\,\eta)}{2\pi\ i} = \frac{1}{\pi} \int_0^\infty \frac{\eta\psi(t)}{(t-\xi)^2 + \eta^2}\, dt$$

The integral is known as Poisson's integral for the half-plane or as Cauchy's singular integral ([7], p. 30).

The result (6) may also be written symbolically as

$$\frac{f(x\cdot e^{-i\pi}) - f(x\cdot e^{i\pi})}{2\pi\ i} = \psi(x) \tag{7}$$

In fact it is sufficient to compare the equation 11.8.4 in ([7], p. 318).

That being stated in ([4], §6) the author characterizes the holomorphy (**piecewise analytic**) of the special function ش$(s)$ in the cut plane $A = \mathbf{C} \setminus \left[-2/3, -1/3\right]$.

In the close interval $[-1, 0]$ results:

$$ش(s) \;=\; \left(1+\frac{1}{3s+1}\right)^{2s+1} \;=\; \exp\left((2s+1)\cdot Log\left(1+\frac{1}{3s+1}\right)\right) \tag{8}$$

where $Log$ denotes the principal branch of the logarithm in the interest of obtaining a single value mapping.

In effects the special function ش possesses two branch points at $s = -\frac{1}{3}$ and $s = -\frac{2}{3}$ and therefore a cut, joining the two branch points, would prevent $s$ to circulate around them, and the special function ش $(s)$ can be treated as a holomorphic function.

We observe that, in our particular case, the results (7) and (6) with (8) can be written and compute in the following way:

$$\frac{ش(x\cdot e^{-i\pi}) - ش(x\cdot e^{i\pi})}{2\pi i} = \lim_{y\to 0^+} \frac{ش(-x-iy) - ش(-x+iy)}{2\pi i}$$

$$= \lim_{y\to 0^+}\left\{\frac{\exp\left((2(-x-iy)+1)\cdot Log\left(1+\frac{1}{3(-x-iy)+1}\right)\right) - \exp\left((2(-x+iy)+1)\cdot Log\left(1+\frac{1}{3(-x+iy)+1}\right)\right)}{2\pi i}\right\}$$

$$= \frac{\sin\ (2\pi t)}{2\pi}\cdot\frac{\left((3t-1)^2\right)^{\,t}}{\left((3t-2)^2\right)^{\,t}}\cdot\left\{\left|\frac{3t-2}{3t-1}\right| - \frac{3t-2}{3t-1}\right\} \tag{9}$$

Finally, with (2), (3), (4), (9) and using the identity (the Euler reflection formula):

$$\Gamma(\alpha)\cdot\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}$$

we can prove, for the linearity property of the Laplace transformation, the following exact symbolic formula of the special function ش, in the cut plane $A = \mathbf{C}\setminus\ (-\infty, 0]$:

$$ش(s) = L\left\{\ 2\cdot\delta(t) + L\left\{\ \frac{1}{2\pi i}\cdot\left[\ ش(t\cdot e^{\,-i\pi}) - ش(t\cdot e^{\,i\pi})\right]\ \right\}\ \right\} \tag{10}$$

where $\delta(t)$ stands for the distribution of Dirac and $e$ represents the Euler's number.

In conclusion we give also the three-dimensional graph of absolute value of the special function ش of a complex variable $s = x + i\,y$ (Fig. 1) from *DERIVE* computer algebra system and in APPENDIX we explain the choice of the Arabian letter ش .

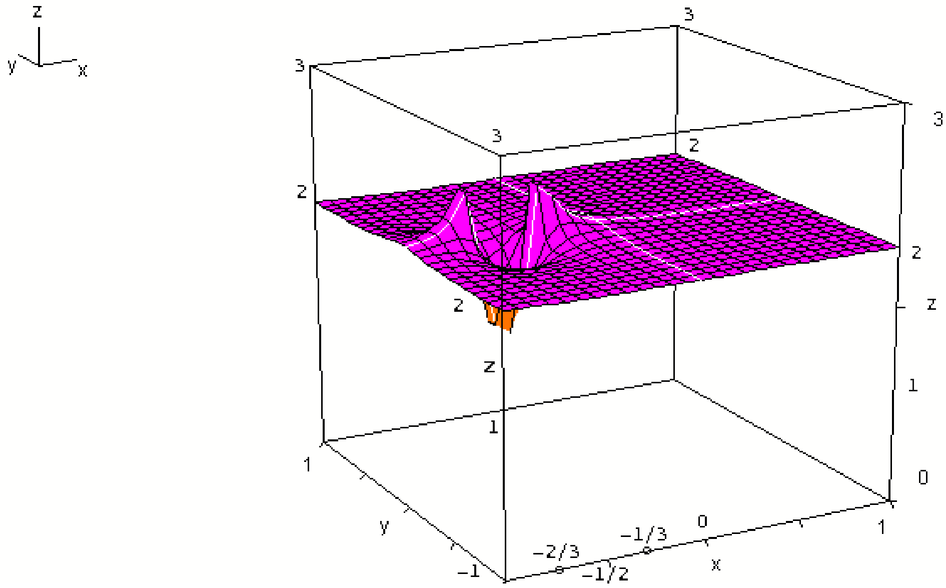


**Fig. 1**

## 2. ADDITIONAL ANALYTIC REMARKS

The beautiful symbolic formula (10) is in practice a consequence of the fact that Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving the distribution of Dirac, sometimes referred to as the symbolic impulse function $\delta(t)$.

Besides that the construction-algorithm ([4], pp. 66-67) of the discrete and special function ش had already evidenced the necessity of an asymmetric impulsive force in order to maintain the set of the points belonging to them all above the horizontal straight line of height 2.

Now, we observe that the asymmetrical impulse function $\delta_+(t)$ is more suitable for use in connection with the one-sided Laplace transformation that the symmetrical impulse function $\delta(t)$.

In effects in the formula (10) the presence of the symbolic impulse function $\delta_+(t)$ would be more appropriated , also because if one applies the Laplace transformation to the "definition" of the impulse function $\delta_+(t)$, one obtains the formal result:

$$L[\delta_+(t)] = 1$$

We recall that the asymmetrical impulse function $\delta_+(t)$ is defined ([3], see Sec. 21.9-6) by:

$$\int_{a+0}^{b} f(\xi)\,\delta_+(\xi - x)\; d\,\xi = \left\{\begin{matrix} 0 & \text{if } x \prec a \text{ or } x \geq b \\ f(x+0) & \text{if } a \leq x \prec b \end{matrix}\right\} \qquad (a \prec b)$$

It is possible to write:

$$\delta_+(t) = \frac{d}{dx} U_+(x)$$

where $U_+(x)$ denotes the asymmetrical unit-step function :

$$U_+(x) = \left\{\begin{matrix} 0 & for\ x \leq 0 \\ 1 & for\ x \succ 0 \end{matrix}\right\}$$

Successively we give the cartesian diagram (Fig. 2) of the unit-step function $U_+(x)$ and of the one related to approximation of the impulse function $\delta_+(t)$.

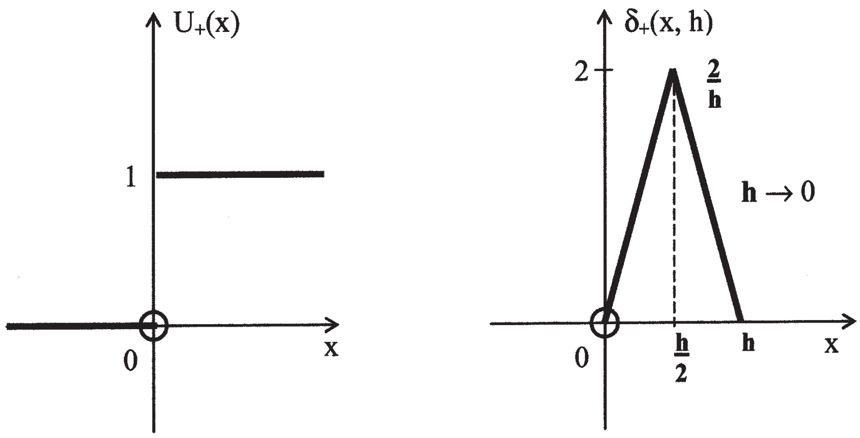


Fig. 2

In the papers [4], [5] and in this paper it has emerged all the importance of the tools like the Laplace transformation and the Stieltjes integral with certain facts from the theory of functions of a complex variable in order to characterizing the analytic properties of the completely monotonic functions.

In particular we recall that the Stieltjes integral is much used in Mechanics and Probability, since it unifies the treatment of the continuous and discrete (and mixed) distributions of mass or probability.

If $\alpha(x)$ is piecewise differentiable, then $d\alpha(x) = \alpha'(x)\,dx$, and the Stieltjes integral is simply in the following form (reduction of a Stieltjes integral to a Riemann integral):

$$\int_a^b f(x)\ \alpha'(x)\,dx$$

For instance a real and similar case to ours is the following: if $\alpha(x)$ is a Heaviside step function, with point masses $m_i$ at $x = x_i$, then

$$d\alpha(x_i) = \lim_{\varepsilon \downarrow 0} [\alpha(x_i + \varepsilon) - \alpha(x_i - \varepsilon)] = m_i, \quad \int_a^b f(x)\,d\,\alpha(x) = \sum_i m_i f(x_i)$$

In this case the integration by parts is usual:

$$\int_a^b f(x)\ dg(x) = f(x) \cdot g(x)\Big|_a^b - \int_a^b g(x)\ df(x)$$

Suppose $\alpha(0) = 0$ , $\alpha(x) = o(e^{cx})$ as $x \to \infty$, and that $\Re s \geq c$, then

$$\int_0^\infty e^{-sx} d\,\alpha(x) = s \cdot \int_0^\infty \alpha(x)\ e^{-sx} dx \tag{11}$$

The integral on the left side represents a Laplace-Stieltjes transform, while the integral on the right side is an ordinary Laplace transform.

More precisely (see [4], compare the equation 14) the second member of (11) is called s-multiplied Laplace transform.

However, we also recall the following result, due to S. Bernstein ([2], pp. 439-440), who was the starting point of many researches in the Probability theory:

**Theorem**: *A function* $\psi(s)$ *on* $(0,\infty)$ *is the Laplace transform of a probability distribution* $F(x)$:

$$\psi(s) = \int_0^\infty e^{-sx} dF(x)$$

*if and only if it is completely monotone in* $(0,\infty)$ with $\psi(0+) = 1$.

In particular, an immediate example for this beautiful theorem is the following function:

$$\psi(s) = \frac{\text{ش}(s)}{2}$$

Notice that the Dirac delta function may be interpreted as a probability density function and that the cumulative distribution function is the Heaviside step function.

It is known that if $X$ is a random variable, the corresponding probability distribution assigns to the interval $[a,\, b]$ the probability $\Pr[a \leq X \leq b]$; for example the probability that the variable $X$ will take a value in the interval $[a,\, b]$.

Now, the probability distribution of the variable $X$ can be uniquely described by its cumulative distribution function $F(x)$, which is defined by:

$$F(x) = \Pr[X \leq x]$$

for any $x \in \mathbf{R}$ and where the right-hand side represent the probability that the variable $X$ takes on a value less than or equal to $x$.

We observe that, in our case, the Heaviside step function is the cumulative distribution function of a random variable which is almost surely 0.

## APPENDIX

The choice of the Arabian letter ش is simply a consequence of the following four reasons:

- the exhaustion of Latin, Greek, Gothic, Jewish and other letters to name a new special function;

- the reference to the strokes of continuous curves (small arcs) that characterize the same function ( [4], see Fig. 3);
- the three dots over the letter that one by one remember the three second-order Eulerian numbers: **1, 120, 494**, inside the Eulerian triangle ( [4], see Fig. 6);
- the meaning of such a letter, that in Al Karaji's algebra (about 1000 A.D.)[2] was declared as the unknown "par excellence", that is absolutely comparable to our "x", but also with the characteristic to form the only conjunction-"ring" between the world of the *unknown* (algebra) and the world of the *known* (arithmetics).

Actually it seems that the origin of the symbol ش , that is pronounced *shin* , is in the Ancient Egypt.

In fact the hieroglyph 𓈙, that is the Egyptian syllable *sha*, is similar and it represents papyrus plants along the Nile.

**Acknowledgements.** This paper and the papers [4], [5] are dedicated to the memory of my father Alessandro Ossicini.

# References

[1] C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Ergebnisse der Math. **87**, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[2] W. Feller, *An Introduction to Probability Theory and Its Application*, vol **2**, 2nd edition, John Wiley & Sons, New York, 1971.

[2] *Al-Karaji Abu Bakr Muhammad ibn al-Hasan (fl. c. 400/1009) : texts and studies* / Collected and repr. by Fuat Sezgin. - Frankfurt/M.: Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 1998. (Publications of the Institute for the History of Arabic-Islamic Science : Islamic mathematics and astronomy; n. 38).

[3] G. Korn and T. Korn, *Mathematical Handbook for Scientists and Engineers*, 2nd edition, Dover, New York, 2000.

[4] A. Ossicini, *The special function* ش , Kragujevac J. Math. **27**, (2005), 63-90.

[5] A. Ossicini, *The special function* ش , II, Kragujevac J. Math. **29**, (2006), 141-150.

[6] T.J. Stieltjes, *Recherches sur les fractions continues*, Annales de la faculté de sciences de Toulouse, vol. **8**, (1894), 1-122.

[7] E.C. Titchmarsh, *Introduction to the Theory of Fourier integrals*, 2nd edition, Oxford University Press, 1948; reprinted New York, Chelsea, 1986.

[8] D.V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, NJ, 1946.