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LIPSCHITZ ESTIMATES FOR MULTILINEAR MARCINKIEWICZ OPERATORS

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Abstract. The purpose of this paper is to establish the boundedness for some multilinear operators generated by Marcinkiewicz operators and Lipschitz functions on Hardy and Herz-Hardy spaces.

1. INTRODUCTION

As the development of singular integral operators T, their commutators and multilinear operators have been well studied(see [2, 3, 4, 5, 6, 7]). From [3, 4, 5, 6, 7], we know that the commutators and multilinear operators generated by T and the BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [1]) proves a similar result when T is replaced by the fractional integral operator. However, it was observed that the commutators and multilinear operators are not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ for $0 < p \leq 1$. But, the boundedness hold if the BMO functions are replaced by the the Lipschitz functions(see [2, 10, 12]). This show the difference of the BMO functions and the Lipschitz functions. The purpose of this paper is to prove the boundedness for some multilinear operators generated by Marcinkiewicz operators and Lipschitz functions on Hardy and Herz-Hardy spaces.

2. PRELIMINARIES AND RESULTS

In this paper, we will consider a class of multilinear operators related to Marcinkiewicz operators, whose definitions are following.

Fix $0 < \gamma \leq 1$ and $\lambda > 1$. Suppose that S^{n-1} is the unit sphere of $R^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1}, i.e.

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0$.

Let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha$$

and

$$Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} D^\alpha A(x)(x - y)^\alpha.$$

We also denote that $\Gamma(x) = \{(y, t) \in R^{n+1}_+ : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz operators are defined by

$$\mu^A_\Omega(f)(x) = \left[\int_0^\infty |F^A_t(f)(x)|^2 \frac{dt}{t^3}\right]^{1/2},$$

$$\mu^A_S(f)(x) = \left[\int\int_{\Gamma(x)} |F^A_t(f)(x, y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\mu^A_\lambda(f)(x) = \left[\int\int_{R^{n+1}_+} \left(\frac{t}{t + |x - y|}\right)^{n\lambda} |F^A_t(f)(x, y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\frac{R_{m+1}(A;x,y)}{|x-y|^m}f(y)dy$$

and

$$F_t^A(f)(x,y)=\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{R_{m+1}(A;x,z)}{|x-z|^m}f(z)dz.$$

The variants of μ_Ω^A, μ_S^A and μ_λ^A are defined by

$$\tilde{\mu}_\Omega^A(f)(x)=\left[\int_0^\infty|\tilde{F}_t^A(f)(x)|^2\frac{dt}{t^3}\right]^{1/2},$$

$$\tilde{\mu}_S^A(f)(x)=\left[\int\int_{\Gamma(x)}|\tilde{F}_t^A(f)(x,y)|^2\frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\tilde{\mu}_\lambda^A(f)(x)=\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|\tilde{F}_t^A(f)(x,y)|^2\frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\frac{Q_{m+1}(A;x,y)}{|x-y|^m}f(y)dy$$

and

$$\tilde{F}_t^A(f)(x,y)=\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{Q_{m+1}(A;x,z)}{|x-z|^m}f(z)dz.$$

Set

$$F_t(f)(y)=\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)dz.$$

We also define that

$$\mu_\Omega(f)(x)=\left(\int_0^\infty|F_t(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x)=\left(\int\int_{\Gamma(x)}|F_t(f)(y)|^2\frac{dydt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x)=\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|F_t(f)(y)|^2\frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators(see [16]).

Note that when $m=0$, μ_Ω^A, μ_S^A and μ_λ^A are just the commutator of Marcinkiewicz operators (see [16]), while when $m>0$, they are non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors(see [2, 3, 4, 5, 7, 11]). In [2], authors obtain the boundedness of multilinear singular integral operators generated by singular integrals and Lipschitz functions on $L^p(p>1)$ and some Hardy spaces. The main purpose of this paper is to discuss the boundedness properties of the multilinear Marcinkiewicz operators on Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [8, 13, 14, 15]). Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, Q will denote a cube of R^n with sides parallel to the axes. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0<p\le 1)$ has the atomic decomposition characterization(see[9]). For $\beta>0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta}=\sup_{\substack{x,h\in R^n\\ h\neq 0}}|f(x+h)-f(x)|/|h|^\beta<\infty.$$

Let $B_k=\{x\in R^n:|x|\le 2^k\}, C_k=B_k\setminus B_{k-1}, k\in Z$. Denote $\chi_k=\chi_{C_k}$ for $k\in Z$ and $\chi_0=\chi_{B_0}$, where χ_E is the characteristic function of the set E.

Definition 1. *Let $0<p,\ q<\infty$, $\alpha\in R$.*

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n)=\{f\in L^q_{loc}(R^n\setminus\{0\}):||f||_{\dot{K}_q^{\alpha,p}}<\infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}}=\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n)=\{f\in L^q_{loc}(R^n):||f||_{K_q^{\alpha,p}}<\infty\},$$

where

$$||f||_{K_q^{\alpha,p}}=\left[\sum_{k=1}^{\infty}2^{k\alpha p}||f\chi_k||_{L^q}^p+||f\chi_0||_{L^q}^p\right]^{1/p}.$$

(3) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

where

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(4) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

where

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 2. *Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if*

1) $Suppa \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

3) $\int a(x)x^\eta dx = 0$ for $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Lemma 1. (see [14]) *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1-1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms(or central (α, q)-atoms of restrict type)a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(\text{ or } ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

□

Now we can state our results as following.

Theorem 1. *Let $0 < \beta \le 1$, $\max(n/(n+\beta), n/(n+\gamma), n/(n+1/2)) < p \le 1$ and $1/p - 1/q = \beta/n$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then μ_Ω^A, μ_S^A and μ_λ^A all map $H^p(R^n)$ continuously into $L^q(R^n)$.* □

Theorem 2. *Let $0 < \beta < \min(1/2, \gamma)$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then $\tilde{\mu}_\Omega^A$, $\tilde{\mu}_S^A$ and $\tilde{\mu}_\lambda^A$ all map $H^{n/(n+\beta)}(R^n)$ continuously into $L^1(R^n)$.* □

Theorem 3. *Let $0 < \beta < \min(1/2, \gamma)$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then μ_Ω^A, μ_S^A and μ_λ^A all map $H^{n/(n+\beta)}(R^n)$ continuously into weak $L^1(R^n)$.* □

Theorem 4. *Let $0 < \beta \le 1$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = \beta/n$ and $n(1-1/q_1) \le \alpha < \min(n(1-1/q_1)+\beta, n(1-1/q_1)+\gamma, n(1-1/q_1)+1/2)$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then μ_Ω^A, μ_S^A and μ_λ^A all map $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ continuously into $\dot{K}_{q_2}^{\alpha,p}(R^n)$.* □

Remark. Theorem 4 also holds for the nonhomogeneous Herz type Hardy space.

3. SOME LEMMAS

We begin with some preliminary lemmas.

Lemma 2. (see [5]) *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$. □

Lemma 3. *Let $0 < \beta \le 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then μ_Ω^A, μ_S^A and μ_λ^A all map $L^p(R^n)$ continuously into $L^q(R^n)$.*

Proof. By Minkowski inequality, for μ_Ω^A, we have

$$\begin{aligned} \mu_\Omega^A(f)(x) &\le \int_{R^n} \frac{|\Omega(x-y)||R_{m+1}(A; x, y)|}{|x-y|^{m+n-1}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \\ &\le C \int_{R^n} \frac{|R_{m+1}(A; x, y)|}{|x-y|^{m+n}} |f(y)| dy; \end{aligned}$$

For μ_S^A, note that $|x-z|\le 2t$, $|y-z|\ge |x-z|-t\ge |x-z|-3t$ when $|x-y|\le t$, $|y-z|\le t$, we obtain

$$\begin{aligned}
&\mu_S^A(f)(x)\\
\le\ &\int_{R^n}\left[\int\int_{|x-y|\le t}\left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-1}|x-z|^m}\right)^2\chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{1/2}dz\\
\le\ &C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\left[\int\int_{|x-y|\le t}\frac{\chi_{\Gamma(z)}(y,t)t^{-n-3}}{(|x-z|-3t)^{2n-2}}dydt\right]^{1/2}dz\\
\le\ &C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+3/2}}\left[\int_{|x-z|/2}^{\infty}\frac{dt}{(|x-z|-3t)^{2n-2}}\right]^{1/2}dz\\
\le\ &C\int_{R^n}\frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n}}|f(z)|dz;
\end{aligned}$$

For μ_λ^A, note that $|x-z|\le t(1+2^{k+1})\le 2^{k+2}t$, $|y-z|\ge |x-z|-2^{k+3}t$ when $|x-y|\le 2^{k+1}t$ and $|y-z|\le t$, we have

$$\begin{aligned}
&\mu_\lambda^A(f)(x)\\
\le&\int_{R^n}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-1}|x-z|^m}\right)^2\chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{1/2}dz\\
\le&C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\\
&\times\left[\int_0^\infty\int_{|x-y|\le t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-3t)^{2n-2}}\frac{dydt}{t^{n+3}}\right]^{1/2}dz\\
&+C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\\
&\times\left[\int_0^\infty\sum_{k=0}^\infty\int_{2^kt<|x-y|\le 2^{k+1}t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-2^{k+3}t)^{2n-2}}\frac{dydt}{t^{n+3}}\right]^{1/2}dz\\
\le&C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\left[\int_{|x-z|/2}^\infty\frac{dt}{(|x-z|-3t)^{2n}}\right]^{1/2}dz\\
&+C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\\
&\times\left[\sum_{k=0}^\infty\int_{2^{-2-k}|x-z|}^\infty 2^{-kn\lambda}(2^kt)^n t^{-n}\frac{2^kdt}{(|x-z|-2^{k+3}t)^{2n}}\right]^{1/2}dz\\
\le&C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n}}dz+C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n}}dz\left[\sum_{k=0}^\infty 2^{kn(1-\lambda)}\right]^{1/2}
\end{aligned}$$

$$= C\int_{R^n}\frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n}}|f(z)|dz.$$

Thus, the lemma follows from [2]. □

4. PROOFS OF THEOREMS

Proof of Theorem 1. To be simply, we denote that $T^A = \mu_\Omega^A$ or μ_S^A or μ_λ^A. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||T^A(a)||_{L^q} \le C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int a(x)x^\eta dx = 0$ for $|\eta| \le [n(1/p-1)]$. We write

$$\int_{R^n}[T^A(a)(x)]^q dx = \left(\int_{2Q}+\int_{(2Q)^c}\right)[T^A(a)(x)]^q dx = I + II.$$

For I, taking $1 < p_1 < n/\beta$ and q_1 such that $1/p_1 - 1/q_1 = \beta/n$, by Holder's inequality and the (L^{p_1}, L^{q_1})-boundedness of T^A(see Lemma 3), we see that

$$I \le C||T^A(a)||_{L^{q_1}}^q |2Q|^{1-q/q_1} \le C||a||_{L^{p_1}}^q |Q|^{1-q/q_1} \le C.$$

To obtain the estimate of II, we need to estimate $T^A(a)(x)$ for $x \in (2Q)^c$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$. Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}(y) = D^\alpha A(y) - (D^\alpha A)_Q$.

For μ_Ω^A, we have, by the vanishing moment of a,

$$\begin{aligned}
&|F_t^A(a)(x)|\\
\le\ &\int_{R^n}\left|\frac{\Omega(x-y)|}{|x-y|^{n+m-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}}\right|\chi_{\Gamma(x)}(y,t)|R_m(\tilde{A};x,y)||a(y)|dy\\
&+\int_{R^n}\frac{\chi_{\Gamma(x)}(y,t)|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}}|R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0)||a(y)|dy\\
&+\left|\int_{R^n}(\chi_{\Gamma(x)}(y,t) - \chi_{\Gamma(x)}(x_0,t))\frac{\Omega(x-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^{n+m-1}}a(y)dy\right|\\
&+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|\int_{\Gamma(x)}\frac{\Omega(x-y)(x-y)^\alpha D^\alpha A(y)}{|x-y|^{n+m-1}}a(y)dy\right|\\
=\ &II_1 + II_2 + II_3 + II_4.
\end{aligned}$$

For II_1, by Lemma 2 and the following inequality, for $b \in Lip_\beta(R^n)$,

$$|b(x) - b_Q| \le \frac{1}{|Q|}\int_Q ||b||_{Lip_\beta}|x-y|^\beta dy \le ||b||_{Lip_\beta}(|x-x_0|+r)^\beta,$$

we get

$$|R_m(\tilde{A};x,y)| \le \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}(|x-y|+r)^{m+\beta},$$

on the other hand, by the following inequality (see [16]):

$$\left|\frac{\Omega(x-y)}{|x-y|^{n+m-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}}\right| \le \left(\frac{r}{|x-x_0|^{n+m}} + \frac{r^\gamma}{|x-x_0|^{n+m+\gamma-1}}\right)$$

and note that $|x-y| \sim |x-x_0|$ for $y \in Q$ and $x \in R^n \setminus Q$, we obtain, similar to the proof of Lemma 3,

$$\begin{aligned}
&\left(\int_0^\infty |II_1|^2 dt/t^3\right)^{1/2}\\
\le\ & C\int_{R^n}\left(\frac{r}{|x-x_0|^{n+m+1}} + \frac{r^\gamma}{|x-x_0|^{n+m+\gamma}}\right)\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}|x-x_0|^{m+\beta}|a(y)|dy\\
\le\ & C\sum_{|\alpha|=m}||D^\alpha A||_{Lip\beta}\left(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n} + \frac{|Q|^{\gamma/n+1-1/p}}{|x-x_0|^{n+\gamma-\beta}}\right).
\end{aligned}$$

For II_2, by the following equality (see [5]):

$$R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0) = \sum_{|\eta|<m}\frac{1}{\eta!}R_{m-|\eta|}(D^\eta\tilde{A};x_0,y)(x-x_0)^\eta$$

we obtain

$$\begin{aligned}
&\left(\int_0^\infty |II_2|^2 dt/t^3\right)^{1/2}\\
\le\ & C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\int_{R^n}\left(\sum_{|\eta|<m}\frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|}}\right)|a(y)|dy\\
\le\ & C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n};
\end{aligned}$$

For II_3, we have

$$\left(\int_0^\infty |II_3|^2 dt/t^3\right)^{1/2}$$

$$
\begin{aligned}
&\le C\int_{R^n}\frac{|R_m(\tilde{A};x,x_0)||a(y)|}{|x-x_0|^{n+m-1}}\left|\int\chi_{\Gamma(x)}(y,t)dt/t^3-\int\chi_{\Gamma(x)}(x_0,t)dt/t^3\right|^{1/2}dy\\
&\le C\int_{R^n}\frac{|R_m(\tilde{A};x,x_0)||a(y)||x_0-y|^{1/2}}{|x-x_0|^{n+m+1/2}}dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\beta}};
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left(\int_0^\infty|II_4|^2dt/t^3\right)^{1/2}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{Lip\beta}\left(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n}+\frac{|Q|^{1+\gamma/n-1/p}}{|x-x_0|^{n+\gamma-\beta}}+\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\beta}}\right).
\end{aligned}
$$

For μ_S^A, we write

$$
\begin{aligned}
&|F_t^A(a)(x,y)|\\
&\le \int_{R^n}\left|\frac{\Omega(y-z)}{|y-z|^{n-1}|x-z|^m}-\frac{\Omega(y-x_0)}{|y-x_0|^{n+m-1}}\right|\chi_{\Gamma(y)}(z,t)|R_m(\tilde{A};x,z)||a(z)|dz\\
&\quad+\int_{R^n}\frac{\chi_{\Gamma(y)}(x_0,t)|\Omega(y-x_0)|}{|y-x_0|^{n+m-1}}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)||a(z)|dz\\
&\quad+\left|\int_{R^n}(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(x_0,t))\frac{\Omega(y-x_0)R_m(\tilde{A};x,x_0)}{|y-x_0|^{n+m-1}}a(z)dz\right|\\
&\quad+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|\int_{\Gamma(y)}\frac{\Omega(y-z)(x-z)^\alpha D^\alpha\tilde{A}(z)}{|y-z|^{n-1}|x-z|^m}a(z)dz\right|\\
&= II_1+II_2+II_3+II_4.
\end{aligned}
$$

We obtain, similar to the proof of Lemma 3 and μ_Ω^A,

$$
\begin{aligned}
&\left[\int\int_{\Gamma(x)}|II_1|^2dydt/t^{n+3}\right]^{1/2}\\
&\le C\int_{R^n}\left(\frac{r}{|x-x_0|^{n+m+1}}+\frac{r^\gamma}{|x-x_0|^{n+m+\gamma}}\right)\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}|x-x_0|^{m+\beta}|a(y)|dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{Lip\beta}\left(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n}+\frac{|Q|^{\gamma/n+1-1/p}}{|x-x_0|^{n+\gamma-\beta}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[\int\int_{\Gamma(x)} |II_2|^2 dydt/t^{n+3}\right]^{1/2} \\
\leq\ & C\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}\int_{R^n}\left(\sum_{|\eta|<m}\frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|}}\right)|a(y)|dy \\
\leq\ & C\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n};
\end{aligned}
$$

For II_3, note that $|x+y-z|\sim|x_0+y-z|$ for $x\in Q$ and $z\in R^n\setminus 2Q$, we obtain, similar to the estimate of II_1,

$$
\begin{aligned}
&\left[\int\int_{\Gamma(x)} |II_3|^2 dydt/t^{n+3}\right]^{1/2} \\
&\leq C\int_{R^n}\left(\int_{R^{n+1}_+}\left[\frac{|a(z)|\chi_{\Gamma(x)}(y,t)|R_m(\tilde{A};x,x_0)|}{|y-x_0|^{n+m-1}}(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(x_0,t))\right]^2 dydt/t^{n+3}\right)^{1/2} dz \\
&\leq C\int_{R^n}|R_m(\tilde{A};x,x_0)||a(z)| \\
&\quad\times\left|\int\int_{|x-y|\leq t}\frac{t^{-n-3}\chi_{\Gamma(y)}(z,t)dydt}{|y-x_0|^{2n+2m-2}}-\int\int_{|x-y|\leq t}\frac{t^{-n-3}\chi_{\Gamma(y)}(x_0,t)dydt}{|y-x_0|^{2n+2m-2}}\right|^{1/2} dz \\
&\leq C\int_{R^n}|R_m(\tilde{A};x,x_0)||a(z)| \\
&\quad\times\left(\int\int_{|x-y|\leq t,|y|\leq t}\left|\frac{1}{|y+z-x_0|^{2n+2m-2}}-\frac{1}{|y|^{2n+2m-2}}\right|dydt/t^{n+3}\right)^{1/2} dz \\
&\leq C\int_{R^n}|R_m(\tilde{A};x,x_0)||a(z)|\left(\int\int_{|x-y|\leq t,|y|\leq t}\frac{|z-x_0|}{|y+z-x_0|^{2n+2m-1}}dydt/t^{n+3}\right)^{1/2} dz \\
&\leq C\int_{R^n}\frac{|R_m(\tilde{A};x,x_0)||a(z)||z-x_0|^{1/2}}{|x-x_0|^{n+m+1/2}}dz \\
&\leq C\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\beta}}
\end{aligned}
$$

and

$$
II_4 \leq C\sum_{|\alpha|=m} ||D^\alpha A||_{Lip\beta}\left(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n}+\frac{|Q|^{1+\gamma/n-1/p}}{|x-x_0|^{n+\gamma-\beta}}+\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\beta}}\right).
$$

For μ_λ^A, similarly,

$$\mu_\lambda^A(a)(x) \le C \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \left[\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n} + \frac{|Q|^{\gamma/n+1-1/p}}{|x-x_0|^{n+\gamma-\beta}} + \frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\beta}} \right].$$

Thus

$$\begin{aligned} & II \\ \le & \sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} [T^A(a)(x)]^q dx \\ \le & C\left(\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}\right)^q \sum_{k=1}^\infty \left[2^{kqn(1/p-(n+\beta)/n)} + 2^{kqn(1/p-(n+\gamma)/n)} + 2^{kqn(1/p-(n+1/2)/n)}\right] \\ \le & C\left(\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}\right)^q, \end{aligned}$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1. $\square$

Proof of Theorem 2. We only give the proof of $\tilde{\mu}_\Omega^A$ and omit the proof of $\tilde{\mu}_S^A$ and $\tilde{\mu}_\lambda^A$ for their similarity. It suffices to show that there exists a constant $C > 0$ such that for every $H^{n/(n+\beta)}$-atom a supported on $Q = Q(x_0, r)$, we have

$$||\tilde{\mu}_\Omega^A(a)||_{L^1} \le C.$$

We write

$$\int_{R^n} \tilde{\mu}_\Omega^A(a)(x)dx = \left[\int_{2Q} + \int_{(2Q)^c}\right] \tilde{\mu}_\Omega^A(a)(x)dx := J_1 + J_2.$$

For J_1, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$

we have, similar to the proof of Lemma 3,

$$\tilde{\mu}_\Omega^A(a)(x) \le \mu_\Omega^A(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)|dy,$$

thus, $\tilde{\mu}_\Omega^A$ is (L^p, L^q)-bounded by Lemma 3 and [10], where $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. We see that

$$J_1 \le C||\tilde{\mu}_\Omega^A(a)||_{L^q} |2Q|^{1-1/q} \le C||a||_{L^p} |Q|^{1-1/q} \le C|Q|^{1+1/p-1/q-(n+\beta)/n}.$$

To obtain the estimate of J_2, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2Q} x^\alpha$. Then $Q_m(A; x, y) = Q_m(\tilde{A}; x, y)$. We write, by the vanishing moment of a and $Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$\begin{aligned}
&\tilde{F}_t^A(a)(x) \\
= &\int_{\Gamma(x)} \frac{\Omega(x-y) R_m(\tilde{A}; x, y)}{|x-y|^{n+m-1}} a(y)dy - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\Gamma(x)} \frac{\Omega(x-y) D^\alpha \tilde{A}(x)(x-y)^\alpha}{|x-y|^{n+m-1}} a(y)dy \\
= &\int \left[\frac{\chi_{\Gamma(x)}(y,t)\Omega(x-y) R_m(\tilde{A}; x, y)}{|x-y|^{n+m-1}} - \frac{\chi_{\Gamma(x)}(x_0,t)\Omega(x-x_0) R_m(\tilde{A}; x, x_0)}{|x-x_0|^{n+m-1}} \right] a(y)dy \\
&- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left[\frac{\chi_{\Gamma(x)}(y,t)\Omega(x-y)(x-y)^\alpha}{|x-y|^{n+m-1}} - \frac{\chi_{\Gamma(x)}(x_0,t)\Omega(x-x_0)(x-x_0)^\alpha}{|x-x_0|^m} \right] \\
&\qquad \times D^\alpha \tilde{A}(x) a(y)dy,
\end{aligned}$$

thus, similar to the proof of Theorem 1, we obtain, for $x \in (2Q)^c$

$$\begin{aligned}
&|\tilde{\mu}_\Omega^A(a)(x)| \\
\leq\ & C|Q|^{-\beta/n} \sum_{|\alpha|=m} [||D^\alpha A||_{Lip_\beta} \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\beta}} + \frac{|Q|^{1/(2n)}}{|x-x_0|^{n+1/2-\beta}} + \frac{|Q|^{\gamma/n}}{|x-x_0|^{n+\gamma-\beta}} \right) \\
&+ |D^\alpha \tilde{A}(x)| \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1}} + \frac{|Q|^{1/(2n)}}{|x-x_0|^{n+1/2-\beta}} + \frac{|Q|^{\gamma/n}}{|x-x_0|^{n+\gamma}} \right)],
\end{aligned}$$

so that,

$$J_2 \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \sum_{k=1}^\infty \left[2^{k(\beta-1)} + 2^{k(\beta-1/2)} + 2^{k(\beta-\gamma)} \right] \leq C,$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2. □

Proof of Theorem 3. We only give the proof of μ_Ω^A. By the following equality

$$R_{m+1}(A; x, y) = Q_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y))$$

and similar to the proof of Lemma 3, we get

$$\mu_\Omega^A(f)(x) \leq \tilde{\mu}_\Omega^A(f)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |f(y)|dy,$$

from Theorem 1 and 2 with [10], we obtain

$$\begin{aligned}
&|\{x\in R^n : \mu_\Omega^A(f)(x)>\eta\}| \\
\le\ & |\{x\in R^n : \tilde{\mu}_\Omega^A(f)(x)>\eta/2\}| \\
&+\left|\left\{x\in R^n : \sum_{|\alpha|=m}\int_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^n}|f(y)|dy>C\eta\right\}\right| \\
\le\ & C\eta^{-1}||f||_{H^{n/(n+\beta)}}.
\end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. We only give the proof of μ_Ω^A. Let $f\in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. We write

$$\begin{aligned}
&||\mu_\Omega^A(f)||_{\dot{K}_q^{\alpha,p}}^p \\
\le\ & \sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||\mu_\Omega^A(a_j)\chi_k||_{L^{q_2}}\right)^p+\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||\mu_\Omega^A(a_j)\chi_k||_{L^{q_2}}\right)^p \\
=\ & L_1+L_2.
\end{aligned}$$

For L_2, by the (L^{q_1},L^{q_2}) boundedness of μ_Ω^A(see Lemma 3), we have

$$\begin{aligned}
L_2 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p \\
&\le \begin{cases} C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right), & 0<p\le 1 \\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2}\right)\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p'/2}\right)^{p/p'}, & p>1 \end{cases} \\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\le C||f||_{H\dot{K}_{q_1}^{\alpha,p}}^p.
\end{aligned}$$

For L_1, similar to the proof of Theorem 1, we have, for $x\in C_k$, $j\le k-3$,

$$\begin{aligned}
&\mu_\Omega^A(a_j)(x) \\
\le\ & C\left(\frac{|B_j|^{\beta/n}}{|x|^n}+\frac{|B_j|^{1/(2n)}}{|x|^{n+1/2-\beta}}+\frac{|B_j|^{\gamma/n}}{|x|^{n+\gamma-\beta}}\right)\int|a_j(y)|dy \\
\le\ & C\Big(2^{j(\beta+n(1-1/q_1)-\alpha)}|x|^{-n} \\
&\quad +2^{j(1/2+n(1-1/q_1)-\alpha)}|x|^{\beta-n-1/2}+2^{j(\gamma+n(1-1/q_1)-\alpha)}|x|^{\beta-n-\gamma}\Big),
\end{aligned}$$

thus

$$
\begin{aligned}
&||\mu_\Omega^A(a_j)\chi_k||_{L^{q_2}}\\
\le\ & C2^{-k\alpha}\left(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)}+2^{(j-k)(1/2+n(1-1/q_1)-\alpha)}+2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& L_1\\
\le\ & C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|\Big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)}\\
&\qquad +2^{(j-k)(1/2+n(1-1/q_1)-\alpha)}+2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\Big)\Big)^p\\
\le\ & \begin{cases} C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}\Big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)}\\ \qquad +2^{(j-k)(1/2+n(1-1/q_1)-\alpha)}+2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\Big)^p, & 0<p\le 1\\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}\Big(2^{(j-k)p(\beta+n(1-1/q_1)-\alpha)/2}\\ \qquad +2^{(j-k)p(1/2+n(1-1/q_1)-\alpha)/2}+2^{(j-k)p(\gamma+n(1-1/q_1)-\alpha)/2}\Big), & p>1\end{cases}\\
\le\ & C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\\
\le\ & C||f||^p_{H\dot K^{\alpha,p}_{q_1}}.
\end{aligned}
$$

This finishes the proof of Theorem 4. □

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