Kragujevac J. Math. 31 (2008) 85–93.

TRIANGULATIONS OF SOME CASES OF POLYHEDRA WITH A SMALL NUMBER OF TETRAHEDRA

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(Received November 30, 2005)

Abstract. Considering the hypothesis that there exists a polyhedron with a minimal triangulation by $2n-10$ tetrahedra, earlier results show that such polyhedra can have only vertices of order 5, 6 or separated vertices of order 4. Other polyhedra have minimal triangulation with a smaller number of tetrahedra. This paper presents the examples of polyhedra with the mentioned property and with the triangulation by $2n-11$ tetrahedra.

1. INTRODUCTION

It is known that it is possible to divide any polygon with $n-3$ diagonals into $n-2$ triangles without gaps and overleaps. This division is called triangulation. Many different practical applications require computer programs, which solve this problem. Examples of such algorithms are given by Seidel [9], Edelsbrunner [5] and Chazelle [2]. The most interesting aspect of the problem is to design algorithms, which are as optimal as possible.

The generalization of this process to higher dimensions is also called a triangulation. It consists of dividing polyhedra (polytop) into tetrahedra (simplices). Besides the fastness of algorithm, there are new problems in higher dimensions. It is proved that it is impossible to triangulate some of nonconvex polyhedra [8] in a three-dimensional space, and it is also proved that different triangulations of the same polyhedron may have different numbers of tetrahedra [1], [6], [10]. Considering the smallest and the largest number of tetrahedra in triangulation (the minimal and the maximal triangulation), these authors obtained values, which linearly, i.e. squarely depend on the number of vertices. Interesting triangulations are described in the papers of Edelsbrunner, Preparata, West [6] and Sleator, Tarjan, Thurston [10]. Some characteristics of triangulation in a three-dimensional space are given by Lee [7], Chin, Fung, Wang [3], Develin [4] and Stojanović [11, 12]. This problem is also related with the problems of triangulation of a set of points in a three-dimensional space [1, 6] and rotatory distance (in a plane) [10].

In this paper we consider polyhedra which have minimal triangulation with a big number of tetrahedra. Some of the previous results are presented in Section 2 and new results are given in Section 3. We will consider convex polyhedra in which each 4 vertices are noncoplanar and all faces are triangular. Furthermore, all considered triangulations are face to face. The number of edges from the same vertex will be called the order of vertex.

2. PREVIOUS RESULTS

One of the triangulations, which gives a small number of tetrahedra, is the cone triangulation [8] described as follows.

One of the vertices is the common apex, which builds one tetrahedron with each of (triangular) faces of the polyhedra, except with these containing it.

By Eulers theorem, a polyhedron with n vertices has $2n-4$ faces if all of them are triangular. So, the number of tetrahedra in triangulation is at most $2n-10$, since, for $n \geq 12$, each polyhedron has at least one vertex of order 6 or more. Sleator, Tarjan and Thurston in [10] considered some cases of "bad" polyhedra, which need a large number of tetrahedra for triangulation. It is proved, using hyperbolic geometry, that the minimal number of tetrahedra, necessary for triangulating such polyhedra, is close to $2n - 10$. That value is tight for one series of polyhedra, which exists for a sufficiently great n. Computer investigation of the equivalent problem of rotatory distance confirms, for $12 \leq n \leq 18$, that there exist polyhedra, with the smallest necessary number of tetrahedra equal to $2n - 10$. This was the reason why the authors gave a hypothesis that the same statement is true for any $n \geq 12$. To prove this hypothesis, it would be good to check when the cone triangulation of polyhedra gives the smallest number of tetrahedra and how it is possible to improve it in other cases. With this aim, in [10] the authors give an example of the polyhedron, which has vertices of great order and for which there exists a triangulation better than the cone one. They also give the advice on how to improve the method in this and some similar cases. The polyhedra with vertices of great order anyway give less than $2n-10$ tetrahedra in the cone triangulation, so, vertices of small order are considered in [11, 12]. The obtained results are as follows:

Theorem 2.1. *Let V be one of the vertices of a polyhedron P whose order is maximal. If the polyhedron P has a vertex of order 3 different and not connected with V, or a sequence of at least 2 vertices of order 4 connected between themselves into a chain, each of them different and not connected with V, then the cone triangulation of P with apex V will not give the smallest number of tetrahedra.*

Remark. *When V is connected with a vertex at the end of chain the cone triangulation is not the minimal one whenever the chain contains at least 3 vertices.*

Theorem 2.2. *For $n \geq 14$ there exists a polyhedron with n vertices which are either of order 5 or 6.*

3. NEW RESULTS

From everything mentioned before, it is clear that candidates for the minimal triangulation with the number of tetrahedra equal to $2n - 10$, are polyhedra with all vertices of order 5 or 6 and separated (not connected between themselves) vertices of order 4. Here we consider such polyhedra with the aim to prove that this condition is necessary but not sufficient.

Let us define the following:

- A *circle of p vertices* of the polyhedron P is a space $p-$gon $A_1, A_2, ..., A_p$ where A_i $(i = 1, \ldots, p)$ are different vertices of P and A_iA_{i+1} $(i = 1, \ldots, p-1)$, A_pA_1 are edges of *P.*

- Let c be a circle of p vertices A_i $(i = 1, \ldots, p)$ on the polyhedron P, and M and N two vertices of P different from A_i. If all paths on P with end points M and N pass through some of the vertices A_i then we say that the circle c separates vertices M and N and call it the separating circle. We also say that M and N are on the different sides of c. If the circle c does not separate the vertices M and N, they are on the same side of the circle.

Theorem 3.1. *If a polyhedron contains a circle of p vertices, separates vertices A and B of order $\nu(A)$ and $\nu(B)$, where $\nu(A) \geq \nu(B) > p$, then the cone triangulation with the apex A is not the minimal one.*

Proof. In this case the cone triangulation has $T_{cone} = 2n - 4 - \nu(A)$ tetrahedra. Let us call *bicone triangulation*, the one which consists of tetrahedra obtained in cone triangulation of the subpolyhedra P_A and P_B, with the apices A and B, respectively, where P_A (i.e. P_B) contains vertices of the separating circle, all the vertices on the same side of the circle as the apex A (i.e. B) and vertex B (i.e. A). Observe that the bipyramid with vertices A, B and these one of the circle, is the common part of polyhedra P_A and P_B. So, if n_A is the number of vertices of P_A and n_B of P_B, the number of tetrahedra in the bicone triangulation is $T_{bicone} = [2n_A - 4 - \nu(A)] + [2n_B - 4 - \nu(B)] - p$. Since $n = n_A + n_B - p - 2$, it holds that

$$T_{bicone} = 2n - 4 - \nu(A) - \nu(B) + p > 2n - 4 - \nu(A) = T_{cone}.$$

When $p = 3$ there exists even better triangulation T^*, because p is splitting the polyhedron into two subpolyhedra P_{A^*} and P_{B^*} with only one common (triangular) face. The triangulation T^* is the union of cone triangulation of polyhedra P_{A^*} and P_{B^*} with $2n_{A^*} - 4 - \nu(A)$ and $2n_{B^*} - 4 - \nu(B)$ tetrahedra, respectively. In this case

$n = n_{A^*} + n_{B^*} - 3$ so, $T_{cone} = 2n_{A^*} + 2n_{B^*} - 4 - \nu(A) - 6$ and $T^* = 2n_{A^*} + 2n_{B^*} - 4 - \nu(A) - 4 - \nu(B)$. Since $\nu(B) \geq 4$, it holds that $T_{cone} \geq T^* + 2$.

Lemma 3.2. *For each $n \geq 13$ there exists a polyhedron with n vertices, containing a circle of 5 vertices, which separates two vertices of order 6, such that all the vertices of polyhedron are of order 5 or 6, or separated vertices of order 4.*

Proof. There are considered three different arrangements R_1, R_2 and R_3 of vertices on one side of the circle of 5 vertices. The vertices of the external circles of all arrangements are noted with A_1, A_2, A_3, A_4, A_5.

In the case of the arrangement R_1 (Figure 1.a) vertices B_1, B_2, B_3, B_4 are connected between themselves by edges B_1B_2, B_2B_3, B_3B_4, B_4B_1, B_1B_3 and with the circle by A_1B_1, A_2B_1, A_2B_2, A_3B_2, A_3B_3, A_4B_3, A_4B_4, A_5B_4, A_5B_1. Here, the vertices B_2 and B_4 are of order 4.

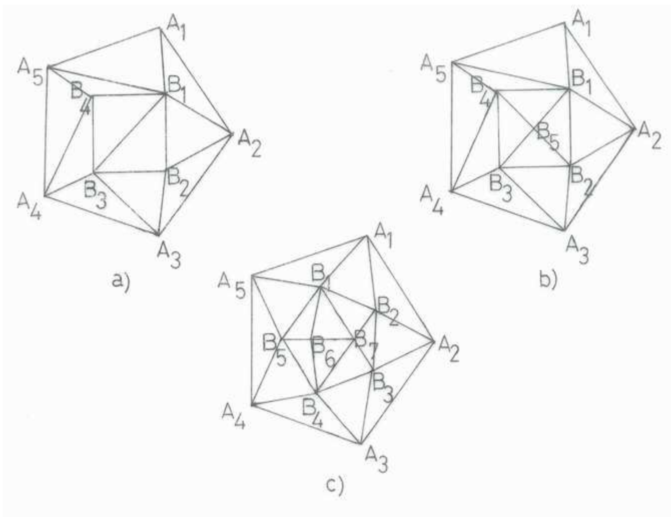

Figure 1

In the case of the arrangement R_2 (Figure 1.b) vertices B_1, B_2, B_3, B_4, B_5 are connected between themselves by B_5B_1, B_5B_2, B_5B_3, B_5B_4, B_1B_2, B_2B_3, B_3B_4, B_4B_1 and with the circle by A_1B_1, A_2B_1, A_2B_2, A_3B_2, A_3B_3, A_4B_3, A_4B_4, A_5B_4, A_5B_1. The vertex B_5 is of order 4.

In the case of the arrangement R_3 (Figure 1.c) vertices B_1, B_2, B_3, B_4, B_5, B_6, B_7 are connected between themselves by B_1B_2, B_2B_3, B_3B_4, B_4B_5, B_5B_1, B_6B_1, B_6B_4,

B_6B_5, B_6B_7, B_7B_1, B_7B_2, B_7B_3, B_7B_4 and with the circle by A_1B_1, A_1B_2, A_2B_2, A_2B_3, A_3B_3, A_3B_4, A_4B_4, A_4B_5, A_5B_5, A_5B_1. Here vertex B_6 is of order 4.

In all cases the vertex B_1 and the vertex B_4 in arrangement R_3 are of order 6. All other non-mentioned vertices B_i are of order 5.

Combining R_i and R_j, $i, j \in \{1, 2, 3\}$ as different sides of the same circle, or adding new circles of 5 vertices between themselves, it is possible to obtain a polyhedron P with n vertices of order 5 or 6, or separated vertices of order 4, for each $n \geq 13$.

If the polyhedron P is a combination of R_i and R_j with the common circle then gluing the vertex noted with A_1 of the external circle of R_i with the vertex A_1 of R_j, $i, j \in \{1, 2\}$ is giving the vertex of order 4 of the common circle. Gluing the vertex A_1 of R_i, $i \in \{1, 2\}$ with A_k, $k \neq 1$ of R_j, $j \in \{1, 2\}$, or with any A_l of R_3 is giving the vertex of order 5. All other vertices of the common circle of P, are of order 6. So, the common circle (with five vertices) is separating the vertices noted with B_1 in the arrangements R_i, R_j, $i, j \in \{1, 2, 3\}$ which are of order 6. In such polyhedra there are no connected vertices of order 4, even when the vertices of the common circle are such, since the vertex A_1 of R_1 (i.e. R_2) is not connected with the vertices B_2, B_4 (i.e. B_5) of order 4. Also the vertices B_2 and B_4 of R_1 are not connected.

The polyhedron P obtained as mentioned has n vertices where:

- $n = 13$ if P is a combination of two R_1;
- $n = 14$ if P is a combination of R_1 and R_2;
- $n = 15$ if P is a combination of two R_2;
- $n = 16$ if P is a combination of R_1 and R_3;
- $n = 17$ if P is a combination of R_2 and R_3;

The external circles $A_1^0A_2^0A_3^0A_4^0A_5^0$ of R_i and $A_1^1A_2^1A_3^1A_4^1A_5^1$ of R_j, $i, j \in \{1, 2, 3\}$ can be connected between themselves, instead of gluing, by e.g. edges $A_1^0A_1^1$, $A_1^0A_2^1$, $A_2^0A_2^1$, $A_2^0A_3^1$, $A_3^0A_3^1$, $A_3^0A_4^1$, $A_4^0A_4^1$, $A_4^0A_5^1$, $A_5^0A_5^1$, $A_5^0A_1^1$. Such new vertices A_k^0, A_l^1 are of order 6, except A_1^0 for $i \in \{1, 2\}$ and A_1^1 for $j \in \{1, 2\}$ which are of order 5. So, the polyhedron is then with $n \in \{18, 19, 20, 21, 22\}$ vertices. Between the external circles it is possible to add the new circle A_1^2, A_2^2, A_3^2, A_4^2, A_5^2 with 5 vertices, connected with both external circles in such a way that polyhedron P has edges $A_1^kA_1^2$, $A_1^kA_2^2$, $A_2^kA_2^2$, $A_2^kA_3^2$, $A_3^kA_3^2$, $A_3^kA_4^2$, $A_4^kA_4^2$, $A_4^kA_5^2$, $A_5^kA_5^2$, $A_5^kA_1^2$, $k \in \{0,1\}$. Similarly, it is possible to add more circles with 5 vertices. Orders of new vertices are 6. In such a way, it is possible to obtain the polyhedron P with n vertices for each $n \geq 13$.

Lemma 3.3. *For $n = 22 + 5k$, $k \in N$ there exists a polyhedron P with n vertices of order 5 or 6, containing circle of 5 vertices, which separates two vertices of order 6.*

Proof. Here, for $k = 1$ P is a combination of two arrangements R (Figure 2.), for $k = 2$ the external circles of the arrangements are connected as in the previous lemma and for $k \geq 3$ we add $k-2$ new circles as before. The arrangement R besides the external circle with vertices A_1, A_2, A_3, A_4, A_5 has 11 inside vertices B_i ($i \in \{1,\ldots,11\}$). The inside vertices are connected between themselves by B_iB_{i+1} for $i \in \{1,2,3,4\} \cup \{6,7,8,9\}$, B_5B_1, $B_{10}B_6$, B_jB_{j+5}, B_jB_{j+6} for $j \in \{1,2,3,4\}$, B_5B_{10}, B_5B_6, $B_{11}B_k$ for $k \in \{6,7,8,9,10\}$, and with the circle by A_lB_l, A_lB_{l+1} for $l \in \{1,2,3,4\}$, A_5B_5, A_5B_1.

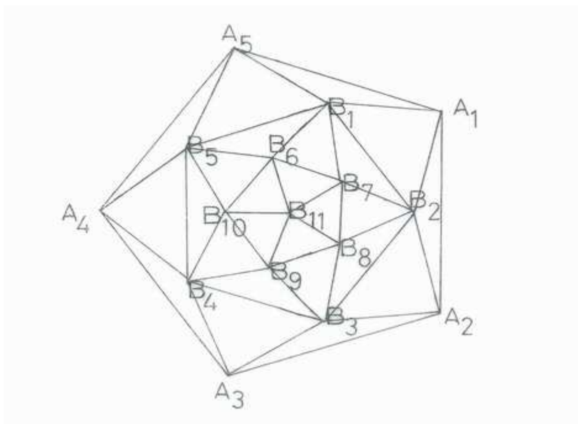

Figure 2

After "gluing" and adding circles, all vertices A_i^j, $i \in \{1,\ldots,5\}$, $j \in \{0,1,\ldots,k-1\}$ are of order 6. The vertices B_i, $i \in \{1,\ldots,5\}$ are of order 6 and B_j, $j \in \{6,\ldots,11\}$ of order 5.

The consequence of theorem 3.1. is that for the polyhedra P in lemma 3.2. and lemma 3.3. the cone triangulation is not the minimal one, since their bicone triangulation gives $2n-11$ tetrahedra. So, we will state the following theorems.

Theorem 3.4. *For each $n \geq 13$ there exists a polyhedron with n vertices of order 5 or 6, or separated vertices of order 4, whose minimal triangulation has less than $2n-10$ tetrahedra.*

Theorem 3.5. *For $n = 22+5k$ there exists a polyhedron with n vertices of order 5 or 6, whose minimal triangulation has less than $2n-10$ tetrahedra.*

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