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ONE PARAMETER LORENTZIAN MOTIONS IN LORENTZ 3-SPACE

Dedicated to Prof. Dr. Bernard ROTH

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Abstract. In this paper, first of all we have given the relations between the absolute, relative and sliding velocities and the relation between the sliding and instantaneous rotations for one parameter Lorentzian motion in 3-dimensional Lorentzian space. In addition to that we have also given the relations between absolute, relative, Coriolis and sliding accelerations of the motion. We have also noted the relations between Coriolis acceleration, instantaneous rotation axis and relative velocity of the motion.

Lastly, we have investigated the acceleration centres when $\left\| \overrightarrow{w} \wedge \overset{\bullet}{\overrightarrow{w}} \right\|^2 \neq 0$ and the acceleration axis when $\left\| \overrightarrow{w} \wedge \overset{\bullet}{\overrightarrow{w}} \right\|^2 = 0$, where the vector $\vec{w}$ is the instantaneous rotation axis of the motion. Furthermore, we have given theorems related to these cases.

1. INTRODUCTION

The determination of a point or a set of points such that its velocity norm vanishes or that is a minimum has always aroused interest among kinematicians. The explanation of this is two-fold: points whose velocity, or acceleration, vanishes are important for they allow one to write simplified equations for the velocity and acceleration of any other point of the rigid body; and a point or a set of points with a minimum velocity norm locates the connecting place of a kinematic pair, in general a helicoidal pair, that connects the rigid body to the reference body. This connection produces a motion with the same characteristics, at least up to the first derivative of the original motion of the rigid body.

Indeed, the search for points of a rigid body with a minimum velocity norm has led to the description of the velocity of a rigid body in terms of infinitesimal screws, or helicoidal fields, and therefore to the definition of the instantaneous screw axis.

By taking Lorentzian plane L^2 instead of Euclidean plane E^2, Ergin [9] has introduced 1-parameter planar motion in Lorentzian plane. Furthermore he gave the relation between the velocities, accelerations and pole curves of these motions. In reference [11] T. Ikawa gave the Euler-Savary formula in Lorentzian plane L^2.

In spherical kinematics, Bottema [6] found the existence of three acceleration axes, straight lines that pass through the fixed point, whose points do not posses acceleration. Further, Bottema found conditions that determine whether all three acceleration axes are real or only one is real. Later, Meyer Zur Capellen and Dittrich [13] extended Bottema's results by studying the acceleration distribution in a rigid body subjected to spherical motion.

Finally, in spatial kinematics Beyer [3] completed a thorough analysis of the acceleration distribution in a rigid body. Moreover, Beyer formulated a system of linear equations in a special coordinate system, whose solution leads to the location of the acceleration center. However, Beyer did not obtain a closed form solution even for this special coordinate system. Finally, Beyer mentioned a previous work by Schell for the analysis of the special cases associated with the determination of the acceleration center.

Bottema and Roth [7] obtained the equation for the acceleration distribution in a rigid body in a special coordinate system. By equating the acceleration to the zero vector and solving the resulting system, the coordinates of the acceleration center are obtained. The coordinate system used by Beyer [3] has some similarities with that employed by Bottema and Roth.

Considering one and two parameters spherical motions in Euclidean space, Muller [14] has given the relations for absolute, sliding, relative velocities and pole curves of these motions. In addition to that he has expressed the corresponding formula of Euler-Savary formula related to the trajectory curves of these 1-parameter spherical motions.

To investigate the geometry of the motion of a line or a point in the motion of space is important in the study of space kinematics or spatial mechanisms or in physics. The geometry of such a motion of a point or a line has a number of applications in geometric modelling and model-based manufacturing of the mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces, [8, 10, 15].

This paper is organized as follows. In this first part, basic concepts have been given in Minkowski space IR_1^3. In the second part, 1-parameter Lorentzian motions in Lorentz 3-space are defined and the relations between the absolute, relative and sliding velocities of this motion have been expressed [16]. Additionally it has been show that the sliding velocity of point X is perpendicular to the instantaneous rotation axis of motion in time t. In the third part, we have given the relations between absolute, relative and Coriolis accelerations of the motion. Furthermore, it has been shown that in a time t Coriolis acceleration of point X is perpendicular to both the relative velocity and instantaneous rotation axis of the motion. In the last part of this study we have worked out the acceleration centres of the motion when $\left\| \overrightarrow{w} \wedge \overset{\bullet}{\overrightarrow{w}} \right\|^2 \neq 0$ and the acceleration axis of the motion when $\left\| \overrightarrow{w} \wedge \overset{\bullet}{\overrightarrow{w}} \right\|^2 = 0$, where $\vec{w}$ represent instantaneous rotation axis of 1-parameter Lorentzian motion in Lorentz 3-space.

We hope that these results will contribute to the study of space kinematics and physics applications.

2. PRELIMINARIES

We start with preliminaries on the geometry of Lorentz 3-space. Let L^3 be a Lorentz 3-space endowed with Lorentzian inner product g of signature $(-,+,+)$. A vector $\vec{X} = (x_1, x_2, x_3)$ of L^3 is said to be time-like if $g\left(\vec{X}, \vec{X}\right) < 0$, space-like if $g\left(\vec{X}, \vec{X}\right) > 0$ and light-like (or null) if $g\left(\vec{X}, \vec{X}\right) = 0$. The set of all vector $\vec{X}$ such that $g\left(\vec{X}, \vec{X}\right) = 0$ is called the light-like (or null) cone and is denoted by Γ. The norm of a vector $\vec{X}$ is defined to be $\left\|\vec{X}\right\| = \sqrt{\left|g\left(\vec{X}, \vec{X}\right)\right|}$. Time orientation is defined as follows: A time-like vector $\vec{X} = (x_1, x_2, x_3)$ is future pointing (respectively past pointing) if and only if $x_1 > 0$ (respectively $x_1 < 0$), [4, 5].

Let $\vec{X}$ be a future pointing time-like unit vector, also $\vec{Y}$ be a future pointing time-like unit vector. If the angle between $\vec{X}$ and $\vec{Y}$ is θ then we may have

$$g\left(\vec{X}, \vec{Y}\right) = -\cosh\theta.$$

As in the case of Euclidean 3-space, the Lorentzian cross product of $\vec{X}$ and $\vec{Y}$ is defined by

$$\vec{X} \wedge \vec{Y} = (y_2x_3 - y_3x_2, y_1x_3 - y_3x_1, y_2x_1 - y_1x_2)$$

where $\vec{X} = (x_1, x_2, x_3)$ and $\vec{Y} = (y_1, y_2, y_3)$ are the vectors of the space L^3, [1].

The matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix}, \quad \begin{bmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{bmatrix}, \quad \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is called the Lorentzian rotation matrix in L^3, where $\theta \in IR$. These matrices are similar to the rotation matrices, which are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in E^3 [4].

Lemma 1. *Time-like vectors are transformed to time-like vectors and space-like vectors are transformed to space-like vectors by Lorentzian orthogonal matrix A. That is, the matrix A conserves the orientation, [4].*

3. LORENTZIAN MOTIONS, THEIR VELOCITIES AND INSTANTANEOUS ROTATION AXIS

One-parameter motion of a body in 3-dimensional Lorentz space, L^3 is given by the transformation

$$\begin{bmatrix} x' \\ 1 \end{bmatrix} = \begin{bmatrix} A & u' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}. \tag{1}$$

where, A is a Lorentzian positive orthogonal matrix, (i.e. $A^{-1} = \varepsilon A^t \varepsilon$ and $\det A = 1$ where ε is a sign matrix). x, x' and u' are 3x1 type real matrices, A and u' are differentiable functions of C^∞ class of a parameter t. x and x' are the position vectors of same points at moving K and fixed space K' with respect to the Lorentzian orthonormal coordinate frame, respectively. Let, these orthonormal coordinate frames be $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{O; \vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ corresponding to moving and fixed spaces, (where base vectors $\vec{e}_2, \vec{e}_3$; $\vec{e}'_2, \vec{e}'_3$ are space-like and the vectors $\vec{e}_1, \vec{e}'_1$ are time-like). Therefore,

$$g(\vec{e}_i, \vec{e}_j) = g\left(\vec{e}'_i, \vec{e}'_j\right) = \varepsilon_i \delta_{ij} \quad , \quad \varepsilon_i = \begin{cases} 1, & \text{are } \vec{e}_i \text{ or } \vec{e}'_i \text{ space} - \text{like} \\ -1, & \text{are } \vec{e}_i \text{ or } \vec{e}'_i \text{ time} - \text{like} \end{cases} \quad , \ 1 \leq i, j \leq 3$$

Suppose that coordinate frames in K and K' are coincident initially at time $t = t_0$. Transformation given by equation (1) is a Lorentzian motion, [12]. From equation (1) we get

$$x' = Ax + u'. \tag{2}$$

Since there are the relations $u' = -Au$ and $u = -A^{-1}u'$ between vectors u' and u, we reach

$$x = A^{-1}x' + u. \tag{3}$$

Equation (2) and (3) express the coordinate transformations between the two system.

If we compute the derivative of equation (2) with respect to t we find

$$\dot{x}' = \dot{A}\,x + A\,\dot{x} + \dot{u}'\,. \tag{4}$$

Here,

$$V_a' = \dot{x}'$$

is absolute velocity, whereas

$$V_f' = \dot{A}\,x + \dot{u}' \tag{5}$$

is sliding velocity and

$$V_r' = A\,\dot{x} \tag{6}$$

is relative velocity. In addition to that, from $V_a = A^{-1}V_a'$, $V_f = A^{-1}V_f'$ and $V_r = A^{-1}V_r'$ following relation is hold between the velocities

$$V_a = V_f + V_r.$$

Now, we investigate the points remaining constant (i.e., pole points) at the same time in moving space K and fixed space K'. Since these points are the points where the sliding velocities are equal to zero we can write

$$\dot{x}' = \dot{A}\,x + \dot{u}' = 0$$

and

$$A^{-1}\,\dot{A}\,x = A^{-1}\,\dot{A}\,u + \dot{u}\,. \tag{7}$$

(where from equation $u' = -Au$, $\dot{u}' = -\dot{A}\,u - A\,\dot{u}$). The solution for this linear equation system is not single.

As $A^{-1}A = AA^{-1} = I_3$,

$$\left(\dot{A^{-1}}\right)A + A^{-1}\,\dot{A} = 0 \quad , \quad \dot{A}\,A^{-1} + A\left(\dot{A^{-1}}\right) = 0. \tag{8}$$

If we take $S = (\dot{A^{-1}})\,A$ and $R = A\,(\dot{A^{-1}})$ then the matrices S and R are anti-symmetric matrices in the sense of Lorentzian, [2]. That is $S^t = -\varepsilon S\varepsilon$ and $R^t = -\varepsilon R\varepsilon$, where ε is a sign matrix. Let assume that w_{ij} $\left(\text{resp. } \overline{w}'_{ij}\right)$ are the elements of S

matrix (resp. R). Let's denote the permutations of the indices $i, j, k = 1, 2, 3;\ 2, 3, 1;\ 3, 1, 2$ by $w_{ij} = w_k \left(\bar{w}'_{ij} = \bar{w}'_k\right)$. Then we can easily get that

$$S = \begin{bmatrix} 0 & w_3 & -w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \quad , \quad R = \begin{bmatrix} 0 & \bar{w}'_3 & -\bar{w}'_2 \\ \bar{w}'_3 & 0 & -\bar{w}'_1 \\ -\bar{w}'_2 & \bar{w}'_1 & 0 \end{bmatrix}. \tag{9}$$

As we know that S and R matrices are anti-symmetric, their determinants are equal to zero, i.e.

$$\det(S) = 0 \quad , \quad \det(R) = 0.$$

For any vectors of $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $\bar{w}' = \begin{bmatrix} \bar{w}'_1 \\ \bar{w}'_2 \\ \bar{w}'_3 \end{bmatrix}$

$$Sw = 0 \quad , \quad R\bar{w}' = 0. \tag{10}$$

If we consider the equation $S = (\overset{\bullet}{A^{-1}})\, A$ and $A^{-1} = \varepsilon A^t \varepsilon$, then we find

$$\begin{aligned} \det(S) &= \det\left((\overset{\bullet}{A^{-1}})\, A\right) = \det(\overset{\bullet}{A^{-1}}) \det A \\ &= \det\left((\overset{\bullet}{A^{t}})\right) = \det \overset{\bullet}{A} = 0. \end{aligned}$$

This means that $\overset{\bullet}{A}$ is singular and

$$rank\,(S) = rank\left(\overset{\bullet}{A}\right).$$

On the other hand, we know that the rank of an anti-symmetric matrix in the sense of Lorentzian is absolutely an even number,[17]. Thus,

$$rank\left(\overset{\bullet}{A}\right) = 2.$$

As a result the equation system (7) is soluble if and only if

$$rank\left(A^{-1}\overset{\bullet}{A}, \dot{u}\right) = rank\left(\overset{\bullet}{A}\right) = 2.$$

From this condition we reach

$$w^t \varepsilon\, \overset{\bullet}{u} = 0 \tag{11}$$

where we suppose that $\vec{w} \neq 0$.

Under these conditions, the equation system of (7) is soluble and gives a linear equation of line [8]. This line is an instantaneous rotation axis which remains fixed in the systems. Let y be an arbitrary point on this axis. Now we rewrite equation (7) as

$$-Sy = -Su + \overset{\bullet}{u}$$

where $A^{-1}\overset{\bullet}{A} = -S$, (from equation $A^{-1}A = I_3$ we get $A^{-1}\overset{\bullet}{A} = -S$). From the solution of the system we understand that the direction of instantaneous rotation axis is parallel to vector $\vec{w}$, [8]. Vector $\vec{w}$ is also called angular velocity. To calculate the components of this angular velocity we need to express Lorentzian orthogonal matrix A in terms of Euler angles. We now go further from moving system to fixed system with the following three rotational motions without slide.

If one makes

1°) rotation of angle φ about $\vec{e}_3$-axis

2°) rotation of angle θ about $\vec{e}_1'$-axis

3°) rotation of angle ψ about $\vec{e}_3'$-axis

then these rotations correspond to the following matrices

$$A_1 = \begin{bmatrix} \cosh\varphi & \sinh\varphi & 0 \\ \sinh\varphi & \cosh\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, A_3 = \begin{bmatrix} \cosh\psi & \sinh\psi & 0 \\ \sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we obtain

$$A = A_3A_2A_1 =$$

$$\begin{bmatrix} \cosh\psi\cosh\varphi + \sinh\psi\sinh\varphi\cos\theta & \cosh\psi\sinh\varphi + \sinh\psi\cosh\varphi\cos\theta & -\sinh\psi\sin\theta \\ \sinh\psi\cosh\varphi + \cosh\psi\sinh\varphi\cos\theta & \sinh\psi\sinh\varphi + \cosh\psi\cosh\varphi\cos\theta & -\cosh\psi\sin\theta \\ \sin\theta\sinh\varphi & \sin\theta\cosh\varphi & \cos\theta \end{bmatrix}.$$

If one calculates $S = (\overset{\bullet}{A^{-1}})\,A$ matrice out of these and compare the result with equation(9), the following equation is reached

$$\begin{aligned} w_1 &= \overset{\bullet}{\psi}\sinh\varphi\sin\theta - \overset{\bullet}{\theta}\cosh\varphi \\ w_2 &= -\overset{\bullet}{\psi}\cosh\varphi\sin\theta + \overset{\bullet}{\theta}\sinh\varphi \\ w_3 &= -\overset{\bullet}{\varphi} - \overset{\bullet}{\psi}\cos\theta. \end{aligned} \tag{12}$$

In addition to that, from $w' = Aw$ we find

$$\begin{aligned} w'_1 &= \dot{\varphi}\sinh\psi\sin\theta - \dot{\theta}\cosh\psi \\ w'_2 &= \dot{\varphi}\cosh\psi\sin\theta - \dot{\theta}\sinh\psi \\ w'_3 &= -\dot{\psi} - \dot{\varphi}\cos\theta. \end{aligned} \tag{13}$$

Similarly, if one calculates $R = A\,(\dot{A}^{-1})$ matrice out of these and compare the result with equation (9), the following equation is reached

$$\begin{aligned} \bar{w}'_1 &= \dot{\varphi}\sinh\psi\sin\theta - \dot{\theta}\cosh\psi \\ \bar{w}'_2 &= \dot{\varphi}\cosh\psi\sin\theta - \dot{\theta}\sinh\psi \\ \bar{w}'_3 &= -\dot{\psi} - \dot{\varphi}\cos\theta. \end{aligned}$$

Hence, it is seen that w' is equal to $\bar{w}'$. Rearranging equation (5) as

$$V_f = A^{-1}V'_f = \varepsilon S^t\varepsilon x - \varepsilon S^t\varepsilon u - \dot{u}$$

and multiplying it by $w^t\varepsilon$ from left gives

$$w^t\varepsilon V_f = w^t S^t\varepsilon x - w^t S^t\varepsilon u - w^t\varepsilon\,\dot{u}$$

For this last equation, by considering equations (10) and (11), we get

$$w^t\varepsilon V_f = 0.$$

So, we can give the following theorem.

Theorem 2. *Sliding velocity of point X at time t in one-parameter Lorentzian motion in Lorentz space L^3 is perpendicular to instantaneous rotation axis.*

4. ACCELERATIONS AND ACCELERATION CENTRE

If we consider equation(4) we find

$$\ddot{x}' = \ddot{A}\,x + \ddot{u}' + 2\,\dot{A}\,\dot{x} + A\,\ddot{x}\,. \tag{14}$$

Here

$$\gamma_a' = \overset{\bullet\bullet}{x'} \tag{15}$$

is absolute acceleration and

$$\gamma_f' = \overset{\bullet\bullet}{A}\, x + \overset{\bullet\bullet}{u'} \tag{16}$$

is sliding acceleration,

$$\gamma_c' = 2\, \overset{\bullet}{A}\, \overset{\bullet}{x} \tag{17}$$

is called Coriolis acceleration and

$$\gamma_r' = A\, \overset{\bullet\bullet}{x} \tag{18}$$

is called relative acceleration. In between these the following rotation is hold

$$\gamma_a = \gamma_f + \gamma_c + \gamma_r.$$

From equation(17) we write

$$\gamma_c = A^{-1}\gamma_c' = -2S\, \overset{\bullet}{x} = 2\varepsilon S^t \varepsilon\, \overset{\bullet}{x}$$

and considering equation (6) leads us to

$$\vec{\gamma}_c = 2\left(\vec{w} \wedge \vec{V}_r\right).$$

So, we can give following theorem.

Theorem 3. *Coriolis acceleration of point X at time t in one-parameter Lorentzian motion in Lorentz space L^3 is perpendicular to both relative velocity and instantaneous rotation axis.*

Now we investigate the points where sliding acceleration is zero at time t. From equation (16) we can write that

$$\overset{\bullet\bullet}{A}\, x + \overset{\bullet\bullet}{u'} = 0 \tag{19}$$

for the points where sliding velocity is equal to zero.

If we consider $\overset{\bullet}{(A^{-1})}\,A = S$, $A^{-1}\overset{\bullet}{A} = -S$ and instantaneous rotation axis $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, we reach the following equations:

$$A^{-1}\overset{\bullet\bullet}{A} = S^2 - \overset{\bullet}{S} \quad , \quad \det\left(S^2 - \overset{\bullet}{S}\right) = \left\| \vec{w} \wedge \overset{\bullet}{\vec{w}} \right\|^2 \tag{20}$$

equation system of (19) changes to

$$\left(S^2 - \overset{\bullet}{S}\right) x = \left(S^2 - \overset{\bullet}{S}\right) u - 2S\,\overset{\bullet}{u} + \overset{\bullet\bullet}{u}\,. \tag{21}$$

If

$$\left\| \vec{w} \wedge \overset{\bullet}{\vec{w}} \right\|^2 \neq 0 \tag{22}$$

then the equation system of (19) can be solved uniquely and the following acceleration centres can be found

$$p_1 = -\left(\overset{\bullet\bullet}{A}\right)^{-1}\overset{\bullet\bullet}{u'} \quad , \quad p_1' = u' - A\left(\overset{\bullet\bullet}{A}\right)^{-1}\overset{\bullet\bullet}{u'}\,. \tag{23}$$

Sliding velocity given by equation (16) can be rewritten in terms of these coordinates

$$\gamma_f = A^{-1}\overset{\bullet\bullet}{A}\,(x - p_1) \quad , \quad \gamma_f' = \overset{\bullet\bullet}{A}\,A^{-1}\,(x' - p_1')\,. \tag{24}$$

Therefore, we can give the following theorem.

Theorem 4. *Sliding velocity of one-parameter Lorentzian motion in 3-dimensional Lorentz space L^3 in terms of the coordinates p_1, p_1' is*

$$\gamma_f = A^{-1}\overset{\bullet\bullet}{A}\,(x - p_1) \quad , \quad \gamma_f' = \overset{\bullet\bullet}{A}\,A^{-1}\,(x' - p_1')\,.$$

5. ACCELERATION AXIS

Now we suppose that the following relation is hold

$$\det\left(S^2 - \overset{\bullet}{S}\right) = \left\| \vec{w} \wedge \overset{\bullet}{\vec{w}} \right\|^2 = 0. \tag{25}$$

Therefore,

$$rank\left(S^2-\overset{\bullet}{S}\right)\leq 2. \tag{26}$$

So,

$$\vec{w}=\vec{k}$$

or

$$\overset{\bullet}{\vec{w}}=c\vec{w}$$

where c is an arbitrary constant and $\vec{k}$ is a constant vector. Now we search these two situations separately.

First, we suggest that

$$\vec{w}=\vec{k}. \tag{27}$$

Since $\vec{w}\neq 0$, there is at least one w_i which satisfies the condition

$$w_i=k_i\neq 0.$$

Under these conditions

$$\overset{\bullet}{S}=0$$

and

$$rank\left(S^2-\overset{\bullet}{S}\right)=rank\left(S^2\right). \tag{28}$$

On the other hand, at least one of minor of $\det\left(S^2\right)$ is not zero. Thus,

$$rank\left(S^2\right)=2.$$

So, the equation(21) can be solved if and only if

$$rank\left(S^2,S^2u-2S\overset{\bullet}{u}+\overset{\bullet\bullet}{u}\right)=2.$$

This condition is same as the following one

$$w^t\varepsilon\overset{\bullet\bullet}{u}=0. \tag{29}$$

Therefore equation system of (21) is soluble under the condition of equation (29). If one try to solve these equations under these conditions, a linear equation of line to instantaneous rotation axis obtained and is called linear acceleration axis.

Now we assume that

$$\overset{\bullet}{\vec{w}} = c\vec{w}.$$

In this case, at least one component of w_i is not constant, otherwise, as

$$\overset{\bullet}{w}_i = cw_i$$

we reach

$$\overset{\bullet}{\vec{w}} = c\vec{w} = 0.$$

So motion becomes pure sliding. If we consider the minors of det $\left(S^2 - \overset{\bullet}{S}\right)$ we see that at last one of these is not zero. Therefore,

$$rank\left(S^2 - \overset{\bullet}{S}\right) = 2.$$

Thus, equation(21) to be solved if and only if

$$rank\left(S^2 - \overset{\bullet}{S}, \left(S^2 - \overset{\bullet}{S}\right)u - 2S\overset{\bullet}{u} + \overset{\bullet\bullet}{u}\right) = 2.$$

This condition is expressed as

$$w^t\varepsilon\overset{\bullet\bullet}{u} = 0 \tag{30}$$

Equation(21) is soluble under the condition of equation (30) and again we get a linear equation of line parallel to instantaneous rotation axis. In addition to this, since there is no possible case other than $\vec{w} = \vec{k}$ or $\overset{\bullet}{\vec{w}} = c\vec{w}$ (with the equation (25) is satisfied) the following condition is not valid

$$rank\left(S^2 - \overset{\bullet}{S}\right) = 1.$$

Therefore equation system of (21) does not represent a planar equation system. So, we can give the following theorem.

Theorem 5. *In Lorentz space L^3, geometrical placement of points, which are perpendicular to instantaneous rotation axis, of sliding acceleration vector of moving system at time t in one-parameter Lorentzian motion is a plane perpendicular to vector $\vec{w} \wedge \overset{\bullet}{\vec{w}}$. This plane hits the point O if there is an acceleration pole at the time* *whereas it hits point O' when there is an acceleration axis. If there is point p_1, then the scalar product of $\vec{w}$ and sliding velocity of equations(24) becomes*

$$w^t \left(S^2 - \overset{\bullet}{S}\right)(x - p_1) = 0.$$

If there is an acceleration axis, from equation(11), (19) and (30) we get either

$$w^t \left[\varepsilon \left(S^2 - \overset{\bullet}{S}\right)(x - u) - 2S^t \varepsilon \overset{\bullet}{u} - \varepsilon \overset{\bullet\bullet}{u}\right] = 0$$

or

$$w^t \varepsilon \overset{\bullet}{S}(x - u) = 0.$$

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