*Kragujevac J. Math.* 31 (2008) 111–129.

# SOME CONVERGENCE RESULTS FOR FIXED POINTS OF HEMICONTRACTIVE OPERATORS IN SOME BANACH SPACES

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*(Received October 31, 2006)*

**Abstract.** In this manuscript, we establish strong convergence results for the improved iteration methods introduced by Owojori and Imoru [7], to fixed points of hemicontractive operators. Our results in this paper are extensions of the results of: Ishikawa [4], Deng and Ding [3], Chidume [1], Chidume and Osilike [2], Owojori and Imoru [6], Qihou [8], Liu [5] and Xu [10] from the Mann and Ishikawa iteration methods, (with and without errors), to more general iteration methods and from Lipschitz or continuous pseudocontractive operators to slightly more general continuous hemicontractive operators.

## 1. INTRODUCTION

Xu [10] introduced suitable Mann and Ishikawa iteration schemes with errors for approximations of fixed points and solutions of nonlinear operators in Banach spaces. Owojori and Imoru [6] introduced a three-step iteration scheme and obtained some convergence results to the fixed points of continuous hemicontractive mappings in Hilbert spaces.

Owojori and Imoru [7] introduced an improved three-step iteration method which contains the one introduced earlier by the authors in [6], as well as the Mann and Ishikawa iteration methods as special cases. It is defined for arbitrary $x_1 \in K$ - a closed bounded convex subset of a Banach space $B$ , by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n S x_n \\ y_n &= a_n^{'} x_n + b_n^{'} S z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1 \tag{1.1}$$

Two special cases of (1.1) are given respectively by:

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n T x_n \\ y_n &= a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1 \tag{1.2}$$

and

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n \\ y_n &= a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1 \tag{1.3}$$

where $S, T$ are nonlinear uniformly continuous self-mappings of $K$ satisfying some contractive definitions and $\{u_n\}$, $\{v_n\}$, $\{\omega_n\}$ are bounded sequences in $K$. Also $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0, 1]$ satisfying:

1°) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

2°) $\sum b_n = \infty$.

**Remark.** *It was observed that the iteration schemes given by* (1.1), (1.2) *and* (1.3) *are all well defined and* (1.2) *is a slight generalization of* (1.3).

Let $H$ be a Hilbert space and $K$ a nonempty subset of $H$. An operator $T : K \to K$ is called hemicontractive if $F(T)$, the fixed point set of $T$, is nonempty and for all $x^* \in F(T)$, the inequality

$$\|Tx - x^*\|^2 \;\leq\; \|x - x^*\|^2 \;+\; \|x - Tx\|^2 \tag{1.4}$$

holds for all $x \in K$.

Our purpose in this manuscript is to establish the convergence of the iteration methods (1.3), (1.2), and (1.1) to the fixed points and common fixed points of the general hemicontractive operators in arbitrary Hilbert spaces.

## 2. MAIN RESULTS

Chidume and Osilike [2] established the following result:

**Lemma 2.1.** *Let $B$ be a uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Then for all $x, y, z \in B$ and $\lambda \in [0,1]$, the following inequality*

$$\|\lambda x+(1-\lambda)y-z\|^q \leq [1-\lambda(q-1)]\|y-z\|^q+\lambda c\|x-z\|^q-\lambda[1-\lambda^{q-1}c]\|x-y\|^q \quad (2.1)$$

*holds, where $c$ is a positive constant.*

**Remark.** *It is known that for a Hilbert space (which is a special Banach space ), $q = 2$ and $c = 1$. Therefore, in a Hilbert space $H$, say, (2.1) reduces to*

$$\|\lambda x + (1-\lambda)y - z\|^2 \leq [1-\lambda]\|y-z\|^2 + \lambda\|x-z\|^2 - \lambda[1-\lambda]\|x-y\|^2 \quad (2.2)$$

*for all $x, y, z \in H$ and $\lambda \in [0,1]$ .*

Weng [9] established the following fundamental result which has become a useful tool in obtaining convergence results.

**Lemma 2.2.** *Let $\{\Phi_n\}$ be a nonnegative sequence of real numbers satisfying:*

$$\Phi_{n+1} \;\leq\; (1-\delta_n)\Phi_n \;+\; \sigma_n \quad (2.3)$$

*where $\delta_n \in [0,1]$ , $\sum \delta_i = \infty$ , and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty} \Phi_n = 0$.*

Our result is the following.

**Theorem 2.3.** *Let $B$ be an arbitrary Hilbert space and $K$ be a nonempty closed bounded and convex subset of $B$. Suppose $T$ is a continuous hemicontractive self–*

*mapping of $K$. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by:*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n \\ y_n &= a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1$$

*where $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ are bounded sequences in $K$ and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$,$\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0, 1]$ satisfying*

1°) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$ ,

2°) $\sum b_n = \infty$ ,

3°) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n^{'} = \lim_{n\to\infty} b_n^{''} = 0$ ,

4°) $\alpha_n := b_n + c_n,\ \beta_n := b_n{'} + c_n{'},\ \gamma_n := b_n" + c_n"$ ,

5°) $\sum \alpha_n\beta_n\gamma_n = \infty$.

*Then the sequence $\{x_n\}$ converges strongly to the fixed point of $T$.*

**Proof.** Since $T$ is hemicontractive, then $F(T)$ - the fixed point set of $T$ is nonempty. Let $x^* \in F(T)$. From the hypothesis we have:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|a_n x_n + b_n T y_n + c_n u_n - x^*\|^2 \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*) - c_n(Ty_n - u_n)\|^2 \\ &\leq (1-\alpha_n)\|(x_n - x^*) - c_n(Ty_n - u_n)\|^2 \\ &\quad +\alpha_n\|(Ty_n - x^*) - c_n(Ty_n - u_n)\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|(Ty_n - x^*) - (x_n - x^*)\|^2 \ . \end{aligned}$$

Since $\alpha_n(1-\alpha_n) \geq 0$, then we have,

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|(x_n - x^*) - c_n(Ty_n - u_n)\|^2 + \alpha_n\|(Ty_n - x^*) - c_n(Ty_n - u_n)\|^2 \ .$$

Expanding further and observing that $\|a - b\|^2 \leq \|a\|^2 + \|b\|^2$, where $a, b$ are real numbers, we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ \leq\ & (1-\alpha_n)[\|x_n - x^*\|^2 + c_n^2\|Ty_n - u_n\|^2] \\ & +\alpha_n[\|Ty_n - x^*\|^2 + c_n^2\|Ty_n - u_n\|^2] - 2c_n < Ty_n - u_n, j(Ty_n - x^*) > \\ \leq\ & (1-\alpha_n)[\|(x_n - x^*)\|^2 + c_n^2\|Ty_n - u_n)\|^2] + \alpha_n[\|Ty_n - x^*\|^2 + c_n^2\|Ty_n - u_n\|^2] \\ \leq\ & (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|Ty_n - x^*\|^2 + c_n^2\|Ty_n - u_n)\|^2 \ . \end{aligned} \tag{2.4}$$

Since $T$ is hemicontractive, then

$$\|Ty_n - x^*\|^2 \ \leq \ \|y_n - x^*\|^2 + \|y_n - Ty_n\|^2 \ . \tag{2.5}$$

We also have the following estimates:

$$\begin{aligned}
& \|y_n - x^*\|^2 \\
= \ & \|a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n - x^*\|^2 \\
= \ & \|(1-\beta_n)(x_n - x^*) + \beta_n(Tz_n - x^*) - c_n^{'}(Tz_n - v_n)\|^2 \\
\leq \ & (1-\beta_n)\|(x_n - x^*) - c_n^{'}(Tz_n - v_n)\|^2 + \beta_n\|(Tz_n - x^*) - c_n^{'}(Tz_n - v_n)\|^2 \\
& -\beta(1-\beta_n)\|(Tz_n - x^*) - (x_n - x^*)\|^2 \ .
\end{aligned}$$

Observe that $\beta(1-\beta_n) \geq 0$, therefore

$$\begin{aligned}
\|y_n - x^*\|^2 \ & \leq (1-\beta_n)[\|(x_n - x^*)\|^2 + {\beta_n}^2\|Tz_n - v_n)\|^2] \\
& \quad +\beta_n[\|Tz_n - x^*\|^2 + {\beta_n}^2\|Tz_n - v_n\|^2] \\
& = (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tz_n - x^*\|^2 + {\beta_n}^2\|Tz_n - v_n\|^2 \ .
\end{aligned} \tag{2.6}$$

$T$ is hemicontractive, therefore

$$\|Tz_n - x^*\|^2 \leq \|z_n - x^*\|^2 + \|z_n - Tz_n\|^2 \ . \tag{2.7}$$

Substituting (2.7) into (2.6), yields

$$\begin{aligned}
\|y_n - x^*\|^2 \ & \leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 \\
& \quad +{\beta_n}^2\|Tz_n - v_n\|^2 + \beta_n\|z_n - Tz_n\| \ .
\end{aligned} \tag{2.8}$$

Substitute (2.8) into (2.5), we obtain:

$$\begin{aligned}
\|Ty_n - x^*\|^2 \ & \leq \ (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 \\
& \quad +{\beta_n}^2\|Tz_n - v_n\|^2 + \beta_n\|z_n - Tz_n\| + \|y_n - Ty_n\|^2 \ .
\end{aligned} \tag{2.9}$$

Further estimates gives the following,

$$\begin{aligned}
& \|z_n - x^*\|^2 \\
= \ & \|a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n - x^*\|^2 \\
= \ & \|(1-\gamma_n)(x_n - x^*) + \gamma_n(Tx_n - x^*) - c_n^{''}(Tx_n - \omega_n)\|^2 \\
\leq \ & (1-\gamma_n)\|(x_n - x^*) - c_n^{''}(Tx_n - \omega_n)\|^2 + \gamma_n\|(Tx_n - x^*) - c_n^{''}(Tx_n - \omega_n)\|^2 \\
\leq \ & (1-\gamma_n)\|(x_n - x^*)\|^2 + \gamma_n\|Tx_n - x^*\|^2 \\
& +{\gamma_n}^2(1-\gamma_n)\|Tx_n - \omega_n\|^2 + {\gamma_n}^2\gamma_n\|Tx_n - \omega_n\|^2 \\
= \ & (1-\gamma_n)\|(x_n - x^*)\|^2 + \gamma_n\|Tx_n - x^*\|^2 + {\gamma_n}^2\|Tx_n - \omega_n\|^2 \ .
\end{aligned} \tag{2.10}$$

By continuity of $T$ and boundedness on $K$, there exists a real numbers $M_1 < \infty$ such that

$$\|Tx_n - x^*\|^2 \leq M_1 \text{ and } \|Tx_n - \omega_n\|^2 \leq M_1 \ .$$

Observe that $c_n^{''} < \gamma_n$ and $\gamma_n{}^2 < \gamma_n$ for all $n$. Then (2.10) reduces to:

$$\|z_n - x^*\|^2 \leq (1 - \gamma_n)\|(x_n - x^*)\|^2 + 2\gamma_n M_1 \ . \tag{2.11}$$

We now substitute (2.11) into (2.9), to get:

$$\begin{aligned} \|Ty_n - x^*\|^2 \ &\leq \ (1-\beta_n)\|x_n - x^*\|^2 + \beta_n[(1-\gamma_n)\|x_n - x^*\|^2 + 2\gamma_n M_1] \\ &\quad +\beta_n{}^2\|Tz_n - v_n\|^2 + \beta_n\|z_n - Tz_n\|^2 + \|y_n - Ty_n\|^2 \\ &\leq (1-\beta_n\gamma_n)\|x_n - x^*\|^2 + 2\beta_n\gamma_n M_1 \\ &\quad +\beta_n{}^2\|Tz_n - v_n\|^2 + \beta_n\|z_n - Tz_n\|^2 + \|y_n - Ty_n\|^2 \ . \end{aligned} \tag{2.12}$$

By continuity of $T$ on the bounded set $K$, there exists a real number $M_2 < \infty$ such that

$$\|z_n - Tz_n\|^2 \leq M_2, \quad \|y_n - Ty_n\|^2 \leq M_2 \quad \text{and} \quad \|Tz_n - v_n\|^2 \leq M_2 \ .$$

Therefore, (2.12) reduces to:

$$\|Ty_n - x^*\|^2 \ \leq \ (1-\beta_n\gamma_n)\|x_n - x^*\|^2 + 2\beta_n\gamma_n M_1 + [2\beta_n + 1]M_2 \ . \tag{2.13}$$

Substitute (2.13) into (2.4), we have :

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n[(1-\beta_n\gamma_n)\|x_n - x^*\|^2 \\ &\quad +2\beta_n\gamma_n M_1 + [2\beta_n + 1)M_2] + c_n^2\|Ty_n - u_n)\|^2 \\ &= [1 - \alpha_n\beta_n\gamma_n]\|x_n - x^*\|^2 + 2\alpha_n\beta_n\gamma_n M_1 + [2\alpha_n\beta_n + \alpha_n]M_2 \\ &\quad +c_n^2\|Ty_n - u_n)\|^2 \ . \end{aligned} \tag{2.14}$$

Continuity of $T$ on $K$ also implies that there exists real numbers $M_3 < \infty$ , such that $\|Ty_n - u_n\|^2 \leq M_3$. Let $M_6 = max[M_1, M_2, M_3]$. Therefore, from (2.14) and the fact that $c_n^2 < \alpha_n{}^2 < \alpha_n$, we have:

$$\|x_{n+1} - x^*\|^2 \leq [1 - \alpha_n\beta_n\gamma_n]\|x_n - x^*\|^2 + [2\alpha_n\beta_n\gamma_n + 2\alpha_n\beta_n + 2\alpha_n]M_6 \ . \tag{2.15}$$

Now, put $\Phi_n = \|x_n - x^*\|^2$ and $\delta_n = \alpha_n\beta_n\gamma_n$ and

$$\sigma_n = [2\alpha_n\beta_n\gamma_n + 2\alpha_n\beta_n + 2\alpha_n]M_6 \ .$$

Then, (2.15) becomes:

$$\Phi_{n+1} \leq (1-\delta_n)\Phi_n + \sigma_n, \quad n \geq 1 \; .$$

Clearly, $\sigma_n = o(\delta_n)$ , $\sum \delta_n = \infty$ and $0 \leq \delta_n \leq 1$. Hence, by Weng [13], $\Phi_n \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to $x^*$. This completes the proof.

**Remark.** *Theorem 2.3. above is an extension of the result of Ishikawa [4] to the larger class of hemicontractive operators and to the more general three-step iteration scheme (1.3). Theorem 2.3 also generalizes those of Chidume [1], Deng and Ding [3].*

Observe that the condition imposed on the parameters is simple and less complicated. For a prototype for parameters, we may consider:

$$\alpha_n = \tfrac{1}{n}, \quad \beta_n = \tfrac{n}{n+1} \quad \text{and} \quad \gamma_n = \tfrac{n+1}{n+2}$$

which obviously satisfy the conditions of our result.

We now investigate the convergence the slightly more general iteration scheme (1.2) for fixed points of continuous hemicontractive operators in Hilbert spaces.

**Theorem 2.4.** *Let $K$ be a closed bounded convex nonempty subset of a Hilbert space $H$. Let $T$ be a completely continuous hemicontractive selfmapping of $K$ . Define a sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by:*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n T x_n \\ y_n &= a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1$$

*where $\{v_n\}$, $\{\omega_n\}$ are bounded sequences in $K$ and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$,$\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying the following conditions:*

1°) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1,$

2°) $\sum b_n = \infty$

3°) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_n^{'} = \lim\limits_{n\to\infty} b_n^{''} = 0$

4°) $\alpha_n = b_n + c_n, \beta_n = b_n^{'} + c_n^{'}, \gamma_n = b_n^{''} + c_n^{''}$

5°) $\sum \alpha_n \beta_n \gamma_n = \infty.$

*Then the sequence $\{x_n\}$ converges strongly to the fixed point of $T$.*

**Proof.** $T$ is hemicontractive implies that $F(T)$ - the fixed point set of $T$ - is nonempty. Let $x^* \in F(T)$. Then from our hypothesis we have the following estimates:

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
= \;& \|a_n x_n + b_n T y_n + c_n T x_n - x^*\|^2 \\
= \;& \|(1-\alpha_n)(x_n - x^*) + \alpha_n (T y_n - x^*) - c_n (T y_n - T x_n)\|^2 \\
\leq \;& (1-\alpha_n)\|(x_n - x^*) - c_n (T y_n - T x_n)\|^2 + \alpha_n \|(T y_n - x^*) - c_n (T y_n - u_n)\|^2 \\
& -\alpha_n (1-\alpha_n)\|(T y_n - x^*) - (x_n - x^*)\|^2 \; .
\end{aligned}$$

Expanding further and observing that $\alpha_n(1-\alpha_n) \geq 0$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \;& \leq (1-\alpha_n)[\|(x_n - x^*)\|^2 + c_n^2 \|T y_n - T x_n)\|^2] \\
& \quad +\alpha_n [\|T y_n - x^*\|^2 + c_n^2 \|T y_n - T x_n\|^2] \\
& \leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|T y_n - x^*\|^2 + c_n^2 \|T y_n - T x_n\|^2 \; .
\end{aligned}$$

Since $c_n^2 \leq {\alpha_n}^2 \leq \alpha_n$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \;& \leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|y_n - x^*\|^2 \\
& \quad +\alpha_n \|y_n - T y_n\|^2 + \alpha_n \|T y_n - T x_n\|^2 \; .
\end{aligned} \tag{2.16}$$

$T$ is completely continuous on the bounded set $K$ implies that there exists a real number $M_4 < \infty$ such that :

$$\|T y_n - T x_n\|^2 \leq M_4 \quad \text{and} \quad \|y_n - T y_n\|^2 \leq M_4 \; .$$

Substituting into (2.16) yields

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|y_n - x^*\|^2 + 2\alpha_n M_4 \; . \tag{2.17}$$

From our hypothesis, we also have the following,

$$\begin{aligned}
& \|y_n - x^*\|^2 \\
= \;& \|a_n' x_n + b_n' T z_n + c_n' v_n - x^*\|^2 \\
= \;& \|(1-\beta_n)(x_n - x^*) + \beta_n (T z_n - x^*) - c_n' (T z_n - v_n)\|^2 \\
\leq \;& (1-\beta_n)\|(x_n - x^*) - c_n' (T z_n - v_n)\|^2 + \beta_n \|(T z_n - x^*) - c_n' (T z_n - v_n)\|^2 \\
& -\beta (1-\beta_n)\|(T z_n - x^*) - (x_n - x^*)\|^2 \\
\leq \;& (1-\beta_n)[\|x_n - x^*\|^2 + {\beta_n}^2 \|T z_n - v_n\|^2] + \beta_n [\|T z_n - x^*\|^2 + {\beta_n}^2 \|T z_n - v_n\|^2] \\
= \;& (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|T z_n - x^*\|^2 + {\beta_n}^2 \|T z_n - v_n\|^2 \\
\leq \;& (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \beta_n \|z_n - T z_n\|^2 + {\beta_n}^2 \|T z_n - v_n\|^2 \; .
\end{aligned} \tag{2.18}$$

Continuity of $T$ on $K$ implies that there exists a real number $M_5 < \infty$ such that $\|z_n - Tz_n\|^2 \leq M_5$ and $\|Tz_n - v_n\|^2 \leq M_5$. Then (2.18) reduces to :

$$\|y_n - x^*\|^2 \leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 + 2\beta_n M_5 \ . \tag{2.19}$$

By similar estimates, we have:

$$\|z_n - x^*\|^2 \leq (1-\gamma_n)\|x_n - x^*\|^2 + 2\gamma_n M_7 \tag{2.20}$$

for some real number $M_7 < \infty$.
Substituting (2.20) into (2.19) yields :

$$\begin{aligned} \|y_n - x^*\|^2 \ &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n(1-\gamma_n)\|x_n - x^*\|^2 \\ &\quad +2\beta_n\gamma_n M_7 + 2\beta_n M_5 \\ &\leq (1-\beta_n\gamma_n)\|x_n - x^*\|^2 + \beta_n(1+\gamma_n)M_8 \end{aligned} \tag{2.21}$$

where $M_8 = \max\ [2M_5\ ,\ 2M_7]$ .
Substitute (2.21) into (2.17), we obtain:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(1-\beta_n\gamma_n)\|x_n - x^*\|^2 \\ &\quad +\alpha_n\beta_n(1+\gamma_n)M_8 + 2\alpha_n M_4 \ . \end{aligned}$$

Let $M_9 = \max[2M_4\ ,\ M_8]$. Then we have:

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n\beta_n\gamma_n)\|x_n - x^*\|^2 + \alpha_n[1+\beta_n(1+\gamma_n)]M_9 \ . \tag{2.22}$$

Now, put

$$\Phi_n = \|x_n - x^*\|^2 \ \text{ and } \ \delta_n = \alpha_n\beta_n\gamma_n \ .$$

Also let

$$\sigma_n = \alpha_n[1+\beta_n(1+\gamma_n)]M_9 \ .$$

Then, (2.22), reduces to:

$$\Phi_{n+1} = (1-\delta_n)\Phi_n + \sigma_n \ .$$

Observe that

$$0 \leq \delta_n < 1 \quad \text{ and } \quad \sum \delta_n = \infty \quad \text{ and } \quad \sigma_n = o(\delta_n) \ .$$

Therefore, by Lemma 2.2, $\Phi_n \to 0$ as $n \to \infty$. This implies that the sequence $\{x_n\}$ converges strongly to $x^*$. This completes the proof.

**Remark.** *Theorem 2.4 is slightly more general than Theorem 2.3. Hence it also generalizes the results of Liu [5], Xu [10], Deng [3], Chidume [1], Chidume and Osilike [2] and others to the generalized three-step iteration procedure.*

We now investigate the convergence of the generalized Ishikawa type iteration procedure for fixed points of hemicontractive operators in Hilbert spaces. Our result is the following.

**Theorem 2.5.** *Let $K$ be a nonempty closed bounded convex subset of a Hilbert space $H$. Suppose $S, T$ are uniformly continuous selfmappings of $K$ and $T$ is hemicontractive on $K$. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by:*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n S x_n \\ y_n &= a_n^{'} x_n + b_n^{'} S z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1$$

*where $\{v_n\}$, $\{\omega_n\}$ are bounded sequences in $K$ and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying the following:*

1°) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

2°) $\sum b_n = \infty$ ,

3°) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_n^{'} = \lim\limits_{n\to\infty} b_n^{''} = 0$,

4°) $\alpha_n =: b_n + c_n, \beta_n =: b_n^{'} + c_n^{'}, \gamma_n =: b_n^{''} + c_n^{''}$

5°) $\sum \alpha_n \beta_n = \infty$.

*Then the sequence $\{x_n\}$ converges strongly to the fixed point of $T$.*

**Proof.** Since $T$ is hemicontractive, then $F(T)$ is nonempty. Let $x^* \in F(T)$. Then, from our hypothesis, we have:

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ = \; & \|a_n x_n + b_n T y_n + c_n S x_n - x^*\|^2 \\ = \; & \|(1-\alpha_n)(x_n - x^*) + \alpha_n (T y_n - x^*) - c_n (T y_n - S x_n)\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\|(x_n-x^*)-c_n(Ty_n-Sx_n)\|^2+\alpha_n\|(Ty_n-x^*)-c_n(Ty_n-Sx_n)\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|(Ty_n-x^*)-(x_n-x^*)\|^2 \\
&\leq (1-\alpha_n)[\|x_n-x^*\|^2+c_n^2\|Ty_n-Sx_n\|^2]+\alpha_n[\|Ty_n-x^*\|^2+c_n^2\|Ty_n-Sx_n\|^2] \\
&= (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|Ty_n-x^*\|^2+c_n^2\|Ty_n-Sx_n\|^2 \\
&= (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|y_n-x^*\|^2+\alpha_n\|y_n-Ty_n\|^2+{\alpha_n}^2\|Ty_n-Sx_n\|^2 \ .
\end{aligned} \tag{2.23}
$$

Continuity of $S, T$ implies that there exists real numbers $q_1, q_2 < \infty$ such that :

$$\|y_n-Ty_n\|^2 \leq q_1 \qquad \text{and} \qquad \|Ty_n-Sx_n\|^2 \leq q_2 \ .$$

Let $q_3 = \max[q_1, q_2]$. Then (2.23) yields:

$$\|x_{n+1}-x^*\|^2 = (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|y_n-x^*\|^2+2\alpha_n q_3 \tag{2.24}$$

We also have the following estimates:

$$
\begin{aligned}
&\|y_n-x^*\|^2 \\
&= \|a_n'x_n+b_n'Sz_n+c_n'v_n-x^*\|^2 \\
&= \|(1-\beta_n)(x_n-x^*)+\beta_n(Sz_n-x^*)-c_n'(Sz_n-v_n)\|^2 \\
&\leq (1-\beta_n)\|(x_n-x^*)-c_n'(Sz_n-v_n)\|^2+\beta_n\|(Tz_n-x^*)-c_n'(Sz_n-v_n)\|^2 \\
&\quad -\beta(1-\beta_n)\|(Sz_n-x^*)-(x_n-x^*)\|^2 \\
&\leq (1-\beta_n)[\|x_n-x^*\|^2+{\beta_n}^2\|Sz_n-v_n\|^2]+\beta_n[\|Sz_n-x^*\|^2+{\beta_n}^2\|Sz_n-v_n\|^2] \\
&= (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|Sz_n-x^*\|^2+{\beta_n}^2\|Sz_n-v_n\|^2 \ .
\end{aligned} \tag{2.25}
$$

Continuity of $S$ on $K$ implies there exists a real number $q_4 < \infty$ such that

$$\|Sz_n-x^*\|^2 \leq q_4 \qquad \text{and} \qquad \|Sz_n-v_n\|^2 \leq q_4.$$

Then, from (2.25), we have:

$$\|y_n-x^*\| \ \leq \ (1-\beta_n)\|x_n-x^*\|^2+2\beta_n q_4 \ . \tag{2.26}$$

Substituting (2.26) into (2.24), we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&= (1-\alpha_n)\|(x_n-x^*)\|^2+\alpha_n[(1-\beta_n)\|x_n-x^*\|^2+2\beta_n q_4]+2\alpha_n q_3 \ .
\end{aligned} \tag{2.27}
$$

Let $q_5 = \max[q_3, q_4]$. Then

$$\|x_{n+1}-x^*\|^2 = (1-\alpha_n\beta_n)\|(x_n-x^*)\|^2+2\alpha_n(\beta_n+1)q_5 \ . \tag{2.28}$$

Putting

$$\rho_n = \|x_n - x^*\|^2$$

and

$$\delta_n = \alpha_n\beta_n, \quad \sigma_n = 2\alpha_n(\beta_n + 1)q_5 \ .$$

Then (2.28) reduces to:

$$\rho_{n+1} \ = \ (1-\delta_n)\rho_n \ + \ \sigma_n \ .$$

Clearly,

$$0 \leq \delta_n \leq 1 \ , \sum \delta_n = \infty \ , \text{ and } \sigma_n = o(\delta_n).$$

Hence by Lemma 2.2, $\rho_n \to 0$ as $n \to \infty$ . This implies that $\{x_n\}$ converges strongly to $x^*$. This completes the proof.

We now consider the situation when the operators $T$ and $S$ are both hemicontractive and investigate the convergence of the sequence generated by (1.1) to the common fixed point of $T$ and $S$, when it exists. We have the following result.

**Theorem 2.6.** *Let $K$ be a nonempty closed bounded convex subset of a Hilbert space $H$. Suppose $S, T$ are uniformly continuous hemicontractive selfmappings of $K$. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n S x_n \\ y_n &= a_n^{'} x_n + b_n^{'} S z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{aligned}\right\} n \geq 1$$

*where $\{v_n\}$, $\{\omega_n\}$ are arbitrary sequences in $K$ and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying*

1°) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1,$

2°) $\sum b_n = \infty$ *and* $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n^{'} = \lim_{n\to\infty} b_n^{''} = 0,$

3°) $\alpha_n := b_n + c_n$, $\beta_n := b_n^{'} + c_n^{'}$ , $\gamma_n := b_n^{''} + c_n^{''}$ *and* $\sum \alpha_n\beta_n\gamma_n = \infty.$

*If $S, T$ have a common fixed point in $K$, then the sequence $\{x_n\}$ converges strongly to the common fixed point of $S$ and $T$.*

**Proof.** Since $S, T$ are hemicontractive, then the fixed point sets $F(S)$ and $F(T)$ are nonempty. Let $p$ be a common fixed point of $S$ and $T$. By our hypothesis and

Lemma 2.2., we have the following estimates:

$$\begin{aligned}
& \|x_{n+1} - p\|^2 \\
= \ & \|a_n x_n + b_n T y_n + c_n S x_n - p\|^2 \\
= \ & \|(1-\alpha_n)(x_n - p) + \alpha_n (T y_n - p) - c_n (T y_n - S x_n)\|^2 \\
\leq \ & (1-\alpha_n)\|(x_n - p) - c_n (T y_n - S x_n)\|^2 + \alpha_n \|(T y_n - p) - c_n (T y_n - S x_n)\|^2 \\
& - \alpha_n (1-\alpha_n)\|(T y_n - p) - (x_n - p)\|^2 \ .
\end{aligned}$$

Observe that $\alpha_n(1-\alpha_n) \geq 0$ and ${c_n}^2 \leq {\alpha_n}^2 \leq \alpha_n$. Therefore, expanding the above further, yields

$$\begin{aligned}
\|x_{n+1} - p\|^2 & \leq (1-\alpha_n)\|(x_n - p) - c_n (T y_n - S x_n)\|^2 \\
& \quad + \alpha_n \|(T y_n - p) - c_n (T y_n - S x_n)\|^2 \\
& \leq (1-\alpha_n)[\|x_n - p\|^2 + c_n^2 \|T y_n - S x_n\|^2] \\
& \quad + \alpha_n [\|T y_n - p\|^2 + c_n^2 \|T y_n - S x_n\|^2] \qquad (2.29) \\
& = (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|T y_n - p\|^2 + c_n^2 \|T y_n - S x_n\|^2 \\
& \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|y_n - p\|^2 \\
& \quad + \alpha_n \|y_n - T y_n\|^2 + \alpha_n \|T y_n - S x_n\|^2 \ .
\end{aligned}$$

Since $S, T$ are uniformly continuous on the bounded set $K$, there exists a positive real number $M_1 < \infty$ such that

$$\|y_n - T y_n\|^2 \leq M_1 \text{ and } \|T y_n - S x_n\|^2 \leq M_1 \ .$$

Therefore, (2.29) yields

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|y_n - p\|^2 + 2\alpha_n M_1 \ . \qquad (2.30)$$

We also have the following estimates,

$$\begin{aligned}
\|y_n - x^*\|^2 & = \|a_n' x_n + b_n' S z_n + c_n' v_n - p\|^2 \\
& = \|(1-\beta_n)(x_n - p) + \beta_n (S z_n - p) - c_n' (S z_n - v_n)\|^2 \\
& \leq (1-\beta_n)\|(x_n - p) - c_n'(S z_n - v_n)\|^2 + \beta_n \|(S z_n - p) - c_n'(S z_n - v_n)\|^2 \\
& \quad - \beta_n (1-\beta_n)\|(S z_n - p) - (x_n - p)\|^2 \ .
\end{aligned}$$

Since $\beta_n(1-\beta_n) \geq 0$, expanding further, we have

$$\begin{aligned}
\|y_n - p\|^2 & \leq (1-\beta_n)[\|x_n - p\|^2 + (1-\beta_n) c_n' \|S z_n - v_n\|^2] \\
& \quad + \beta_n [\|S z_n - p\|^2 + \beta_n c_n' \|S z_n - v_n\|^2] \qquad (2.31) \\
& = (1-\beta_n)\|x_n - p\|^2 + \beta_n \|S z_n - p\|^2 + c_n' \|S z_n - v_n\|^2 \ .
\end{aligned}$$

Observe that $c_n^{'} \leq \beta_n$ and $S$ is hemicontractive. Also, continuity of $S$ on $K$ implies that there exist a positive real number $M_2 < \infty$ such that

$$\|z_n - Sz_n\|^2 \leq M_2 \text{ and } \|Sz_n - v_n\|^2 \leq M_2 \ .$$

Then from (2.31) we have:

$$\begin{aligned} \|y_n - p\| &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 \\ &\quad + \beta_n\|z_n - Sz_n\|^2 + \|Sz_n - v_n\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 + 2\beta_n M_2 \ . \end{aligned} \tag{2.32}$$

We also have the following estimates:

$$\begin{aligned} \|z_n - p\|^2 &= \|a_n^{''}x_n + b_n^{''}Tx_n + c_n^{''}\omega_n - p\|^2 \\ &= \|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p) - c_n^{''}(Tx_n - \omega_n)\|^2 \\ &\leq (1-\gamma_n)\|(x_n - p) - c_n^{''}(Tx_n - \omega_n)\|^2 \\ &\quad + \|(Tx_n - p) - c_n^{''}(Tx_n - \omega_n)\|^2 \ . \end{aligned}$$

Expanding further, we have

$$\begin{aligned} \|z_n - p\|^2 &\leq (1-\gamma_n)[\|x_n - p\|^2 + c_n^{''}\|Tx_n - \omega_n\|^2] \\ &\quad + \gamma_n[\|Tx_n - p\|^2 + c_n^{''}\|Tx_n - \omega_n\|^2] \\ &= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|Tx_n - p\|^2 + c_n^{''}\|Tx_n - \omega_n\|^2 \ . \end{aligned}$$

Continuity of $T$ on the bounded set $K$ implies that there exists a positive real number $M_3 < \infty$ such that

$$\|Tx_n - x^*\|^2 \leq M_3 \quad \text{and} \quad \|Tx_n - \omega_n\|^2 \leq M_3 \ .$$

Therefore, since $c_n^{''} \leq \gamma_n$, we have

$$\|z_n - p\|^2 \leq (1-\gamma_n)\|x_n - p\|^2 + 2\gamma_n M_3 \ . \tag{2.33}$$

Substituting (2.33) into (2.32) yields,

$$\begin{aligned} \|y_n - p\|^2 &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n(1-\gamma_n)\|x_n - p\|^2 + 2\beta_n\gamma_n M_3 + 2\beta_n M_2 \\ &= (1-\beta_n\gamma_n)\|x_n - p\|^2 + 2\beta_n(\gamma_n M_+ M_2) \ . \end{aligned} \tag{2.34}$$

Substitution (2.34) into (2.30), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n(1-\beta_n\gamma_n)\|x_n-p\|^2\\ &\quad +2\alpha_n\beta_n(\gamma_n M_3+M_2)+2\alpha_n M_1 \qquad (2.35)\\ &= (1-\alpha_n\beta_n\gamma_n)\|x_n-p\|^2+2\alpha_n M_1+2\alpha_n\beta_n(\gamma_n M_3+M_2)\ .\end{aligned}$$

Let $M = \max\,[2M_1, 2M_2, 2M_3]$, then we have

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n\beta_n\gamma_n)\|x_n-p\|^2+\alpha_n[1+\beta_n(\gamma_n+1)]M\ . \qquad (2.36)$$

Now, setting

$$\rho_n = \|x_n - x^*\|^2, \quad \delta_n = \alpha_n\beta_n\gamma_n, \quad \sigma_n = \alpha_n[1+\beta_n(\gamma_n+1)]M,$$

then (2.36) reduces to:

$$\rho_{n+1} \ = \ (1-\delta_n)\rho_n \ + \ \sigma_n$$

Clearly,

$$0 \leq \delta_n \leq 1\ , \quad \sum\delta_n = \infty \quad \text{and} \quad \sigma_n = o(\delta_n)\ .$$

Hence by Lemma 2.2, $\rho_n \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to $p$ - the common fixed point of $S$ and $T$. This completes the proof.

**Remark.** *Theorem 2.6. is a refinement of some previous relevant results for hemicontractions including the result of Owojori and Imoru [6], to common fixed points of hemicontractive operators in Hilbert spaces.*

We now consider the situation when in (1.1) the operator $S$ is nonexpansive and $T$ is hemicontractive. Our result is the following ...

**Theorem 2.7.** *Let $B, K, T$ and the sequence $\{x_n\}$ be as defined in Theorem 2.6. Suppose $S : K \to K$ is nonexpansive. Replace the condition $3°$) on the parameters replaced with*

$$\sum\alpha_n\beta_n\gamma_n \ = \ \infty \quad \textit{and} \ \ 2{\alpha_n}^2 \leq (1+\alpha_n)\alpha_n\beta_n\gamma_n \leq 1+2{\alpha_n}^2.$$

*And suppose all other conditions are satisfied. If $S, T$ have a common fixed point in $K$, then the sequence $\{x_n\}$ converges strongly to the common fixed point of $T$ and $S$.*

**Proof.** Since $K$ is uniformly convex and $S$ is nonexpansive, then, $F(S)$ is nonempty. Also, $T$ is hemicontractive implies that $F(T)$ is nonempty. Let $x^*$ be the common fixed point of $S$ and $T$. From our hypothesis and by Lemma 2.1, we have the following estimates,

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|a_nx_n+b_nTy_n+c_nSx_n-x^*\|^2\\ &= \|(1-\alpha_n)(x_n-x^*)+\alpha_n(Ty_n-x^*)-c_n(Ty_n-Sx_n)\|^2\\ &\le (1-\alpha_n)\|(x_n-x^*)\|^2+\alpha_n\|Ty_n-x^*\|^2+c_n^2\|Ty_n-Sx_n)\|^2\ .\end{aligned}$$

But $T$ is hemicontractive and $c_n^2 \le {\alpha_n}^2$. Therefore,

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|y_n-x^*\|^2\\ &\quad+\alpha_n\|y_n-Ty_n\|^2+{\alpha_n}^2\|Ty_n-Sx_n\|^2\\ &= (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|y_n-x^*\|^2\\ &\quad+\alpha_n\|y_n-Ty_n\|^2+{\alpha_n}^2\|(Ty_n-x^*)-(Sx_n-x^*)\|^2\\ &\le (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|y_n-x^*\|^2+\alpha_n\|y_n-Ty_n\|^2\\ &\quad+{\alpha_n}^2\|Ty_n-x^*\|^2+{\alpha_n}^2\|Sx_n-x^*)\|^2\ .\end{aligned}$$

Since $T$ is hemicontractive and $S$ is nonexpansive, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|y_n-x^*\|^2+\alpha_n\|y_n-Ty_n\|^2\\ &\quad+{\alpha_n}^2\|y_n-x^*\|^2+{\alpha_n}^2\|y_n-Ty_n\|^2+{\alpha_n}^2\|x_n-x^*\|^2\\ &= [1-\alpha_n(1-\alpha_n)]\|x_n-x^*\|^2+\alpha_n(1+\alpha_n)\|y_n-x^*\|^2\\ &\quad+\alpha_n(1+\alpha_n)\|y_n-Ty_n\|^2\ .\end{aligned} \tag{2.37}$$

Estimate (2.31) is also valid here. Observing that $c_n \le \beta_n$ we have

$$\|y_n-x^*\|^2 \le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|Sz_n-x^*\|^2+\beta_n\|Sz_n-v_n\|^2\ . \tag{2.38}$$

Since $S$ is nonexpansive, then

$$\|Sz_n-x^*\|^2 \le \|z_n-x^*\|^2\ .$$

Also continuity of $S$ on the bounded set $K$ implies that there exists a real number $R_2 < \infty$ such that $\|Sz_n-u_n\|^2 \le R_2$. Substituting into (2.38) yields

$$\|y_n-x^*\|^2 \le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|z_n-x^*\|^2+\beta_nR_2\ . \tag{2.39}$$

Substituting the equation (2.33) i.e.

$$\|z_n-x^*\|^2 \le (1-\gamma_n)\|x_n-x^*\|^2+2\gamma_nM_3$$

(which also holds in this case) into (2.39), we obtain

$$\begin{aligned} \|y_n - x^*\|^2 &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n(1-\gamma_n)\|x_n - x^*\|^2 + 2\beta_n\gamma_n M_3 + \beta_n R_2 \\ &\leq (1-\beta_n\gamma_n)\|x_n - x^*\|^2 + \beta_n(\gamma_n+1)R_3 \end{aligned} \tag{2.40}$$

where $R_4 < \infty$ is a real number such that $R_4 = \max\,[M_3, R_2]$.
Substitute (2.40) into (2.37) and observing that $T$ is continuous on the bounded set $K$ implies that there exists a positive real number $R_5 < \infty$ such that $\|y_n - Ty_n\|^2 \leq R_5$ we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq [(1-\alpha_n(1-\alpha_n))\|x_n - x^*\|^2 \\ &\quad + \alpha_n(1+\alpha_n)[(1-\beta_n\gamma_n)\|x_n - x^*\|^2 + \beta_n(\gamma_n+1)R_4] \\ &\quad + \alpha_n(1+\alpha_n)R_5 \\ &= [1-\alpha_n(1-\alpha_n) + \alpha_n(1+\alpha_n)(1-\beta_n\gamma_n)]\|x_n - x^*\|^2 \\ &\quad + \alpha_n(1+\alpha_n)[\beta_n(\gamma_n+1)+1]R_6 \\ &= [1-\alpha_n\{(1+\alpha_n)\beta_n\gamma_n - 2\alpha_n\}]\|x_n - x^*\|^2 \\ &\quad +\alpha_n(1+\alpha_n)[\beta_n(\gamma_n+1)+1]R_6 \end{aligned} \tag{2.41}$$

where $R_6 < \infty$ is a real number such that $R_6 = \max[R_4, R_5]$ .
Now, set

$$\Phi_n = \|x_n - x^*\|^2 \ , \quad \delta_n \ = \ \alpha_n\{(1+\alpha_n)\beta_n\gamma_n - 2\alpha_n\}$$

and

$$\sigma_n \ = \ \alpha_n(1+\alpha_n)[\beta_n(\gamma_n+1)+1]R_6 \ .$$

Then (2.14) reduces,

$$\Phi_{n+1} \ = \ (1-\delta_n)\Phi_n \ + \ \sigma_n \ .$$

Observe that $0 \leq \delta_n \leq 1$ and $\sigma_n = o(\delta_n)$. Also $\sum \delta_n = \infty$ from hypothesis.
Hence by Lemma 2.2, $\Phi_n \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to $x^*$ - the common fixed point of $S$ and $T$. This completes the proof.

**Remark** *We observe that our results in this manuscript can be readily extended to uniformly smooth Banach spaces, especially the $L_p$ spaces, $p \geq 2$. This is evident by adopting similar techniques as in our proofs above and applying the equivalence of the inequality in Lemma 1.1 for $L_p$ spaces in the expansions of expressions which are* *likely to occur in the appropriate proofs. The equivalent inequality in $L_p$ spaces, $p \geq 2$ is given by*

$$\|\lambda x + (1-\lambda)y - z\|^2 \leq [1-\lambda]\|y-z\|^2 + \lambda(p-1)\|x-z\|^2 - \lambda[1-\lambda]\|x-y\|^2$$

*for all $x, y, z \in L_p$ and $\lambda \in [0,1]$.*

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