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SHARP WEIGHTED INEQUALITY FOR MULTILINEAR COMMUTATOR OF STRONGLY SINGULAR INTEGRAL OPERATOR

Xiaosha Zhou

College of Mathematics
Changsha University of Science and Technology
Changsha 410077, P.R. of China
(e-mail: zhouxiaosha57@126.com)

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Abstract. In this paper, we prove a sharp inequality for the multilinear commutator of strongly singular integral operator. By using this inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

1. INTRODUCTION AND RESULTS

As the development of singular integral operators, the commutators of the singular integral operators have been well studied (see [5, 6, 7, 8].) In this paper, we will study the multilinear commutator of the strongly singular integral operator as following.

Let $0 < b < 1$ and $\theta(\xi)$ be a smooth radial cut-off function on R^n such that $\theta(\xi) = 1$ if $|\xi| \geq 1$ and $\theta(\xi) = 0$ if $|\xi| \leq 1/2$. The strongly singular integral operator is a multiplier operator which is defined by

$$(T(f))\hat{}(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^{nb/2}}\hat{f}(\xi).$$

The kernel K for T is very singular. Roughly speaking, it looks like $K(x) = \vartheta(x)e^{i|x|^{-b'}}/|x|^n$, where $b' = b/(1-b)$ and $supp\ \vartheta \subset \{x \in R^n : |x| \le 2\}$. In fact, we know that

$$T(f)(x) = p.v. \int_{R^n} K(x-y)f(y)dy$$

and for $|x| \ge 2|y|$ (see[1])

$$|K(x-y) - K(x)| \le C|y||x|^{-n-b'-1}.$$

Let $b_j (j = 1, ..., m)$ be the fixed locally integrable functions on R^n. The the multilinear commutator related to T is defined by

$$T_{\tilde{b}}(f)(x) = p.v. \int_{R^n} \left[\prod_{j=1}^{m} (b_j(x) - b_j(y)) \right] K(x-y)f(y)dy.$$

Note that when $b_1 = \cdots = b_m$, $T_{\tilde{b}}$ is just the m order commutator (see [3, 7]). The strongly singular integral operator has been studied by several authors (see[1, 2, 10]). In [3], the weighted norm inequality for the commutator of strongly singular integral operator is obtained by using a sharp estimate. In [8], Perez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator of singular integral operator. The main purpose of this paper is to prove a sharp inequality for the multilinear commutator of strongly singular integral operator. As the application, we obtain the weighted L^p-norm inequality for the multilinear commutator.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with side parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [4, 9])

$$f^{\#}(x) = \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^\#||_{L^\infty}$. Given functions $b_j (j = 1, ..., m)$ and $0 < r < \infty$, denote that

$$C_{\tilde{b}}^r(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q \prod_{j=1}^m |b_j(x) - b_j(y)|^r |f(y)| dy,$$

we write $C_{\tilde{b}}^r(f) = C_{\tilde{b}}(f)$ if $r = 1$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)| dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, denote that $\sigma^c = \{1, \cdots, m\} \setminus \sigma$; For $\tilde{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, denote $\tilde{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\tilde{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$. We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$(see [4]).

We shall prove the following theorems.

Theorem 1. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $1 < r < \infty$ and $x \in R^n$,*

$$(T_{\tilde{b}}(f))^\#(x)$$
$$\le C\left[\prod_{j=1}^m ||b_j||_{BMO} M_r(f)(x) + M((C_{\tilde{b}}^r(|f|^r))^{1/r})(x) + \sum_{j=1}^m \sum_{\sigma\in C_j^m} ||\tilde{b}_\sigma||_{BMO} M_r(T_{\tilde{b}_{\sigma^c}}(f))(x)\right].$$

Theorem 2. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then $T_{\tilde{b}}$ is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_p$, that is*

$$||T_{\tilde{b}}(f)||_{L^p(w)} \le C||f||_{L^p(w)}.$$

2. PROOF OF THEOREMS

To prove the theorems, we need the following lemmas.

Lemma 1. ([1]) *Let T be the strongly singular integral operator. Then T is bounded on $L^p(w)$ for $w \in A_p$ and $1 < p < \infty$.*

Lemma 2. ([1]) *Let $0 < b < 1$, $b' = b/(1-b)$, $1 < p < \infty$, $1/p + 1/p' = 1$, $(2+b')/p \le 1$ and*

$$\tilde{K}(x) = \frac{e^{i|x|^{-b'}}}{|x|^{n(b'+2)/p}}.$$

Then

$$||\tilde{K} * f||_{L^p} \le C||f||_{L^{p'}}.$$

Lemma 3. ([4]) *Suppose that $1 \le r < p < \infty$ and $w \in A_p$. Then*

$$||M_r(f)||_{L^p(w)} \le C||f||_{L^p(w)}.$$

Lemma 4. ([3]) *Suppose that $b_j \in BMO(R^n)(j = 1, ..., m)$, $1 \le r < \infty$, $1 < p < \infty$ and $w \in A_p$. Then*

$$||C_{\tilde{b}}^r(f)||_{L^p(w)} \le C||f||_{L^p(w)}.$$

The proof of Lemma 4 is similar to the proof of Theorem 2.3 and 2.4 in [3], we omit the detail.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned} &\frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f)(x) - C_0|dx \\ \le\ & C\left[M_r(f)(x) + M((C_{\tilde{b}}^r(|f|^r))^{1/r})(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} ||\tilde{b}_\sigma||_{BMO} M_r(T_{\tilde{b}_{\sigma^c}}(f))(\tilde{x})\right]. \end{aligned}$$

Fix a cube $Q = Q(x_0, d)$ such that $\tilde{x} \in Q$. Let d_0 be a positive number satisfying $4d_0 = d_0^{1/(1+b')}$, $\tilde{Q} = Q(x_0, d^{1/(1+b')})$. We first prove the case $m = 1$, we only prove that

$$\frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f)(x) - C_0|dx \le C\left[M_r(f)(x) + M((C_{\tilde{b}}^r(|f|^r))^{1/r})(\tilde{x}) + M_r(T(f))(\tilde{x})\right].$$

Consider the following two cases:

Case 1. $d < d_0$. We write, for $f_1 = f\chi_{4Q}$, $f_2 = f\chi_{\tilde{Q}\setminus 4Q}$ and $f_3 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f)(x) - T_{\tilde{b}}(f_3)(x_0)|\,dx \\
\le\ & \frac{1}{|Q|}\int_Q |b(x) - b_{2Q}||T(f)(x)|dx + \frac{1}{|Q|}\int_Q |T((b-b_{2Q})f_1(x)|dx \\
&+\frac{1}{|Q|}\int_Q |T((b-b_{2Q})f_2(x)|dx + \frac{1}{|Q|}\int_Q |T((b-b_{2Q})f_3(x) - T((b-b_{2Q})f_3(x_0)|dx \\
=\ & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3 and I_4, respectively. First, by the L^r-boundedness of T(Lemma 1), we obtain, by Hölder's inequality,

$$\begin{aligned}
I_1 &\le C\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q} |b(x) - b_{2Q}|^{r'} dx\right)^{1/r'} \\
&\le C||b||_{BMO} M_r(T(f))(\tilde{x});
\end{aligned}$$

For I_2, by the L^p-boundedness of T for $1 < p < r$, we get

$$\begin{aligned}
I_2 &\le C\left(\frac{1}{|Q|}\int_{R^n} |T((b-b_{2Q})f_1(x))(x)|^p dx\right)^{1/p} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n} |b(x) - b_{2Q}||f_1(x))(x)|^p dx\right)^{1/p} \\
&\le C\left(\frac{1}{|2Q|}\int_{2Q} |f(x)|^r dx\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q} |b(x) - b_{2Q}|^{pr/(r-p)} dx\right)^{(r-p)/pr} \\
&\le C||b||_{BMO} M_r(f)(\tilde{x});
\end{aligned}$$

For I_3, following Chanillo's argument(see [1]), assume that $(2+b')/r' < 1$, we write

$$\begin{aligned}
& T((b-b_{2Q})f_2(x) \\
:= & \int_{R^n} \frac{\vartheta(x-y)e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}} \left(\frac{1}{|x-y|^{n(1-(2+b')/r')}} - \frac{1}{|x_0-y|^{n(1-(2+b)/r')}} \right) \\
& \times (b(y)-b_{2Q})f_2(y)dy \\
+ & \int_{R^n} \frac{\vartheta(x-y)e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}} \left(\frac{1}{|x-y|^{n(1-(2+b')/r')}} - \frac{1}{|x_0-y|^{n(1-(2+b')/r')}} \right) \\
& \times (b(y)-b_{2Q})f_2(y)dy \\
= & I_3^{(1)}(x) + I_3^{(2)}(x);
\end{aligned}$$

Note that $|b_{2Q} - b_{2^{k+1}Q}| \le k||b||_{BMO}$. Then

$$\begin{aligned}
& |I_3^{(1)}(x)| \\
\le & C \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(y)-b_{2Q}||f(y)|dy \\
\le & C \sum_{k=1}^{\infty} 2^{-k} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(x)|^r dx \right)^{1/r} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(x)-b_{2Q}|^{r'} dx \right)^{1/r'} \\
\le & C||b||_{BMO} \sum_{k=1}^{\infty} k 2^{-k} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(x)|^r dx \right)^{1/r} \\
\le & C||b||_{BMO} M_r(f)(\tilde{x});
\end{aligned}$$

Taking k_0 such that $2^{k_0} d < d^{1/(1+b')} \le 2^{k_0+1} d$, by Lemma 2 and Minkowski' inequality, we get

$$\begin{aligned}
& \frac{1}{Q} \int_Q |I_3^{(2)}(x)|dx \\
\le & C|Q|^{-1/r'} \left(\int_{R^n} \frac{|b(y)-b_{2Q}|^r |f_2(y)|^r}{|x_0-y|^{nr(1-(2+b')/r')}} dy \right)^{1/r} \\
\le & C|Q|^{-1/r'} \left(\sum_{k=1}^{k_0} (2^k d)^{n(r-1)(1+b')} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|^r |b(y)-b_{2Q}|^r dy \right)^{1/r} \\
\le & C|Q|^{-1/r'} \\
\times & \left(\sum_{k=1}^{k_0} (2^k d)^{n(r-1)(1+b')} \left[\frac{1}{|2Q|} \int_{2Q} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|^r |b(y)-b(z)|^r dy \right)^{1/r} dz \right]^r \right)^{1/r} \\
\le & CM((C_{\tilde{b}}^r(|f|^r))^{1/r})(\tilde{x});
\end{aligned}$$

Thus

$$I_3 \le C\left[||b||_{BMO} M_r(f)(\tilde{x}) + M((C_b^r(|f|^r))^{1/r})(\tilde{x})\right];$$

For I_4, note that $|x-y| \approx |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition of K,

$$\begin{aligned} I_4 &\le C\frac{1}{|Q|}\int_Q\int_{R^n}\frac{|x-x_0|}{|x_0-y|^{n+b'+1}}|f_3(y)||b(y)-b_{2Q}|dydx \\ &\le C\frac{1}{|Q|}\int_Q\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+b'+1}}|f(y)||b(y)-b_{2Q}|dydx \\ &\le C\sum_{k=0}^{\infty}2^{-k}\frac{1}{|2^k\tilde{Q}|}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|f(y)||b(y)-b_{2Q}|dy \\ &\le C\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2Q|}\int_{2Q}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)||b(y)-b(z)|dy\right)dz \\ &\le CM(C_{\tilde{b}}(|f|))(\tilde{x}). \end{aligned}$$

Case 2. $d \ge d_0$. We do not subtract the constant C_0. Let $l = 4d_0^{-1}$ and $f_1 = f\chi_{lQ}$, by the location of the support of K, we have $T_{\tilde{b}}(f) = T_{\tilde{b}}(f_1)$, thus

$$\begin{aligned} \frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f)(x)|\,dx &\le \frac{1}{|Q|}\int_Q|b(x)-b_{lQ}||T(f_1)(x)|dx + \frac{1}{|Q|}\int_Q|T((b-b_{lQ})f_1(x)|dx \\ &= J_1 + J_2; \end{aligned}$$

Similar to the proof of I_1 and I_2 for **Case 1**, we get

$$\begin{aligned} J_1 &\le C\left(\frac{1}{|Q|}\int_Q|T(f_1)(x)|^r dx\right)^{1/r}\left(\frac{1}{|lQ|}\int_Q|b(x)-b_{lQ}|^{r'}dx\right)^{1/r'} \\ &\le C\left(\frac{1}{|lQ|}\int_{lQ}|f(x)|^r dx\right)^{1/r}\left(\frac{1}{|lQ|}\int_Q|b(x)-b_{lQ}|^{r'}dx\right)^{1/r'} \\ &\le C||b||_{BMO}M_r(f)(\tilde{x}); \\ J_2 &\le C\left(\frac{1}{|Q|}\int_{R^n}|T((b-b_{lQ})f_1(x))(x)|^p dx\right)^{1/p} \\ &\le C\left(\frac{1}{|Q|}\int_{R^n}||b(x)-b_{lQ}||f_1(x))(x)|^p dx\right)^{1/p} \\ &\le C\left(\frac{1}{|lQ|}\int_{lQ}|f(x)|^r dx\right)^{1/r}\left(\frac{1}{|lQ|}\int_{lQ}|b(x)-b_{lQ}|^{pr/(r-p)}dx\right)^{(r-p)/pr} \\ &\le C||b||_{BMO}M_r(f)(\tilde{x}), \end{aligned}$$

which proves the case 1.

Now we turn to the case $m \geq 2$. Also consider the following two cases:

Case 1. $d < d_0$. Following [8], we write, for $f_1 = f\chi_{4Q}$, $f_2 = f\chi_{\tilde{Q}\setminus 4Q}$, $f_3 = f\chi_{R^n\setminus\tilde{Q}}$ and $b = (b_1, ..., b_m)$,

$$\begin{aligned}
T_{\tilde{b}}(f)(x) &= \int_{R^n}\left[\prod_{j=1}^{m}(b_j(x)-b_j(y))\right]K(x-y)f(y)dy \\
&= (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f)(x) \\
&\quad +(-1)^m T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x) \\
&\quad +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-b(x))_{\sigma^c}K(x-y)f(y)dy \\
&= (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f)(x) \\
&\quad +(-1)^m T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x) \\
&\quad +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma T_{\tilde{b}_{\sigma^c}}(f)(x),
\end{aligned}$$

thus

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q|T_{\tilde{b}}(f)(x)-T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_3)(x_0)|dx \\
\leq\ &\frac{1}{|Q|}\int_Q|(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f)(x)|dx \\
&+\frac{1}{|Q|}\int_Q\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|(b(x)-(b)_{2Q})_\sigma T_{\tilde{b}_{\sigma^c}}(f)(x)|dx \\
&+\frac{1}{|Q|}\int_Q|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|dx \\
&+\frac{1}{|Q|}\int_Q|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)|dx \\
&+\frac{1}{|Q|}\int_Q|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_3)(x) \\
&\qquad -T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_3)(x_0)|dx \\
=\ &L_1+L_2+L_3+L_4+L_5;
\end{aligned}$$

Similar to the proof of $m = 1$, we get, for $1 < p_1, \cdots, p_m < \infty$, $1 < p < r$, $1 < q_1, \cdots, q_m < \infty$, $1/r + 1/p_1 + \cdots + 1/p_m = 1$ and $p/r + 1/q_1 + \cdots + 1/q_m = 1$,

$$
\begin{aligned}
L_1 &\le C\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r}\left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|^{p_1}dx\right)^{1/p_1}\\
&\quad \times\cdots\times\left(\frac{1}{|2Q|}\int_{2Q}|b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m}\\
&\le C\prod_{j=1}^m ||b_j||_{BMO}M_r(T(f))(\tilde{x});\\
L_2 &\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\tilde{b}_\sigma||_{BMO}M_r(T_{\tilde{b}_{\sigma^c}}(f))(\tilde{x});\\
L_3 &\le C\left(\frac{1}{|Q|}\int_{R^n}|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1(x))(x)|^p dx\right)^{1/p}\\
&\le C\left(\frac{1}{|Q|}\int_{R^n}(|b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||f_1(x)|)^p dx\right)^{1/p}\\
&\le C\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{pq_1}dx\right)^{1/pq_1}\\
&\quad \times\cdots\times\left(\frac{1}{|2Q|}\int_{2Q}|b_m(x)-(b_m)_{2Q}|^{pq_m}dx\right)^{1/pq_m}\\
&\le C\prod_{j=1}^m ||b_j||_{BMO}M_r(f)(\tilde{x});
\end{aligned}
$$

Similarly, for L_4, we get, for $1 < p_1,\cdots,p_m < \infty$ and $1/r + 1/p_1 + \cdots + 1/p_m = 1$,

$$
\begin{aligned}
L_4 &\le \frac{1}{Q}\int_Q\left|\int_{R^n}\frac{\vartheta(x-y)e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}}\left(\frac{1}{|x-y|^{n(1-(2+b')/r')}}-\frac{1}{|x_0-y|^{n(1-(2+b)/r')}}\right)\right|\\
&\quad \times|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_2(y)|dydx\\
&\quad +\frac{1}{Q}\int_Q\left|\int_{R^n}\frac{\vartheta(x-y)e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}}\left(\frac{1}{|x-y|^{n(1-(2+b')/r')}}-\frac{1}{|x_0-y|^{n(1-(2+b')/r')}}\right)\right|\\
&\quad \times|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_2(y)|dydx\\
&\le C\sum_{k=1}^\infty 2^{-k}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\prod_{j=1}^m|b_j(y)-(b_j)_{2Q}||f(y)|dy\\
&\quad +C|Q|^{-1/r'}\left(\int_{R^n}\frac{\prod_{j=1}^m|b_j(y)-(b_j)_{2Q}|^r|f_2(y)|^r}{|x_0-y|^{nr(1-(2+b')/r')}}dy\right)^{1/r}
\end{aligned}
$$

$$\leq C\sum_{k=1}^{\infty}2^{-k}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(x)|^r dx\right)^{1/r}$$
$$\times\prod_{j=1}^{m}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_j(x)-(b_j)_{2Q}|^{p_j'}dx\right)^{1/p_j'}$$
$$+C|Q|^{-1/r'}\left(\sum_{k=1}^{k_0}(2^k d)^{n(r-1)(1+b')}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}|^r dy\right)^{1/r}$$
$$\leq C\prod_{j=1}^{m}||b_j||_{BMO}\sum_{k=1}^{\infty}k^m 2^{-k}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(x)|^r dx\right)^{1/r}$$
$$+C|Q|^{-1/r'}\left(\sum_{k=1}^{k_0}(2^k d)^{n(r-1)(1+b')}\right)^{1/r}$$
$$\times\left(\left[\frac{1}{|2Q|}\int_{2Q}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r\prod_{j=1}^{m}|b_j(y)-(b_j)(z)|^r dy\right)^{1/r}dz\right]^r\right)^{1/r}$$
$$\leq C\left(\prod_{j=1}^{m}||b_j||_{BMO}M_r(f)(\tilde{x})+M((C^r_{\tilde{b}}(|f|^r))^{1/r})(\tilde{x})\right);$$

For L_5, we get

$$\begin{aligned}
L_5 &\leq C\frac{1}{|Q|}\int_Q\int_{R^n}\frac{|x-x_0|}{|x_0-y|^{n+b'+1}}|f_3(y)|\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}|dydx\\
&\leq C\frac{1}{|Q|}\int_Q\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+b'+1}}|f(y)|\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}|dydx\\
&\leq C\sum_{k=0}^{\infty}2^{-k}\frac{1}{|2^k\tilde{Q}|}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|f(y)|\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}|dy\\
&\leq C\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2Q|}\int_{2Q}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|\prod_{j=1}^{m}|b_j(y)-b_j(z)|dy\right)dz\\
&\leq CM(C_{\tilde{b}}(|f|))(\tilde{x}).
\end{aligned}$$

Similarly, for

Case 2. $d\geq d_0$, we get

$$\frac{1}{|Q|}\int_Q|T_{\tilde{b}}(f)(x)|\,dx\leq C\prod_{j=1}^{m}||b_j||_{BMO}M_r(f)(\tilde{x})+C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\tilde{b}_\sigma||_{BMO}M_r(T_{\tilde{b}_{\sigma^c}}(f))(\tilde{x}).$$

These complete the proof of Theorem 1. □

Proof of Theorem 2. We choose $1 < r < p$ in Theorem 1, and by using Lemma 3, 4 and induction on m, we get

$$||T_{\tilde{b}}(f)||_{L^p(w)} \leq C||(T_{\tilde{b}}(f))^{\#}||_{L^p(w)} \leq C||f||_{L^p(w)}.$$

This finishes the proof. □

References

[1] S. Chanillo, *Weighted norm inequalities for strongly singular convolution operators*, Trans. Ammer. Math. Soc., **281** (1984), 77-107.

[2] C. Feffeman and E. M. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137-193.

[3] J. Garcia-Cuerva, E. Harboure, C. Segovia and J. L. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J., **40** (1991), 1397-1420.

[4] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math.**16**, Amsterdam, 1985.

[5] C. Pérez, *Endpoint estimate for commutators of singular integral operators*, J. Func. Anal., **128** (1995), 163-185.

[6] C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, J. Fourier Anal. Appl., **3** (1997), 743-756.

[7] C. Pérez and G. Pradolini, *Sharp weighted endpoint estimates for commutators of singular integral operators*, Michigan Math. J., **49** (2001), 23-37.

[8] C. Pérez and R. Trujillo-Gonzalez, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc., **65** (2002), 672-692.

[9] E. M. Stein, *Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[10] S. Wainger, *Special trigonometric series in k-dimensions*, Mem. Amer. Math. Soc., **59** (1965).