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WEIGHTED BOUNDEDNESS FOR MULTILINEAR MARCINKIEWICZ OPERATORS ON SOME HARDY SPACES

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Abstract. In this paper, the weighted boundedness for the multilinear Marcinkiewicz operators on certain Hardy and Hardy-Block spaces are obtained.

1. INTRODUCTION AND DEFINITIONS

Fix $\lambda > 1$. Suppose that S^{n-1} is the unit sphere of $R^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on $S^{n-1} (0 < \gamma \leq 1)$, i.e.

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} S(x')dx' = 0$.

Let m be a positive integer and A be a function on R^n. We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}(y,t)$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

and

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [10]).

Note that when $m = 0$, μ_λ^A is just the commutator of Marcinkiewicz operator (see [10]). It is well known that multilinear operators, as a non-trivial extension of commutator, are of great interest in harmonic analysis and have been widely studied by many authors (see [2, 3, 4, 5]). The main purpose of this paper is to study the weighted boundedness of the multilinear Marcinkiewicz operators on certain Hardy and Hardy-Block spaces. Let us first introduce some definitions (see [1, 2, 3, 4, 5, 6]).

Throughout this paper, B will denote a ball of R^n. For a locally integrable function f, set $f_B = |B|^{-1} \int_B f(x)dx$.

Definition 1. *Let A, w be the functions on R^n, $w \in A_1$ (that is $M(w)(x) \le cw(x)$ a.e.) and m be a positive integer. A bounded measurable function a on R^n is said to be a (w, D^mA)-atom if*

i) supp $a \subset B = B(x_0, r)$,

ii) $||a||_{L^\infty(w)} \le w(B)^{-1}$, *iii)* $\int_{R^n} a(y)dy = \int_{R^n} a(y)D^\alpha A(y)dy = 0$ *for* $|\alpha| = m$;

A temperate distribution f *is said to belong to* $H^1_{D^mA}(w)$, *if, in the Schwartz distributional sense, it can be written as*

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where a_j*'s are* (w, D^mA) *atoms,* $\lambda_j \in C$ *and* $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. *Moreover,* $||f||_{H^1_{D^mA}} \sum_{j=1}^{\infty} |\lambda_j|$.

Definition 2. *Let* $w \in A_1$. *A function* f *is said to be in the weighted Block* H^1 *space* $H^1_B(w)$ *if* $f(x)$ *can be written as*

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where a_j'*s are the* w*-atoms, that is all* a_j'*s satisfy conditions i) ii) in Definition 1 and the following condition*

iii)' $\int_{R^n} a(y)dy = 0$ *and* $\lambda_j \in C$ *with*

$$\sum_{j=1}^{\infty} |\lambda_j| \left(1 + \log^+ \frac{1}{|\lambda_j|}\right) < \infty,$$

moreover, $||f||_{H^1_B(w)} \sim \sum_{j=1}^{\infty} |\lambda_j|(1 + \log^+((\sum_i |\lambda_i|)/|\lambda_j|))$.

2. THEOREMS and PROOFS

We begin with some preliminary lemmas.

Lemma 1. [4] *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x - y|$.

Lemma 2. [8] *Let $w \ge 0$ and $\{g_k\}$ be a sequence of measurable functions satisfying*

$$||g_k||_{L^{1,\infty}_w} \le 1.$$

Then, for every numerical sequence $\{\lambda_k\}$, we have

$$||\sum_k \lambda_k g_k||_{L^{1,\infty}_w} \le C\sum_k |\lambda_k|(1+\log(\sum_j |\lambda_j|/|\lambda_k|)).$$

Lemma 3. *Let $w \in A_1$, $1 < p < \infty$ and $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then μ_λ^A is bounded on $L^p(w)$, that is*

$$||\mu_\lambda^A(f)||_{L^p(w)} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^p(w)}.$$

Proof. Note that $|x-z| \le 2t$, $|y-z| \ge |x-z|-t \ge |x-z|-3t$ when $|x-y| \le t$, $|y-z| \le t$, and $|x-z| \le t(1+2^{k+1}) \le 2^{k+2}t$, $|y-z| \ge |x-z| - 2^{k+3}t$ when $|x-y| \le 2^{k+1}t$, $|y-z| \le t$, we have, by Minkowski inequality,

$$\mu_\lambda^A(f)(x)$$

$$\le \int_{R^n}\left[\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-\delta-1}|x-z|^m}\right)^2 \chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{1/2}dz$$

$$\le C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\left[\int_0^\infty\int_{|x-y|\le t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-3t)^{2n-2}}\frac{dydt}{t^{n+3}}\right]^{1/2}dz$$

$$\le C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}$$

$$\times\left[\int_0^\infty\sum_{k=0}^\infty\int_{2^kt<|x-y|\le 2^{k+1}t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-2^{k+3}t)^{2n-2}}\frac{dydt}{t^{n+3}}\right]^{1/2}dz$$

$$\le C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\left[\int_{|x-z|/2}^\infty\frac{dt}{(|x-z|-3t)^{2n}}\right]^{1/2}dz$$

$$+C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\left[\sum_{k=0}^\infty\int_{2^{-2-k}|x-z|}^\infty 2^{-kn\lambda}(2^kt)^n t^{-n}\frac{2^kdt}{(|x-z|-2^{k+3}t)^{2n}}\right]^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n}}dz+C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n}}dz\left[\sum_{k=0}^{\infty}2^{kn(1-\lambda)}\right]^{1/2}$$

$$=C\int_{R^n}\frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n}}|f(z)|dz,$$

thus, the lemma follows from [5]. □

Theorem 1. *Let $w \in A_1$, $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then μ_λ^A is bounded from $H^1_{D^mA}(w)$ to $L^1(w)$.*

Proof. By a standard argument, it suffices to show that there exists a constant $C > 0$ such that for every (w, D^mA) atom a,

$$||\mu_\lambda^A(a)||_{L^1(w)} \leq C.$$

Let a be a (w, D^mA) atom supported on a ball $B = B(x_0, r)$. We write

$$\begin{aligned}&\int_{R^n}\mu_\lambda^A(a)(x)w(x)dx\\ =\ &\int_{|x-x_0|\leq 2r}\mu_\lambda^A(a)(x)w(x)dx+\int_{|x-x_0|>2r}\mu_\lambda^A(a)(x)w(x)dx\\ =\ &I+II.\end{aligned}$$

For I, taking $p > 1$, by Hölder's inequality and the $L^p(w)$-boundedness of μ_λ^A(see Lemma 3), we get

$$I \leq C||\mu_\lambda^A(a)||_{L^p(w)}w(2B)^{1-1/p} \leq C||a||_{L^p(w)}w(B)^{1-1/p} \leq C.$$

To obtain the estimate of II, we need to estimate $\mu_S^A(a)(x)$ for $x \in (2B)^c$. Let $\tilde{B} = 5\sqrt{n}B$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{B}}\cdot x^\alpha$. Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A} = D^\alpha A - (D^\alpha A)_Q$ for all α with $|\alpha| = m$. We write, by the vanishing moment of a,

$$\begin{aligned}F_t^A(a)(x,y)\ =\ &\int_{R^n}\left[\frac{\chi_{\Gamma(z)}(y,t)\Omega(y-z)}{|y-z|^{n-1}|x-z|^m}-\frac{\chi_{\Gamma(x_0)}(y,t)\Omega(y-x_0)}{|y-x_0|^{n-1}|x-x_0|^m}\right]R_m(\tilde{A};x,z)a(z)dz\\ &+\int_{R^n}\frac{\chi_{\Gamma(x_0)}(y,t)\Omega(y-x_0)}{|y-x_0|^{n-1}|x-x_0|^m}[R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)]a(z)dz\\ &-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{\chi_{\Gamma(z)}(y,t)\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m}D^\alpha\tilde{A}(z)a(z)dz,\end{aligned}$$

then

$$\mu_\lambda^A(a)(x)$$

$$\leq \Big[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda}$$

$$\times \left(\int_{R^n} \left|\frac{\chi_{\Gamma(z)}(y,t)\Omega(y-z)}{|y-z|^{n-1}|x-z|^m} - \frac{\chi_{\Gamma(z)}(y,t)\Omega(y-x_0)}{|y-x_0|^{n-1}|x-x_0|^m}\right| |R_m(\tilde{A};x,z)||a(z)|dz\right)^2 \frac{dydt}{t^{n+3}}\Big)^{1/2}$$

$$+\Big[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda}$$

$$\times \left(\int_{R^n} |\chi_{\Gamma(z)}(y,t) - \chi_{\Gamma(x_0)}(y,t)| \frac{|\Omega(y-x_0)|}{|y-x_0|^{n-1}|x-x_0|^m} |R_m(\tilde{A};x,z)||a(z)|dz\right)^2 \frac{dydt}{t^{n+3}}\Big]^{1/2}$$

$$+\Big[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda}$$

$$\times \left(\int_{R^n} \frac{\chi_{\Gamma(x_0)}(y,t)\Omega(y-x_0)}{|y-x_0|^{n-1}|x-x_0|^m} |R_m(\tilde{A};x,z) - R_m(\tilde{A};x,x_0)||a(z)|dz\right)^2 \frac{dydt}{t^{n+3}}\Big]^{1/2}$$

$$+\left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda}\right.$$

$$\left.\times \left|\sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{\chi_{\Gamma(z)}(y,t)\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m} D^\alpha A(z)a(z)dz\right|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

$$= II_1 + II_2 + II_3 + II_4.$$

By Lemma 1, we know, for $z \in B$ and $x \in 2^{k+1}B \setminus 2^k B$,

$$|R_m(\tilde{A};x,z)| \leq Ck|x-z|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO};$$

By the condition of Ω, we get that $|\Omega| \leq C$ and (see [10])

$$\left|\frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-1}}\right| \leq C\left(\frac{|z-x_0|}{|y-x_0|^n} + \frac{|z-x_0|^\gamma}{|y-x_0|^{n-1+\gamma}}\right),$$

noting that $|x-z| \sim |x-x_0|$ for $z \in B$ and $x \in R^n \setminus 2B$, similar to the proof of

Lemma 3, we obtain

$$\begin{aligned} II_1 &\leq C\int_B \left[\frac{r}{|x-x_0|^{m+n+1}}+\frac{r^\gamma}{|x-x_0|^{m+n+\gamma}}\right]|R_m(\tilde{A};x,z)||a(z)|dz \\ &\leq C\left(\frac{r}{|x-x_0|^{n+1}}+\frac{r^\gamma}{|x-x_0|^{n+\gamma}}\right)k|B|w(B)^{-1}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}; \end{aligned}$$

For II_2, similar to the proof of II_1 and Lemma 3, we obtain

$$\begin{aligned} &II_2 \\ &\leq C\int_B \frac{|R_m(\tilde{A};x,z)||a(z)|}{|x-x_0|^m}\left(\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|\chi_{\Gamma(z)}-\chi_{\Gamma(x_0)}|^2\frac{t^{-n-3}dydt}{|y-x_0|^{2n-2}}\right)^{1/2}dz \\ &\leq C\int_B \frac{|R_m(\tilde{A};x,z)||a(z)|}{|x-x_0|^m} \\ &\times\left|\int\int_{\Gamma(z)}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{t^{-n-3}dydt}{|y-x_0|^{2n-2}}-\int\int_{\Gamma(x_0)}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{t^{-n-3}dydt}{|y-x_0|^{2n-2}}\right|^{1/2}dz \\ &\leq C\int_B \frac{|R_m(\tilde{A};x,z)||a(z)|}{|x-x_0|^m}\left(\int\int_{|x-y|\leq t,|y|\leq t}\frac{|z-x_0|t^{-n-3}dydt}{|y+z-x_0|^{2n-1}}\right)^{1/2}dz \\ &\leq C\int_B \frac{|R_m(\tilde{A};x,z)||a(z)||z-x_0|^{1/2}}{|x-x_0|^{m+n+1/2}}dz \\ &\leq C\frac{r^{1/2}}{|x-x_0|^{n+1/2}}k|B|w(B)^{-1}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}; \end{aligned}$$

For II_3, by the formula (see [3]):

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};z,x_0)(x-x_0)^\beta$$

and Lemma 1, we get

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)|\leq C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}||D^\alpha A||_{BMO},$$

so that

$$II_3 \leq C\int_B |x-x_0|^{-(n+m)}\sum_{|\beta|<m}\left|R_{m-|\beta|}(D^\beta\tilde{A};z,x_0)\right||x-x_0|^{|\beta|}|a(z)|dz$$

$$\leq \; C\int_B |x-x_0|^{-(n+m)}\sum_{|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|a(z)|dz$$
$$\leq \; C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_B \frac{|x_0-z|}{|x-x_0|^{n+1}}|a(z)|dz$$
$$\leq \; C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0|^{-n-1}|B|^{1/n+1}w(B)^{-1};$$

For II_4 , we write, by the vanishing moment of a,

$$\int_{R^n}\frac{\chi_{\Gamma(z)}(y,t)\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m}D^\alpha\tilde{A}(z)a(z)dz$$
$$= \int_{R^n}\left[\frac{\chi_{\Gamma(z)}(y,t)\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m}-\frac{\chi_{\Gamma(x_0)}(y,t)\Omega(y-x_0)(x-x_0)^\alpha}{|y-x_0|^{n-1}|x-x_0|^m}\right]D^\alpha\tilde{A}(z)a(z)dz$$
$$= \int_{R^n}\frac{\chi_{\Gamma(z)}(y,t)(x-z)^\alpha}{|x-z|^m}\left[\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-1}}\right]D^\alpha\tilde{A}(z)a(z)dz$$
$$+\int_{R^n}\frac{\chi_{\Gamma(z)}(y,t)\Omega(y-x_0)}{|y-x_0|^{n-1}}\left[\frac{(x-z)^\alpha}{|x-z|^m}-\frac{(x-x_0)^\alpha}{|x-x_0|^m}\right]D^\alpha\tilde{A}(z)a(z)dz$$
$$+\int_{R^n}(\chi_{\Gamma(z)}(y,t)-\chi_{\Gamma(x_0)}(y,t))\frac{\Omega(y-x_0)(x-x_0)^\alpha}{|y-x_0|^{n-1}|x-x_0|^m}D^\alpha\tilde{A}(z)a(z)dz,$$

similar to the estimates of $II_1$and II_2, we obtain

$$II_4$$
$$\leq C\sum_{|\alpha|=m}\left(\frac{r}{|x-x_0|^{n+1}}+\frac{r^\gamma}{|x-x_0|^{n+\gamma}}+\frac{r^{1/2}}{|x-x_0|^{n+1/2}}\right)\int_B|D^\alpha A(z)-(D^\alpha A)_B||a(z)|dz$$
$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|B|w(B)^{-1}\left(\frac{r}{|x-x_0|^{n+1}}+\frac{r^\gamma}{|x-x_0|^{n+\gamma}}+\frac{r^{1/2}}{|x-x_0|^{n+1/2}}\right).$$

Therefore, note that if $w\in A_1$, then

$$\frac{w(B_2)}{|B_2|}\frac{|B_1|}{w(B_1)}\leq C$$

for all balls B_1, B_2 with $B_1\subset B_2$(see [7]), we obtain

$$II \;\leq\; \sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\mu_\lambda^A(a)(x)w(x)dx$$
$$\leq\; C\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}k$$

$$
\begin{aligned}
&\times\left(\frac{|B|^{1+1/n}}{|x-x_0|^{n+1}}+\frac{|B|^{1+1/2n}}{|x-x_0|^{n+1/2}}+\frac{|B|^{1+\gamma/n}}{|x-x_0|^{n+\gamma}}\right)\frac{w(x)}{w(B)}dx\\
\leq\ & C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}k\left(2^{-k}+2^{-k/2}+2^{-k\gamma}\right)\frac{|B|}{w(B)}\frac{w(2^{k+1}B)}{|2^{k+1}B|}\\
\leq\ & C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}k\left(2^{-k}+2^{-k/2}+2^{-k\gamma}\right)\\
\leq\ & C,
\end{aligned}
$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1. □

Theorem 2. *Let $w\in A_1$ and $D^\alpha A\in BMO(R^n)$ for $|\alpha|=m$. Then μ_λ^A is bounded $H_B^1(w)$ to $L_w^{1,\infty}(R^n)$.*

Proof. By Lemma 2, it suffices to show that there exists a constant $c>0$ such that for every w-atom a,

$$||\mu_\lambda^A(a)||_{L^{1,\infty}(w)}\leq C.$$

Let a be a w-atom supported on a ball $B=B(x_0,r)$. We write, for $\eta>0$,

$$
\begin{aligned}
&w(\{x\in R^n:\mu_\lambda^A(a)(x)>\eta\})\leq w(\{x\in 2B:\mu_\lambda^A(a)(x)>\eta\})\\
&+w(\{x\in(2B)^c:\mu_\lambda^A(a)(x)>\eta\})=J+JJ.
\end{aligned}
$$

For J, by the $L^p(w)$-boundedness of μ_λ^A for $p>1$(see Lemma 3), we obtain

$$
\begin{aligned}
J\ &\leq\ \eta^{-1}||\mu_\lambda^A(a)||_{L(w)^1}\leq C\eta^{-1}||\mu_\lambda^A(a)||_{L^p(w)}w(B)^{1-1/p}\\
&\leq\ C\eta^{-1}||a||_{L^p(w)}w(B)^{1-1/p}\leq C\eta^{-1};
\end{aligned}
$$

For JJ, similar to the proof of Theorem 1, we obtain, for $x\in(2B)^c$,

$$\mu_\lambda^A(a)(x)\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}k\left(\frac{|B|^{1+1/n}}{|x-x_0|^{n+1}}+\frac{|B|^{1+1/2n}}{|x-x_0|^{n+1/2}}+\frac{|B|^{1+\gamma/n}}{|x-x_0|^{n+\gamma}}\right)w(B)^{-1},$$

thus

$$JJ\ \leq\ \eta^{-1}\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\mu_\lambda^A(a)(x)w(x)dx$$

$$
\begin{aligned}
&\leq C\eta^{-1}\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}k \\
&\quad\times\left(\frac{|B|^{1+1/n}}{|x-x_0|^{n+1}}+\frac{|B|^{1+1/2n}}{|x-x_0|^{n+1/2}}+\frac{|B|^{1+\gamma/n}}{|x-x_0|^{n+\gamma}}\right)\frac{w(x)}{w(B)}dx \\
&\leq C\eta^{-1}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}k\left(2^{-k}+2^{-k/2}+2^{-k\gamma}\right)\frac{|B|}{w(B)}\frac{w(2^{k+1}B)}{|2^{k+1}B|} \\
&\leq C\eta^{-1}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}k\left(2^{-k}+2^{-k/2}+2^{-k\gamma}\right) \\
&\leq C\eta^{-1}.
\end{aligned}
$$

Combining the estimates of J and JJ, we obtain

$$
w(\{x\in R^n:\mu_\lambda^A(a)(x)>\eta\})\leq C\eta^{-1}.
$$

This complete the proof of Theorem 2. □

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