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THE FIFTH AND SIXTH COEFFICIENTS OF α-CLOSE-TO-CONVEX FUNCTIONS

Kunle Oladeji Babalola

Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, Nigeria
(e-mail: khayrah.babalola@gmail.com)

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Abstract. We obtain sharp bounds on the fifth and sixth coefficients of α-close-to-convex functions introduced in [2].

1. INTRODUCTION

Let A be the class of functions of the form:

$$f(z) = z + a_2 z^2 + ...$$

which are analytic in the unit disk $U = \{z \in \mathbf{C}\colon |\mathbf{z}| < \mathbf{1}\}$. In [2], Chichra considered functions $f \in A$ for which $f(z)f'(z)/z \neq 0$ for $z \in U$, and if for some nonnegative real number α, there exists a starlike function:

$$\phi(z) = z + b_2 z^2 + ...$$

such that

$$Re\left\{(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right\} > 0.$$

The above geometric condition implies that the term in the curly brackets belongs to the class P of analytic functions:

$$p(z) = 1 + p_1 z + ...$$

which have positive real part in U. He called functions of this type α-close-to-convex and denoted the class by C_α. Interestingly, the family of functions includes two well known ones. These are the classes of close-to-convex and convex functions, which respectively correspond to C_0 and C_∞ (see [2]). Among others, he proved the following inequalities:

Theorem 1 [2]. *Let $f(z) \in C_\alpha$. Then*

$$|a_2| \leq \frac{2+\alpha}{1+\alpha};$$

$$|a_3| \leq \frac{9+23\alpha+6\alpha^2}{3(1+\alpha)(1+2\alpha)};$$

$$|a_4| \leq \frac{4+22\alpha+34\alpha^2+6\alpha^3}{4(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

The inequalities are sharp.

He remarked that his method might not be easily employed to obtain bounds on higher coefficients in this family of functions. Apart from his efforts, the present author is not aware of any further development on the higher coefficients of this important family of functions. And as it is well known, the coefficient problem in univalent functions theory is ever demanding attention. In this note we visit the old problem of Chichra and obtain the best possible upper bounds for the fifth and sixth coefficients of functions of the class C_α. Our results are the following inequalities:

Theorem 2. *Let $f(z) \in C_\alpha$. Then*

$$|a_5| \leq \frac{25+238\alpha+755\alpha^2+902\alpha^3+120\alpha^4}{5(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)};$$

$$|a_6| \leq \frac{6+83\alpha+418\alpha^2+951\alpha^3+955\alpha^4+120\alpha^5}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)}.$$

The inequalities are sharp. Equalities in both cases are attained for the function

$$f(z) = \begin{cases} \displaystyle\int_0^z \frac{1}{\alpha} t^{-1/\alpha}(1-t)^{2(1/\alpha-1)} \left(\int_0^t \frac{\xi^{1/\alpha-1}(1+\xi)^2}{(1-\xi)^{2(1/\alpha+1)}} d\xi \right) dt, & \text{if } \alpha \neq 0; \\ \displaystyle\int_0^z \frac{1+t}{(1-t)^3} dt, & \text{if } \alpha = 0. \end{cases}$$

In our proof, which is presented in the next section, we shall depend on the well known inequalities (Caratheodory lemma and coefficient functionals for starlike functions):

$$|p_n| \le 2, \ \ n \ge 1; \qquad |b_n| \le n, \ \ n \ge 2; \qquad |b_3 - \lambda b_2^2| \le 3 - 4\lambda, \ \ \lambda \le \frac{3}{4} \tag{1}$$

(λ real) and the following lemma which concerns bounds on certain other functionals in the coefficient space of the family of starlike functions.

Lemma 1 [1]. *For every starlike function $\phi(z)$ and real numbers μ, ρ, σ, τ and ω, we have the sharp inequalities:*

$$|b_4 - \mu b_2 b_3| \le 4 - 6\mu; \ \ \mu \le \frac{5}{9},$$

$$|b_4 - \mu b_2 b_3 - \rho b_2^3| \le 4 - 6\mu - 8\rho; \ \ 5 - 9\mu - 12\rho \ge 0,$$

$$|b_5 - \sigma b_2^2 b_3| \le 5 - 12\sigma; \ \ \sigma \le \frac{2}{9},$$

$$|b_5 - \tau b_2 b_4 - \omega b_3^2| \le 5 - 8\tau - 9\omega; \ \ 2 - 5\tau - 9\omega \ge 0.$$

The proof of the above lemma, which was presented in [1], made use of the equality $p_2 = \frac{1}{2}p_1^2 + \varepsilon(2 - \frac{1}{2}|p_1|^2)$, $|\varepsilon| \le 1$, which is a consequence of the well known Caratheodory-Toeplitz inequality $|p_2 - \frac{1}{2}p_1^2| \le 2 - \frac{1}{2}|p_1|^2$. The extremal function is the Koebe function (up to rotation): $k(z) = z/(1-z)^2$.

Now the proof of the main result.

2. PROOF OF THE MAIN RESULT

Let $f \in C_\alpha$. Then there exists an analytic function $p \in P$ and a starlike function $\phi(z)$ such that

$$(1-\alpha)zf'(z)\phi'(z) + \alpha\phi(z)(zf'(z))' = p(z)\phi(z)\phi'(z).$$

The right hand side of the above equation gives

$$p(z)\phi(z)\phi'(z) = z + c_2z^2 + ...$$

where

$$c_n = \sum_{k=0}^{n-1} q_{k+1}b_{n-k}; \quad n \geq 2,$$

and

$$q_k = \sum_{j=0}^{k-1}(k-j)p_j b_{k-j}; \quad (p_0 = b_1 = q_1 = 1). \tag{2}$$

Similarly the left hand side yields

$$(1-\alpha)zf'(z)\phi'(z) + \alpha\phi(z)(zf'(z))' = z + d_2z^2 + ...$$

where

$$d_n = \sum_{k=0}^{n-1}(n-k)[(n-2k-1)\alpha + k + 1]b_{k+1}a_{n-k}; \quad n \geq 2.$$

Comparing the coefficients c_n and d_n we obtain the recurrence relation:

$$n[1+(n-1)\alpha]a_n = \sum_{k=0}^{n-1} b_{n-k}q_{k+1} - \sum_{k=1}^{n-1}(n-k)[(n-2k-1)\alpha + k + 1]b_{k+1}a_{n-k},$$

with $q_1 = 1$, $a_1 = b_1 = 1$ and $n \geq 2$. Thus we have

$$2(1+\alpha)a_2 = (\alpha - 1)b_2 + q_2, \tag{3}$$

$$3(1+2\alpha)a_3 = 2(\alpha-1)b_3 + b_2q_2 + q_3 - 4a_2b_2, \tag{4}$$

$$4(1+3\alpha)a_4 = 3(\alpha-1)b_4 + b_3q_2 + b_2q_3 + q_4 + 2(3-\alpha)a_2b_3 - 3(2+\alpha)a_3b_2 \tag{5}$$

$$5(1+4\alpha)a_5 = 4(\alpha-1)b_5 + b_4q_2 + b_3q_3 + b_2q_4 + q_5 + 4(\alpha-2)a_2b_4 - 9a_3b_3 - 8(1+\alpha)a_4b_2, \tag{6}$$

$$\begin{aligned} 6(1+5\alpha)a_6 &= 5(\alpha-1)b_6 + b_5q_2 + b_4q_3 + b_3q_4 + b_2q_5 + q_6 \\ &+ 2(3\alpha-5)a_2b_5 + 3(\alpha-4)a_3b_4 - 4(3+\alpha)a_4b_3 - 5(2+3\alpha)a_5b_2, \quad (7) \end{aligned}$$

and so on, and where from (2), $q_2 = 2b_2 + p_1$, $q_3 = 3b_3 + 2b_2p_1 + p_2$, $q_4 = 4b_4 + 3b_3p_1 + 2b_2p_2 + p_3$, $q_5 = 5b_5 + 4b_4p_1 + 3b_3p_2 + 2b_2p_3 + p_4$ and $q_6 = 6b_6 + 5b_5p_1 + 4b_4p_2 + 3b_3p_3 + 2b_2p_4 + p_5$. Using these in (3) - (7) we get

$$2(1+\alpha)a_2 = (1+\alpha)b_2 + p_1, \quad (8)$$

$$3(1+\alpha)(1+2\alpha)a_3 = (1+\alpha)(1+2\alpha)b_3 + (1+3\alpha)b_2p_1 + (1+\alpha)p_2, \quad (9)$$

$$\begin{aligned} 4(1+\alpha)(1+2\alpha)(1+3\alpha)a_4 &= (1+\alpha)(1+2\alpha)(1+3\alpha)b_4 \\ &+ (1+2\alpha)(1+5\alpha)b_3p_1 + (1+\alpha)(1+5\alpha)b_2p_2 \\ &+ (1+\alpha)(1+2\alpha)p_3 + \alpha(\alpha-1)b_2^2p_1, \quad (10) \end{aligned}$$

$$5A_0a_5 = A_0b_5 + A_1b_4p_1 + A_2b_3p_2 + A_3b_2p_3 + A_4p_4 + A_5b_2b_3p_1 + A_6b_2^2p_2 + A_7b_2^3p_1, \quad (11)$$

where $A_0 = (1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)$, $A_1 = (1+2\alpha)(1+3\alpha)(1+7\alpha)$, $A_2 = (1+\alpha)(1+3\alpha)(1+8\alpha)$, $A_3 = (1+\alpha)(1+2\alpha)(1+7\alpha)$, $A_4 = (1+\alpha)(1+2\alpha)(1+3\alpha)$, $A_5 = 2\alpha(\alpha-1)(2+5\alpha)$, $A_6 = 2\alpha(\alpha^2-1)$ and $A_7 = -A_6$, and

$$\begin{aligned} 6B_0a_6 &= B_0b_6 + B_1b_5p_1 + B_2b_4p_2 + B_3b_3p_3 + B_4b_2p_4 + B_5p_5 + B_6b_4b_2p_1 \\ &+ B_7b_3^2p_1 + B_8b_2^2b_3p_1 + B_9b_2^4p_1 + B_{10}b_3b_2p_2 + B_{11}b_2^3p_2 + B_{12}b_2^2p_3, \quad (12) \end{aligned}$$

with $B_0 = (1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)$, $B_1 = (1+2\alpha)(1+3\alpha)(1+4\alpha)(1+9\alpha)$, $B_2 = (1+\alpha)(1+3\alpha)(1+4\alpha)(1+11\alpha)$, $B_3 = (1+\alpha)(1+2\alpha)(1+4\alpha)(1+11\alpha)$, $B_4 = (1+\alpha)(1+2\alpha)(1+3\alpha)(1+9\alpha)$, $B_5 = (1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)$, $B_6 = 6\alpha(\alpha-1)(1+3\alpha)^2$, $B_7 = 4\alpha(\alpha-1)(1+2\alpha)(1+4\alpha)$, $B_8 = \alpha(1-\alpha)(11+45\alpha+34\alpha^2)$, $B_9 = 2\alpha(\alpha^2-1)(2+3\alpha)$, $B_{10} = -10\alpha - 53\alpha^2 - 73\alpha^3 - 44\alpha^4$, $B_{11} = -B_9$ and $B_{12} = 3\alpha(\alpha^2-1)(1+2\alpha)$.

From (8) - (10) we can obtain the inequalities of Chichra (Theorem 1) using the inequalities (1).

Now for $n = 5$, if $\alpha \in [1, \infty)$, we define $\lambda_1 = -A_6/A_2$ and $\lambda_2 = -A_7/A_5$ and then write (11) as

$$5A_0a_5 = A_0b_5 + A_1b_4p_1 + A_2p_2\{b_3 - \lambda_1 b_2^2\} + A_3b_2p_3 + A_4p_4 + A_5b_2p_1\{b_3 - \lambda_2 b_2^2\},$$

and for $\alpha \in [0, 1]$, we define $\mu = -A_5/A_1$ and rewrite (11) as

$$5A_0a_5 = A_0b_5 + A_1p_1\{b_4 - \mu b_2b_3\} + A_2p_2\{b_3 - \lambda_1 b_2^2\} + A_3b_2p_3 + A_4p_4 + A_7b_2^3p_1.$$

It is easily verified that for all $\alpha \geq 0$, the real numbers λ_1, $\lambda_2 \leq \frac{3}{4}$ in the two equations above. So also $\mu \leq \frac{5}{9}$. Thus using the inequalities (1) and the first of Lemma 1, the two equations above both yield the first inequality of our theorem, that is, the upper bound for a_5. Next we proceed to compute the bound for a_6. We define $\lambda_1 = -B_{12}/B_3$, $\lambda_2 = -B_9/B_8$, $\mu = -B_{10}/B_2$, $\rho = -B_{11}/B_2$, $\sigma = -B_8/B_1$, $\tau = -B_6/B_1$, and $\omega = -B_7/B_1$, and if $\alpha \in [1, \infty)$ we write (12) as

$$\begin{aligned} 6B_0a_6 &= B_0b_6 + B_1p_1\{b_5 - \sigma b_2^2b_3\} + B_2p_2\{b_4 - \mu b_2b_3 - \rho b_2^3\} \\ &+ B_3b_3p_3 + B_4b_2p_4 + B_5p_5 + B_6b_4b_2p_1 + B_7b_3^2p_1 + B_9b_2^4p_1 + B_{12}b_2^2p_3, \end{aligned}$$

and for $\alpha \in [0, 1]$, we rewrite (12) as

$$\begin{aligned} 6B_0a_6 &= B_0b_6 + B_1p_1\{b_5 - \tau b_2b_4 - \omega b_3^2\} + B_2p_2\{b_4 - \mu b_2b_3\} \\ &+ B_3p_3\{b_3 - \lambda_1 b_2^2\} + B_4b_2p_4 + B_5p_5 + B_8b_2^2p_1\{b_3 - \lambda_2 b_2^2\} + B_{11}b_2^3p_2. \end{aligned}$$

It is again easy to verify that for all $\alpha \geq 0$, the real numbers λ_1, λ_2, μ, ρ, σ, τ, and ω defined in the preceding two equations all satisfy (as appropriate) the conditions of Lemma 1 and the inequalities (1). Thus the bound for a_6 follows by appropriately applying the inequalities (1) and Lemma 1.

The extremal function is obtained by choosing $\phi(z) = z/(1-z)^2$ and $p(z) = (1+z)/(1-z)$ in the integral representation formulae:

$$f(z) = \begin{cases} \displaystyle\int_0^z \frac{1}{\alpha} t^{-1}[\phi(t)]^{(1/\alpha - 1)} \left(\int_0^t [\phi(\xi)]^{(1/\alpha-1)} \phi'(\xi)p(\xi)d\xi \right) dt & \text{if } \alpha \neq 0, \\ \displaystyle\int_0^z t^{-1}\phi(t)p(t)dt & \text{if } \alpha = 0. \end{cases}$$

(see [2]). This completes the proof.

3. REMARK

Here we guide readers to the computation of the coefficients of the given extremal function. After simple calculation, we can write the extremal function in series form as:

$$f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\eta_{n-1}}{n} + \frac{c+1}{c+2}\frac{\eta_{n-2}G_1}{n} + ... + \frac{c+1}{c+n}\frac{\eta_1 G_{n-2}}{n} + \frac{c+1}{c+n}\frac{G_{n-1}}{n}\right) z^n$$

where

$$c = \frac{1}{\alpha} - 1,$$

$$\eta_n = \frac{(-1)^n}{n!}\prod_{j=1}^{n}(2c - j + 1),$$

$$G_1 = 2 + m_1, \quad G_n = m_{n-2} + 2m_{n-1} + m_n, \;\; n \geq 2;$$

with

$$m_n = \frac{1}{n!}\prod_{j=1}^{n}(2c + j + 3), \quad m_0 = 1.$$

By careful computation, we find that

$$\begin{aligned} a_5 \;=\; & \frac{c(c-1)(2c-1)(2c-3)}{30} - \frac{c(2c-1)(2c-2)(c+1)(2c+6)}{15(c+2)} \\ + \; & \frac{c(2c-1)(c+1)(2c^2+13c+19)}{5(c+3)} - \frac{2c(c+1)(4c^3+42c^2+134c+132)}{15(c+4)} \\ + \; & \frac{(c+1)(4c^4+60c^3+311c^2+669c+510)}{30(c+5)}. \end{aligned}$$

This gives

$$a_5 = \frac{A - 2B + 6C - 4D + E}{30(c+2)(c+3)(c+4)(c+5)}$$

where

$$A = 4c^8 + 44c^7 + 127c^6 - 85c^5 - 629c^4 + 41c^3 + 858c^2 - 360c,$$

$$B = 8c^8 + 116c^7 + 596c^6 + 1160c^5 + 32c^4 - 1996c^3 - 636c^2 - 720c,$$

$$C = 4c^8 + 72c^7 + 509c^6 + 1769c^5 + 3029c^4 + 1979c^3 - 482c^2 - 760c,$$

$$D = 4c^8 + 86c^7 + 760c^6 + 3572c^5 + 9628c^4 + 14846c^3 + 12072c^2 + 3960c,$$

and

$$E = 4c^8 + 100c^7 + 1051c^6 + 6079c^5 + 21181c^4 + 45505c^3 + 58764c^2 + 41556c + 12240$$

so that

$$a_5 = \frac{25c^4 + 338c^3 + 1619c^2 + 3226c + 2040}{5(c+2)(c+3)(c+4)(c+5)}$$

and finally, setting $c = 1/\alpha - 1$, we have

$$a_5 = \frac{25 + 238\alpha + 755\alpha^2 + 902\alpha^3 + 120\alpha^4}{5(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)}.$$

As above, a_6 can be computed *mutatis mutandis*. Bounds on the higher coefficients of functions in the class C_α can also be found using the recurrence relation we have obtained, after the determination of bounds on relevant emerging functionals in the family of starlike functions, which are associated with the desired higher coefficients.

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