Kragujevac J. Math. 30 (2007) 27–43.

A GAGLIARDO–NIRENBERG INEQUALITY, WITH APPLICATION TO DUALITY-BASED A POSTERIORI ESTIMATION IN THE L^1 NORM

Endre Süli

Oxford University Computing Laboratory, Wolfson Building, Parks Road, Oxford OX1 3QD, United Kingdom
(e-mail: endre.suli@comlab.ox.ac.uk)

(Received October 25, 2006)

Abstract. We prove the Gagliardo–Nirenberg-type multiplicative interpolation inequality

$$\|v\|_{\mathrm{L}^1(\mathbb{R}^n)} \leq C\|v\|^{1/2}_{\mathrm{Lip}'(\mathbb{R}^n)}\|v\|^{1/2}_{\mathrm{BV}(\mathbb{R}^n)} \qquad \forall v \in \mathrm{Lip}'(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n),$$

where C is a positive constant, independent of v. Here $\|\cdot\|_{\mathrm{Lip}'(\mathbb{R}^n)}$ is the norm of the dual to the Lipschitz space $\mathrm{Lip}_0(\mathbb{R}^n) := \mathrm{C}^{0,1}_0(\mathbb{R}^n) = \mathrm{C}^{0,1}(\mathbb{R}^n) \cap \mathrm{C}_0(\mathbb{R}^n)$ and $\|\cdot\|_{\mathrm{BV}(\mathbb{R}^n)}$ signifies the norm in the space $\mathrm{BV}(\mathbb{R}^n)$ consisting of functions of bounded variation on $\mathbb{R}^n$. We then use a local version of this inequality to derive an *a posteriori* error bound in the $\mathrm{L}^1(\Omega')$ norm, with $\bar{\Omega}' \subset \Omega = (0,1)^n$, for a finite element approximation to a boundary-value problem for a first-order linear hyperbolic equation, under the limited regularity requirement that the solution to the problem belongs to $\mathrm{BV}(\Omega)$.

1. INTRODUCTION

The aim of this paper is to establish the following Gagliardo–Nirenberg-type multiplicative interpolation inequality: there exists a constant $C > 0$, such that

$$\|v\|_{\mathrm{L}^1(\mathbb{R}^n)} \leq C\|v\|^{1/2}_{\mathrm{Lip}'(\mathbb{R}^n)}\|v\|^{1/2}_{\mathrm{BV}(\mathbb{R}^n)} \qquad \forall v \in \mathrm{Lip}'(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n), \tag{1}$$

where $\|\cdot\|_{\mathrm{Lip}'(\mathbb{R}^n)}$ is the norm of the dual to the Lipschitz space $\mathrm{Lip}_0(\mathbb{R}^n) := \mathrm{C}_0^{0,1}(\mathbb{R}^n) = \mathrm{C}^{0,1}(\mathbb{R}^n) \cap \mathrm{C}_0(\mathbb{R}^n)$ and $\|\cdot\|_{\mathrm{BV}(\mathbb{R}^n)}$ signifies the norm in the space $\mathrm{BV}(\mathbb{R}^n)$ consisting of functions of bounded variation on $\mathbb{R}^n$. Here, for $k \in \mathbb{N}_0$, $\mathrm{C}_0^k(\mathbb{R}^n)$ denotes the set of k-times continuously differentiable functions with compact support in $\mathbb{R}^n$ and $\mathrm{C}_0^\infty(\mathbb{R}^n) = \bigcap_{k=0}^\infty \mathrm{C}_0^k(\mathbb{R}^n)$; for the sake of notational simplicity, we write $\mathrm{C}_0(\mathbb{R}^n)$ instead of $\mathrm{C}_0^0(\mathbb{R}^n)$.

We refer to text of Meyer [8], particularly Theorems 17 and 18 on p.129 in Section 2.2, for the statement and proof of the closely related *improved Gagliardo–Nirenberg inequality* due to Cohen, Dahmen, Daubechies and DeVore, according to which there exists a constant $C > 0$ such that, for every function v that belongs to the intersection of the homogeneous Besov space $\dot{\mathrm{B}}_\infty^{-1,\infty}(\mathbb{R}^n)$ with $\mathrm{BV}(\mathbb{R}^n)$,

$$\|v\|_{\mathrm{L}^2(\mathbb{R}^n)} \leq C \|v\|^{1/2}_{\dot{\mathrm{B}}_\infty^{-1,\infty}(\mathbb{R}^n)} \|v\|^{1/2}_{\mathrm{BV}(\mathbb{R}^n)}. \tag{2}$$

In (2), compared with the inequality (1) established here, instead of $\|\cdot\|_{\mathrm{L}^1(\mathbb{R}^n)}$ the left-hand side of the inequality includes the $\mathrm{L}^2(\mathbb{R}^n)$ norm, while the right-hand side contains the norm $\|\cdot\|_{\dot{\mathrm{B}}_\infty^{-1,\infty}(\mathbb{R}^n)}$ instead of the dual Lipschitz norm $\|\cdot\|_{\mathrm{Lip}'(\mathbb{R}^n)}$. We note in passing that the dual Lipschitz norm also appears in the articles by Tadmor [11] and Nessyahu & Tadmor [9], for example, in the analysis of numerical methods for scalar hyperbolic partial differential equations.

We begin, in Section 2, by establishing a local version of (1). We then extend this local inequality to the whole of $\mathbb{R}^n$ in Section 3. In the final section, Section 4, we use the local version of the inequality (1) on Ω', where $\bar{\Omega}' \subset \Omega = (0,1)^n$, to derive a residual-based *a posteriori* bound in the $\mathrm{L}^1(\Omega')$ norm on the error between the analytical solution of a boundary-value problem for a first-order linear hyperbolic equation and its finite element approximation, under the limited regularity requirement that the analytical solution to the problem belongs to $\mathrm{BV}(\Omega)$. *A posteriori* error bounds are crucial building blocks of adaptive finite element algorithms, aimed at optimally distributing the computational mesh so as to accurately capture the analytical solution in a certain, prescribed, norm, or a linear or nonlinear functional of the analytical solution. The mathematics of *a posteriori* error estimation is an active field of research. We shall not attempt to survey this thriving and broad subject here; instead, we refer the reader to the survey articles and monographs [1, 3, 4, 5, 6, 12] listed in the bibliography.

2. INTERIOR BOUND

Suppose that $\Omega \subset \mathbb{R}^n$ is either a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary or $\Omega = \mathbb{R}^n$. We begin by deriving an interior version of inequality (1). For this purpose we consider a function $K \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that $K(x) \geq 0$ and $K(-x) = K(x)$ for all x in $\mathbb{R}^n$, $\mathrm{supp}(K)$ is the unit ball $B_1 = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and

$$\int_{\mathbb{R}^n} K(\xi)\,\mathrm{d}\xi = 1.$$

Let us suppose that Ω' is a bounded open subset of Ω with $\bar{\Omega}' \subset \Omega$, let $d \in \mathbb{R}_+ \cup \{\infty\}$ denote the distance between $\partial\Omega'$ and $\partial\Omega$, and suppose that $\delta \in (0, d')$. In particular if $\Omega = \mathbb{R}^n$, then $d' = \infty$; otherwise $0 < d' < \infty$.

We consider the function $K_\delta \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ defined by

$$K_\delta(x) := \frac{1}{\delta^n} K\left(\frac{x}{\delta}\right).$$

For any $\psi \in \mathrm{L}^\infty(\mathbb{R}^n)$, with $\mathrm{supp}(\psi) \subset \bar{\Omega}'$, we define

$$\psi_\delta = \psi * K_\delta,$$

where $*$ signifies convolution over $\mathbb{R}^n$. Observe that since

$$\mathrm{supp}(\psi_\delta) \subset \mathrm{supp}(\psi) + \mathrm{supp}(K_\delta) \subset \bar{\Omega} \qquad \text{for all } \delta \in (0, d'),$$

ψ_δ belongs to $\mathrm{C}_0^\infty(\mathbb{R}^n)$. Since $\mathrm{supp}(\psi_\delta) \subset \bar{\Omega}$, it follows that $\psi_\delta|_{\bar{\Omega}} \in \mathrm{Lip}_0(\Omega) = \mathrm{C}^{0,1}(\bar{\Omega}) \cap \mathrm{C}_0(\bar{\Omega})$, where, for a bounded open set Ω, $\mathrm{C}_0(\bar{\Omega})$ denotes the set of all uniformly continuous functions defined on $\bar{\Omega}$ which vanish on $\partial\Omega$. For $\Omega = \mathbb{R}^n$, $\mathrm{Lip}_0(\mathbb{R}^n)$ has been defined above. We begin by showing the following result.

Lemma 1. *Let* $\psi \in \mathrm{L}^\infty(\mathbb{R}^n)$ *with* $\mathrm{supp}(\psi) \subset \bar{\Omega}'$*, and define* $\psi_\delta(x) = \psi * K_\delta$*. Further, let* B_1 *denote the unit ball in* $\mathbb{R}^n$ *centred at* 0*, and define*

$$C_1 := \int_{B_1} |\nabla K(\xi)|\,\mathrm{d}\xi;$$

then, for any δ *in* $(0, d')$*,*

$$\|\psi_\delta\|_{\mathrm{Lip}(\Omega)} \leq C_1 \delta^{-1} \|\psi\|_{\mathrm{L}^\infty(\Omega')}.$$

Here $\|\cdot\|_{\mathrm{Lip}(\Omega)}$ *is the norm of the space* $\mathrm{Lip}_0(\Omega)$*, defined by*

$$\|w\|_{\mathrm{Lip}(\Omega)} := \sup_{x,\, x' \in \Omega;\ x \neq x'} \frac{|w(x) - w(x')|}{|x - x'|}, \qquad w \in \mathrm{Lip}_0(\Omega).$$

Proof. Recalling the definition of convolution, it is immediate that

$$|\psi_\delta(x) - \psi_\delta(x')| \leq \|\psi\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |K_\delta(x-y) - K_\delta(x'-y)|\,\mathrm{d}y$$

for any pair of points x, x' in Ω. Noting that $K_\delta \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ and applying the Integral Mean Value Theorem it follows that

$$|K_\delta(x-y) - K_\delta(x'-y)| \leq |x - x'| \int_0^1 |\nabla K_\delta(\theta(x'-y) + (1-\theta)(x-y))|\,\mathrm{d}\theta$$

for any pair of points x, x' in Ω and any $y \in \mathbb{R}^n$. Upon integrating both sides of the last inequality with respect to $y \in \mathbb{R}^n$ we deduce that

$$\int_{\mathbb{R}^n} |K_\delta(x-y) - K_\delta(x'-y)|\,\mathrm{d}y \\ \leq |x - x'| \int_{\mathbb{R}^n} \left(\int_0^1 |\nabla K_\delta(x - y + \theta(x'-x))|\,\mathrm{d}\theta \right) \mathrm{d}y,$$

for any pair of points x, x' in Ω. Interchanging the order of integration on the right-hand side and performing the change of variables $y \mapsto z(y) = x - y + \theta(x'-x)$ for θ fixed in $[0,1]$ and x, x' fixed in Ω, recalling the translation-invariance of the Lebesgue measure on $\mathbb{R}^n$ we find that

$$\int_{\mathbb{R}^n} |K_\delta(x-y) - K_\delta(x'-y)|\,\mathrm{d}y \leq |x - x'| \int_{\mathbb{R}^n} |\nabla K_\delta(z)|\,\mathrm{d}z.$$

Now

$$\nabla K_\delta(x) = \delta^{-1-n}(\nabla K)\left(\frac{x}{\delta}\right),$$

and therefore, upon noting that $\mathrm{supp}(K) = B_1$,

$$\int_{\mathbb{R}^n} |\nabla K_\delta(z)| \, \mathrm{d}z = \delta^{-1} C_1,$$

with C_1 as in the statement of the Lemma. Returning to the first inequality in the proof and recalling that $\mathrm{supp}(\psi) \subset \bar{\Omega}'$, we deduce the required result. $\diamond$

Next we prove the following lemma.

Lemma 2. *Suppose that $\varphi \in \mathrm{W}^{1,1}(\Omega)$; then, there exists a positive constant C_2 such that, for any δ in $(0, d')$,*

$$\|\varphi - \varphi_\delta\|_{\mathrm{L}^1(\Omega')} \le C_2 \delta |\varphi|_{\mathrm{W}^{1,1}(\Omega)},$$

where $\varphi_\delta := \varphi * K_\delta$, with φ defined to be identically zero over the set $\mathbb{R}^n \setminus \bar{\Omega}$, and $C_2 := |B_1| \max_{\xi \in B_1} K(\xi)$, with $|B_1|$ denoting the Lebesgue measure of the unit ball B_1 in $\mathbb{R}^n$.

Proof. Suppose, to begin with, that $\varphi \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{C}^\infty(\Omega)$. Given $x \in \Omega'$, we have the following sequence of equalities:

$$\begin{aligned}
\varphi(x) - \varphi_\delta(x) &= \varphi(x) - \int_{\mathbb{R}^n} \varphi(y) K_\delta(x-y) \, \mathrm{d}y \\
&= \varphi(x) - \int_{\mathbb{R}^n} \varphi(x-y) K_\delta(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} [\varphi(x) - \varphi(x-y)] \, K_\delta(y) \, \mathrm{d}y \\
&= \delta^{-n} \int_{|y|<\delta} [\varphi(x) - \varphi(x-y)] \, K(y/\delta) \, \mathrm{d}y \\
&= \int_{B_1} [\varphi(x) - \varphi(x-\delta\xi)] \, K(\xi) \, \mathrm{d}\xi.
\end{aligned}$$

The last equality implies that

$$|\varphi(x) - \varphi_\delta(x)| \le C_3 \int_{B_1} |\varphi(x) - \varphi(x-\delta\xi)| \, \mathrm{d}\xi, \qquad x \in \Omega',$$

where $C_3 = \max_{\xi\in B_1} K(\xi)$. Upon integration over Ω' we deduce that

$$\begin{aligned} \|\varphi - \varphi_\delta\|_{\mathrm{L}^1(\Omega')} &\leq C_3 \int_{\Omega'}\int_{B_1} |\varphi(x) - \varphi(x-\delta\xi)|\,\mathrm{d}\xi\,\mathrm{d}x \\ &= C_3 \int_{B_1} \|\varphi(\cdot) - \varphi(\cdot - \delta\xi)\|_{\mathrm{L}^1(\Omega')}\,\mathrm{d}\xi. \end{aligned} \tag{3}$$

In order to further bound the right-hand side in (3), observe that

$$\varphi(x) - \varphi(x-\delta\xi) = \delta\xi\cdot\int_0^1 (\nabla\varphi)(\theta x + (1-\theta)(x-\delta\xi))\,\mathrm{d}\theta$$

for any x in Ω' and any $\xi \in B_1$. Consequently,

$$|\varphi(x) - \varphi(x-\delta\xi)| \leq \delta \int_0^1 |\nabla\varphi(x-(1-\theta)\delta\xi)|\,\mathrm{d}\theta$$

so that

$$\|\varphi(\cdot) - \varphi(\cdot - \delta\xi)\|_{\mathrm{L}^1(\Omega')} \leq \delta \int_0^1 \int_{\Omega'} |\nabla\varphi(x-(1-\theta)\delta\xi)|\,\mathrm{d}x\,\mathrm{d}\theta. \tag{4}$$

Upon performing the change of variables $x \mapsto z(x) = x - (1-\theta)\delta\xi$ for fixed θ in $[0,1]$ and $\xi \in B_1$, it follows from (4) that

$$\begin{aligned} \|\varphi(\cdot) - \varphi(\cdot - \delta\xi)\|_{\mathrm{L}^1(\Omega')} &\leq \delta \int_0^1 \int_{\Omega'-(1-\theta)\delta\xi} |\nabla\varphi(z)|\,\mathrm{d}z\,\mathrm{d}\theta \\ &\leq \delta \int_\Omega |\nabla\varphi(z)|\,\mathrm{d}z = \delta|\varphi|_{\mathrm{W}^{1,1}(\Omega)}. \end{aligned} \tag{5}$$

Substituting (5) into (3) gives

$$\|\varphi - \varphi_\delta\|_{\mathrm{L}^1(\Omega')} \leq C_2\delta|\varphi|_{\mathrm{W}^{1,1}(\Omega)},$$

where $C_2 = C_3|B_1|$. This proves the desired inequality for $\varphi \in \mathrm{W}^{1,1}(\Omega)\cap \mathrm{C}^\infty(\Omega)$. For $\varphi \in \mathrm{W}^{1,1}(\Omega)$ the inequality then follows by density of $\mathrm{W}^{1,1}(\Omega)\cap\mathrm{C}^\infty(\Omega)$ in the Sobolev space $\mathrm{W}^{1,1}(\Omega)$. $\diamond$

Now we extend this result to functions of bounded variation. Let us suppose for this purpose that $u \in \mathrm{L}^1(\Omega)$; we then put

$$\int_\Omega |Du| := \sup\left\{\int_\Omega u \,\mathrm{div}\, v\,\mathrm{d}x \,:\, v = (v_1,\dots,v_n) \in [\mathrm{C}^1_0(\Omega)]^n,\ |v(x)| \leq 1 \text{ for } x\in\Omega\right\}.$$

A function $u \in \mathrm{L}^1(\Omega)$ is said to be of bounded variation on Ω if

$$|u|_{\mathrm{BV}(\Omega)} := \int_\Omega |Du| < \infty.$$

The linear space of functions of bounded variation of Ω is denoted by $\mathrm{BV}(\Omega)$ and is equipped with the norm $\|\cdot\|_{\mathrm{BV}(\Omega)}$ defined by

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{\mathrm{L}^1(\Omega)} + |u|_{\mathrm{BV}(\Omega)}.$$

Thus, $\mathrm{BV}(\Omega)$ is the set of functions $u \in \mathrm{L}^1(\Omega)$ whose weak gradient Du is a bounded (vector-valued) Radon measure with finite total variation $|u|_{\mathrm{BV}(\Omega)}$.

If $u \in \mathrm{W}^{1,1}(\Omega)$ ($\subset \mathrm{BV}(\Omega)$), then

$$|u|_{\mathrm{BV}(\Omega)} = \int_\Omega |Du| = \|\nabla u\|_{\mathrm{L}^1(\Omega)},$$

where ∇u is the distributional gradient of u (cf. [10]).

We recall the following approximation result of Anzellotti and Giaquinta [2]; see also Theorem 1.17 in Giusti [7] and Theorem 5.3.3 in Ziemer [13].

Theorem 1. *For each function $v \in \mathrm{BV}(\Omega)$ there exists a sequence $\{\varphi_j\}_{j=1}^\infty$ of functions in $\mathrm{W}^{1,1}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ such that*

$$\begin{aligned} \lim_{j\to\infty} \int_\Omega |v - \varphi_j| \,\mathrm{d}x &= 0, \\ \lim_{j\to\infty} \int_\Omega |D\varphi_j| \,\mathrm{d}x &= \int_\Omega |Dv|. \end{aligned}$$

Now, suppose that $v \in \mathrm{BV}(\Omega)$ (extended by zero to the whole of $\mathbb{R}^n$) and consider a sequence $\{\varphi_j\}_{j=1}^\infty$ of functions in $\mathrm{W}^{1,1}(\Omega) \cap \mathrm{C}^\infty(\Omega)$, as in Theorem 1; namely,

$$\lim_{j\to\infty} \|v - \varphi_j\|_{\mathrm{L}^1(\Omega)} = 0$$

and

$$\lim_{j\to\infty} |\varphi_j|_{\mathrm{W}^{1,1}(\Omega)} = |v|_{\mathrm{BV}(\Omega)}.$$

By Lemma 2, we also have, with φ_j defined to be identically 0 in the set $\mathbb{R}^n \setminus \bar{\Omega}$, that

$$\|\varphi_j - \varphi_j * K_\delta\|_{\mathrm{L}^1(\Omega')} \le C_2 \delta |\varphi_j|_{\mathrm{W}^{1,1}(\Omega)}.$$

Passing to the limit as $j \to \infty$, we deduce that

$$\|v - v * K_\delta\|_{\mathrm{L}^1(\Omega')} \leq C_2 \delta \int_\Omega |Dv| = C_2 \delta |v|_{\mathrm{BV}(\Omega)}.$$

Thus we have proved the following lemma.

Lemma 3. *Suppose that $v \in \mathrm{BV}(\Omega)$, extended to $\mathbb{R}^n \setminus \bar{\Omega}$ as the identically zero function; then, for each $\delta \in (0, d')$,*

$$\|v - v_\delta\|_{\mathrm{L}^1(\Omega')} \leq C_2 \delta |v|_{\mathrm{BV}(\Omega)},$$

where $v_\delta = v * K_\delta$ and C_2 is the same positive constant as in Lemma 2.

Now suppose that $v \in \mathrm{Lip}'(\Omega) \cap \mathrm{BV}(\Omega)$ (extended by zero onto the whole of $\mathbb{R}^n$). With $\delta \in (0, d')$ as above, we then write

$$\int_\Omega v(x)\psi(x)\,\mathrm{d}x = \int_\Omega v(x)\psi_\delta(x)\,\mathrm{d}x + \int_\Omega v(x)[\psi(x) - \psi_\delta(x)]\,\mathrm{d}x,$$

where, as before, $\psi \in \mathrm{L}^\infty(\Omega')$ (extended by zero onto the whole of $\mathbb{R}^n$). Further, recalling that $K(-z) = K(z)$ for all $z \in \mathbb{R}^n$, we deduce that

$$\int_\Omega v(x)[\psi(x) - \psi_\delta(x)]\,\mathrm{d}x = \int_\Omega [v(x) - v_\delta(x)]\psi(x)\,\mathrm{d}x.$$

Thereby,

$$\left|\int_\Omega v(x)\psi(x)\,\mathrm{d}x\right| \leq \|v\|_{\mathrm{Lip}'(\Omega)} \|\psi_\delta\|_{\mathrm{Lip}(\Omega)} + \|v - v_\delta\|_{\mathrm{L}^1(\Omega')} \|\psi\|_{\mathrm{L}^\infty(\Omega')}.$$

Recalling Lemmas 1 and 2, we find that

$$\left|\int_\Omega v(x)\psi(x)\,\mathrm{d}x\right| \leq \left\{C_1 \delta^{-1} \|v\|_{\mathrm{Lip}'(\Omega)} + C_2 \delta |v|_{\mathrm{BV}(\Omega)}\right\} \|\psi\|_{\mathrm{L}^\infty(\Omega')}.$$

Let us, in particular, choose $\psi = \chi_{\Omega'}\,\mathrm{sgn}(v)$, where $\chi_{\Omega'}$ is the characteristic function of the open set Ω'. We then deduce that

$$\|v\|_{\mathrm{L}^1(\Omega')} \leq C_1 \delta^{-1} \|v\|_{\mathrm{Lip}'(\Omega)} + C_2 \delta |v|_{\mathrm{BV}(\Omega)} \quad \forall \delta \in (0, d'),\ \forall v \in \mathrm{Lip}'(\Omega) \cap \mathrm{BV}(\Omega). \quad (6)$$

Trivially, (6) implies that

$$\|v\|_{\mathrm{L}^1(\Omega')} \leq C_1 \delta^{-1} \|v\|_{\mathrm{Lip}'(\Omega)} + C_2 \delta \|v\|_{\mathrm{BV}(\Omega)} \quad \forall \delta \in (0, d'),\ \forall v \in \mathrm{Lip}'(\Omega) \cap \mathrm{BV}(\Omega), \quad (7)$$

where $\delta' = \mathrm{dist}(\partial\Omega', \partial\Omega)$, $\bar{\Omega}' \subset \Omega$. We begin by considering the case of $d' < \infty$; the case of $d' = \infty$, corresponding to the choice of $\Omega = \mathbb{R}^n$, will be discussed in the next section.

For $v \in \mathrm{Lip}'(\Omega) \cap \mathrm{BV}(\Omega)$, $v \neq 0$, we consider the mapping

$$\delta \in \mathbb{R}_+ \mapsto f(v;\delta) := C_1 \delta^{-1} \|v\|_{\mathrm{Lip}'(\Omega)} + C_2 \delta \|v\|_{\mathrm{BV}(\Omega)}.$$

The function $f(v;\cdot)$ is strictly positive on $\mathbb{R}_+$ and attains its minimum value at

$$\delta = \delta_0 := \sqrt{\frac{C_1 \|v\|_{\mathrm{Lip}'(\Omega)}}{C_2 \|v\|_{\mathrm{BV}(\Omega)}}}.$$

We shall consider two mutually exclusive cases, depending on the size of $\delta_0 > 0$ relative to $d' < \infty$.

Case 1: $\delta_0 \in (0, d')$. On equilibrating the two terms on the right-hand side of (7) by choosing $\delta = \delta_0$, we get

$$\|v\|_{\mathrm{L}^1(\Omega')} \leq C_4 \|v\|^{1/2}_{\mathrm{Lip}'(\Omega)} \|v\|^{1/2}_{\mathrm{BV}(\Omega)}, \tag{8}$$

where $C_4 = 2\sqrt{C_1 C_2}$. For future reference, we rewrite (8) in the following equivalent form:

$$\|v\|_{\mathrm{L}^1(\Omega')} \leq \sqrt{C_1 C_2} \|v\|^{1/2}_{\mathrm{Lip}'(\Omega)} \|v\|^{1/2}_{\mathrm{BV}(\Omega)} + \sqrt{C_1 C_2} \|v\|^{1/2}_{\mathrm{Lip}'(\Omega)} \|v\|^{1/2}_{\mathrm{BV}(\Omega)}, \tag{9}$$

the inequality being valid for all $v \in \mathrm{Lip}'(\Omega) \cap \mathrm{BV}(\Omega)$ such that $\delta_0 \in (0, d')$. Since

$$\sqrt{\frac{C_2 \|v\|_{\mathrm{BV}(\Omega)}}{C_1 \|v\|_{\mathrm{Lip}'(\Omega)}}} = \frac{1}{\delta_0} = \max\left(\frac{1}{d'}, \frac{1}{\delta_0}\right) = \max\left(\frac{1}{d'}, \sqrt{\frac{C_2 \|v\|_{\mathrm{BV}(\Omega)}}{C_1 \|v\|_{\mathrm{Lip}'(\Omega)}}}\right),$$

we can rewrite (9) as follows:

$$\|v\|_{\mathrm{L}^1(\Omega')} \leq C_1 \|v\|_{\mathrm{Lip}'(\Omega)} \max\left(\frac{1}{d'}, \sqrt{\frac{C_2 \|v\|_{\mathrm{BV}(\Omega)}}{C_1 \|v\|_{\mathrm{Lip}'(\Omega)}}}\right) + \sqrt{C_1 C_2} \|v\|^{1/2}_{\mathrm{Lip}'(\Omega)} \|v\|^{1/2}_{\mathrm{BV}(\Omega)}. \tag{10}$$

Case 2: $\delta_0 \in [d', \infty)$. In this case, the function $f(v;\cdot)$ is strictly monotonic decreasing in the interval $\delta \in (0, d')$, and therefore $\inf_{\delta \in (0,d')} f(v,\delta) = f(v, d')$. Hence,

$$\begin{aligned}
\|v\|_{\mathrm{L}^1(\Omega')} &\leq C_1 (d')^{-1} \|v\|_{\mathrm{Lip}'(\Omega)} + C_2 d' \|v\|_{\mathrm{BV}(\Omega)} \\
&\leq C_1 (d')^{-1} \|v\|_{\mathrm{Lip}'(\Omega)} + C_2 \delta_0 \|v\|_{\mathrm{BV}(\Omega)} \\
&\leq C_1 (d')^{-1} \|v\|_{\mathrm{Lip}'(\Omega)} + \sqrt{C_1 C_2} \|v\|^{1/2}_{\mathrm{Lip}'(\Omega)} \|v\|^{1/2}_{\mathrm{BV}(\Omega)}.
\end{aligned}$$

Now, since $d' \le \delta_0$, we have that

$$\frac{1}{d'} = \max\left(\frac{1}{d'}, \frac{1}{\delta_0}\right) = \max\left(\frac{1}{d'}, \sqrt{\frac{C_2\|v\|_{\mathrm{BV}(\Omega)}}{C_1\|v\|_{\mathrm{Lip}'(\Omega)}}}\right),$$

so, once again,

$$\|v\|_{\mathrm{L}^1(\Omega')} \le C_1\|v\|_{\mathrm{Lip}'(\Omega)} \max\left(\frac{1}{d'}, \sqrt{\frac{C_2\|v\|_{\mathrm{BV}(\Omega)}}{C_1\|v\|_{\mathrm{Lip}'(\Omega)}}}\right) + \sqrt{C_1C_2}\|v\|^{1/2}_{\mathrm{Lip}'(\Omega)}\|v\|^{1/2}_{\mathrm{BV}(\Omega)}. \quad (11)$$

Combining Cases 1 and 2, as expressed by inequalities (10) and (11), we see that, irrespective of the relative magnitudes of δ_0 and d', we have

$$\|v\|_{\mathrm{L}^1(\Omega')} \le C_1\|v\|_{\mathrm{Lip}'(\Omega)} \max\left(\frac{1}{d'}, \sqrt{\frac{C_2\|v\|_{\mathrm{BV}(\Omega)}}{C_1\|v\|_{\mathrm{Lip}'(\Omega)}}}\right) + \sqrt{C_1C_2}\|v\|^{1/2}_{\mathrm{Lip}'(\Omega)}\|v\|^{1/2}_{\mathrm{BV}(\Omega)}, \quad (12)$$

for all $v \in \mathrm{Lip}'(\Omega) \cap \mathrm{BV}(\Omega) \setminus \{0\}$, where $\bar{\Omega}' \subset \Omega$ and where $d' := \mathrm{dist}(\partial\Omega', \partial\Omega) \in (0, \infty)$.

3. GLOBAL BOUND

We shall now consider the case when $\Omega = \mathbb{R}^n$, and extend the result stated in (12) from Ω' to the whole of $\mathbb{R}^n$. On taking $d' = \infty$ in (7) and equilibrating the two terms on the right-hand side of (7) by choosing $\delta = \delta_0 \in (0, d') = (0, \infty)$, we have that

$$\|v\|_{\mathrm{L}^1(\Omega')} \le C_4\|v\|^{1/2}_{\mathrm{Lip}'(\mathbb{R}^n)}\|v\|^{1/2}_{\mathrm{BV}(\mathbb{R}^n)} \qquad \forall v \in \mathrm{Lip}'(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n).$$

Now, let us assume that $\{\Omega_j\}_{j=1}^\infty$ is a nested sequence of bounded open sets, $\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \mathbb{R}^n$ such that $\mathbb{R}^n = \bigcup_{j=1}^\infty \Omega_j$. On taking $\Omega' = \Omega_j$ in the last inequality, we see that

$$\|v\|_{\mathrm{L}^1(\Omega_j)} \le C_4\|v\|^{1/2}_{\mathrm{Lip}'(\mathbb{R}^n)}\|v\|^{1/2}_{\mathrm{BV}(\mathbb{R}^n)} \qquad \forall v \in \mathrm{Lip}'(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n), \quad j = 1, 2, \dots,$$

where $C_4 = 2\sqrt{C_1C_2}$ is independent of Ω_j.

Defining $v_j(x) := \chi_{\Omega_j}(x)v(x)$, $j = 1, 2, \dots$, where χ_{Ω_j} is the characteristic function of Ω_j, we can restate this inequality as follows:

$$\|v_j\|_{\mathrm{L}^1(\mathbb{R}^n)} \le C_4\|v\|^{1/2}_{\mathrm{Lip}'(\mathbb{R}^n)}\|v\|^{1/2}_{\mathrm{BV}(\mathbb{R}^n)} \qquad \forall v \in \mathrm{Lip}'(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n), \quad j = 1, 2, \dots .$$

As $|v(x)| = \lim_{j\to\infty} |v_j(x)|$ for a.e. $x \in \mathbb{R}^n$, it follows by Fatou's Lemma that

$$\|v\|_{\mathrm{L}^1(\mathbb{R}^n)} \le \liminf_{j\to\infty} \|v_j\|_{\mathrm{L}^1(\mathbb{R}^n)}.$$

Consequently, passing to the limit over $j \to \infty$ gives

$$\|v\|_{\mathrm{L}^1(\mathbb{R}^n)} \le C_4 \|v\|^{1/2}_{\mathrm{Lip}'(\mathbb{R}^n)} \|v\|^{1/2}_{\mathrm{BV}(\mathbb{R}^n)} \qquad \forall v \in \mathrm{Lip}'(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n), \tag{13}$$

which is the desired multiplicative interpolation inequality.

4. APPLICATION IN A POSTERIORI ERROR ANALYSIS

Suppose that $\Omega = (0,1)^n$, and let Γ signify the union of all $(n-1)$-dimensional open faces of Ω. Let us denote by ν the unit outward normal vector to Γ. Suppose that $b \in [\mathrm{C}^{0,1}(\bar\Omega)]^n$, $c \in \mathrm{C}^{0,1}(\bar\Omega)$, and $f \in \mathrm{BV}(\Omega)$. We shall suppose that there exists a positive contact c_0 such that $c(x) + \frac{1}{2}\nabla \cdot b(x) \ge c_0$ for all $x \in \bar\Omega$, and that the components b_i, $i = 1, \ldots, n$, of the vector field b are strictly positive on $\bar\Omega$. We then consider the first-order linear hyperbolic partial differential equation

$$\mathcal{L}u := \nabla \cdot (bu) + cu = f \qquad \text{in } \Omega, \tag{14}$$

supplemented by the boundary condition

$$u|_{\Gamma_-} = g, \tag{15}$$

where $\Gamma_- = \{x \in \Gamma : b(x)\cdot\nu(x) < 0\}$ and $g \in \mathrm{L}^1(\Gamma_-)$; analogously, we define $\Gamma_+ = \{x \in \Gamma : b(x)\cdot\nu(x) > 0\}$.

The weak formulation of the boundary-value problem amounts to finding $u \in \mathrm{BV}(\Omega)$ such that

$$-\int_\Omega u\, b\cdot\nabla v \,\mathrm{d}x + \int_{\Gamma_+} (b\cdot\nu)\, u\, v \,\mathrm{d}s + \int_\Omega c\, u\, v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x - \int_{\Gamma_-} (b\cdot\nu)\, g\, v \,\mathrm{d}s \tag{16}$$

for all $v \in \mathrm{C}^{0,1}(\bar\Omega)$.

Now, let $\{\mathcal{T}_h\}_{h>0}$ denote a shape-regular family of partitions of Ω into disjoint open simplices κ whose union is Ω; for $\kappa \in \mathcal{T}_h$ we define $h_\kappa := \mathrm{diam}(\kappa)$ and let $h =$

$\max_{\kappa \in \mathcal{T}_h} h_\kappa$. Further, let $\mathcal{V}_{hp}$ denote the set of all continuous piecewise polynomials of degree $p \geq 1$ defined on $\mathcal{T}_h$.

The finite element approximation of (16) is defined as follows: find $u_h \in \mathcal{V}_{hp}$ such that

$$-\int_\Omega u_h\, b \cdot \nabla v_h \,\mathrm{d}x + \int_{\Gamma_+} (b \cdot \nu)\, u_h\, v_h \,\mathrm{d}s + \int_\Omega c\, u_h\, v_h \,\mathrm{d}x \\ = \int_\Omega f\, v_h \,\mathrm{d}x - \int_{\Gamma_-} (b \cdot \nu) g\, v_h \,\mathrm{d}s \qquad \forall v_h \in \mathcal{V}_{hp}. \quad (17)$$

On denoting the expression on the left-hand side of (17) by $B(u_h, v_h)$, it is easily seen that $(w_h, v_h) \in \mathcal{V}_{hp} \times \mathcal{V}_{hp} \mapsto B(w_h, v_h) \in \mathbb{R}$ is a bilinear functional, and

$$B(v_h, v_h) \geq c_0 \|v_h\|^2_{\mathrm{L}^2(\Omega)} \qquad \forall v_h \in \mathcal{V}_{hp}.$$

Since $\mathcal{V}_{hp}$ is a finite-dimensional linear space it then follows that problem (17) has a unique solution $u_h \in \mathcal{V}_{hp}$.

To derive an *a posteriori* bound on the error $u - u_h$ in the $\mathrm{L}^1(\Omega')$ norm where $\bar{\Omega}' \subset \Omega$, we begin by establishing an *a posteriori* error bound in the $\mathrm{Lip}'(\Omega)$ norm using a duality argument which involves the formal adjoint $\mathcal{L}^* : z \mapsto -b \cdot \nabla z + cz$ of the differential operator $\mathcal{L}$.

Let $\psi \in \mathrm{Lip}_0(\Omega)$, and let $z \in \mathrm{C}^{0,1}(\bar{\Omega})$ denote the corresponding (classical) solution to the hyperbolic boundary-value problem

$$\mathcal{L}^* z = \psi \qquad \text{in } \Omega$$

subject to $z|_{\Gamma_+} = 0$.

Consequently, $z \in \mathrm{C}^{0,1}(\bar{\Omega})$ also satisfies the following identity:

$$-\int_\Omega w\, b \cdot \nabla z \,\mathrm{d}x + \int_{\Gamma_+} (b \cdot \nu) w\, z \,\mathrm{d}s + \int_\Omega c\, w\, z \,\mathrm{d}x = \int_\Omega w\, \psi \,\mathrm{d}x \qquad \forall w \in \mathrm{BV}(\Omega). \quad (18)$$

Thus, for any $z_h \in \mathcal{V}_{hp}$,

$$
\begin{aligned}
&\int_\Omega (u-u_h)\,\psi\,\mathrm{d}x \\
&\quad= -\int_\Omega (u-u_h)\,b\cdot\nabla z\,\mathrm{d}x + \int_{\Gamma_+}(b\cdot\nu)(u-u_h)z\,\mathrm{d}s + \int_\Omega c\,(u-u_h)\,z\,\mathrm{d}x \\
&\quad= -\int_\Omega (u-u_h)\,b\cdot\nabla(z-z_h)\,\mathrm{d}x + \int_{\Gamma_+}(b\cdot\nu)(u-u_h)\,(z-z_h)\,\mathrm{d}s \\
&\qquad+ \int_\Omega c\,(u-u_h)\,(z-z_h)\,\mathrm{d}x \\
&\quad= \int_\Omega f\,(z-z_h)\,\mathrm{d}x - \int_{\Gamma_-}(b\cdot\nu)\,g\,(z-z_h)\,\mathrm{d}x \\
&\qquad- \left[-\int_\Omega u_h\,b\cdot\nabla(z-z_h)\,\mathrm{d}x + \int_{\Gamma_+}(b\cdot\nu)\,u_h\,(z-z_h)\,\mathrm{d}s + \int_\Omega c\,u_h\,(z-z_h)\,\mathrm{d}x\right] \\
&\quad= \int_\Omega (f-\nabla\cdot(bu_h)-cu_h)\,(z-z_h)\,\mathrm{d}x - \int_{\Gamma_-}(b\cdot\nu)\,(g-u_h)\,(z-z_h)\,\mathrm{d}x.
\end{aligned}
$$

On defining the *internal residual* $\mathtt{R}_\Omega = f - \nabla\cdot(bu_h) - cu_h$ on Ω and the *boundary residual* $\mathtt{R}_\Gamma = |b\cdot\nu|(g-u_h)$ on Γ_-, we then deduce the *error representation formula*

$$
\int_\Omega (u-u_h)\,\psi\,\mathrm{d}x = \int_\Omega \mathtt{R}_\Omega\,(z-z_h)\,\mathrm{d}x + \int_{\Gamma_-}\mathtt{R}_\Gamma\,(z-z_h)\,\mathrm{d}x \qquad \forall z_h \in \mathcal{V}_{hp}.
$$

Hence, with $\mathtt{h}$ signifying the positive piecewise constant function defined on $\mathcal{T}_h$ such that $\mathtt{h}(x) = h_\kappa$ for all $x \in \kappa$ and all $\kappa \in \mathcal{T}_h$,

$$
\begin{aligned}
\left|\int_\Omega (u-u_h)\,\psi\,\mathrm{d}x\right| &\le \|\mathtt{h}\mathtt{R}_\Omega\|_{\mathrm{L}^1(\Omega)}\|\mathtt{h}^{-1}(z-z_h)\|_{\mathrm{L}^\infty(\Omega)} + \|\mathtt{h}\mathtt{R}_\Gamma\|_{\mathrm{L}^1(\Gamma_-)}\|\mathtt{h}^{-1}(z-z_h)\|_{\mathrm{L}^\infty(\Gamma_-)} \\
&\le \left(\|\mathtt{h}\mathtt{R}_\Omega\|_{\mathrm{L}^1(\Omega)} + \|\mathtt{h}\mathtt{R}_\Gamma\|_{\mathrm{L}^1(\Gamma_-)}\right)\|\mathtt{h}^{-1}(z-z_h)\|_{\mathrm{L}^\infty(\Omega)} \qquad \forall z_h \in \mathcal{V}_{hp}.
\end{aligned}
$$

Thus,

$$
\left|\int_\Omega (u-u_h)\,\psi\,\mathrm{d}x\right| \le \left(\|\mathtt{h}\mathtt{R}_\Omega\|_{\mathrm{L}^1(\Omega)} + \|\mathtt{h}\mathtt{R}_\Gamma\|_{\mathrm{L}^1(\Gamma_-)}\right)\inf_{z_h\in\mathcal{V}_{hp}}\|\mathtt{h}^{-1}(z-z_h)\|_{\mathrm{L}^\infty(\Omega)}.
$$

Using a standard approximation property of the finite element space $\mathcal{V}_{hp}$ in the $\mathrm{L}^\infty(\Omega)$ norm, we deduce that

$$
\left|\int_\Omega (u-u_h)\,\psi\,\mathrm{d}x\right| \le K_{\mathrm{approx}}\left(\|\mathtt{h}\mathtt{R}_\Omega\|_{\mathrm{L}^1(\Omega)} + \|\mathtt{h}\mathtt{R}_\Gamma\|_{\mathrm{L}^1(\Gamma_-)}\right)|z|_{\mathrm{Lip}(\Omega)},
$$

where K_{approx} is a positive constant, dependent only on the shape-regularity of the family $\{\mathcal{T}_h\}_{h>0}$.

For the sake of simplicity, we shall assume henceforth that b is a constant vector with positive entries. Since $\psi \in \mathrm{Lip}_0(\Omega)$ vanishes on Γ, it follows by hyperbolic regularity theory that

$$c_0 \|z\|_{\mathrm{L}^\infty(\Omega)} \leq \|\psi\|_{\mathrm{L}^\infty(\Omega)}$$

and

$$\begin{aligned} c_0 \|\nabla z\|_{\mathrm{L}^\infty(\Omega)} &\leq \|\nabla\psi\|_{\mathrm{L}^\infty(\Omega)} + \|\nabla c\|_{\mathrm{L}^\infty(\Omega)} \|z\|_{\mathrm{L}^\infty(\Omega)} \\ &\leq \|\nabla\psi\|_{\mathrm{L}^\infty(\Omega)} + c_0^{-1} \|\nabla c\|_{\mathrm{L}^\infty(\Omega)} \|\psi\|_{\mathrm{L}^\infty(\Omega)}. \end{aligned}$$

Furthermore, because $\psi|_\Gamma = 0$, we have that

$$\|\psi\|_{\mathrm{L}^\infty(\Omega)} \leq \mathrm{diam}(\Omega) \|\nabla\psi\|_{\mathrm{L}^\infty(\Omega)},$$

and therefore,

$$|z|_{\mathrm{Lip}(\Omega)} \leq K_{\text{stab}} |\psi|_{\mathrm{Lip}(\Omega)},$$

where

$$K_{\text{stab}} = c_0^{-1} \left(1 + c_0^{-1} \mathrm{diam}(\Omega) \|\nabla c\|_{\mathrm{L}^\infty(\Omega)}\right).$$

Thus,

$$\left| \int_\Omega (u - u_h)\, \psi \,\mathrm{d}x \right| \leq K_{\text{stab}} K_{\text{approx}} \left(\|\mathtt{hR}_\Omega\|_{\mathrm{L}^1(\Omega)} + \|\mathtt{hR}_\Gamma\|_{\mathrm{L}^1(\Gamma_-)} \right) |\psi|_{\mathrm{Lip}(\Omega)}$$

for all $\psi \in \mathrm{Lip}_0(\Omega)$, whereby,

$$\|u - u_h\|_{\mathrm{Lip}'(\Omega)} \leq K_{\text{stab}} K_{\text{approx}} \left(\|\mathtt{hR}_\Omega\|_{\mathrm{L}^1(\Omega)} + \|\mathtt{hR}_\Gamma\|_{\mathrm{L}^1(\Gamma_-)} \right) =: \mathtt{Apost}_{\mathrm{Lip}'}.$$

Furthermore,

$$\|u - u_h\|_{\mathrm{BV}(\Omega)} \leq \|u\|_{\mathrm{BV}(\Omega)} + \|u_h\|_{\mathrm{BV}(\Omega)} \leq K_{\text{stab}} \|f\|_{\mathrm{BV}(\Omega)} + \|u_h\|_{\mathrm{BV}(\Omega)} =: \mathtt{Apost}_{\mathrm{BV}},$$

where we have used that $\|u\|_{\mathrm{BV}(\Omega)} \leq K_{\text{stab}} \|f\|_{\mathrm{BV}(\Omega)}$.

We note that, trivially, $\mathrm{BV}(\Omega) \subset \mathrm{L}^1(\Omega)$ and, when $\Omega \subset \mathbb{R}^n$ is bounded, as is the case in this section, $\mathrm{L}^1(\Omega) \subset \mathrm{Lip}'(\Omega)$; therefore also $\mathrm{BV}(\Omega) \subset \mathrm{Lip}'(\Omega)$ and thereby $\mathrm{Lip}'(\Omega) \cap \mathrm{BV}(\Omega) = \mathrm{BV}(\Omega)$.

It now follows from (12) that

$$\begin{aligned}\|u-u_h\|_{\mathrm{L}^1(\Omega')} &\leq \sqrt{C_1}\|u-u_h\|^{1/2}_{\mathrm{Lip}'(\Omega)} \max\left(\frac{\sqrt{C_1}}{d'}\|u-u_h\|^{1/2}_{\mathrm{Lip}'(\Omega)}, \sqrt{C_2}\|u-u_h\|^{1/2}_{\mathrm{BV}(\Omega)}\right)\\ &\quad+\sqrt{C_1C_2}\|u-u_h\|^{1/2}_{\mathrm{Lip}'(\Omega)}\|u-u_h\|^{1/2}_{\mathrm{BV}(\Omega)}\\ &\leq \sqrt{C_1}\,\mathtt{Apost}^{1/2}_{\mathrm{Lip}'}\max\left(\frac{\sqrt{C_1}}{d'}\,\mathtt{Apost}^{1/2}_{\mathrm{Lip}'}, \sqrt{C_2}\,\mathtt{Apost}^{1/2}_{\mathrm{BV}}\right)\\ &\quad+\sqrt{C_1C_2}\,\mathtt{Apost}^{1/2}_{\mathrm{Lip}'}\,\mathtt{Apost}^{1/2}_{\mathrm{BV}},\end{aligned}$$

in any domain $\Omega' \subset\subset \Omega$, with $d' = \mathrm{dist}(\partial\Omega', \partial\Omega)$.

We note, in particular, that if

$$\frac{\sqrt{C_1}}{d'}\,\mathtt{Apost}^{1/2}_{\mathrm{Lip}'} \leq \sqrt{C_2}\,\mathtt{Apost}^{1/2}_{\mathrm{BV}},$$

that is if

$$\mathtt{Apost}_{\mathrm{Lip}'} \leq \frac{C_2(d')^2}{C_1}\,\mathtt{Apost}_{\mathrm{BV}}, \tag{19}$$

then the following conditional *a posteriori* error bound holds:

$$\|u-u_h\|_{\mathrm{L}^1(\Omega')} \leq 2\sqrt{C_1C_2}\,\mathtt{Apost}^{1/2}_{\mathrm{Lip}'}\,\mathtt{Apost}^{1/2}_{\mathrm{BV}},$$

the *condition* being inequality (19).

Since the quantity $\mathtt{Apost}_{\mathrm{BV}}$ featuring in the right-hand side of (19) is expected to be, at best, of size $\mathcal{O}(1)$ as $h \to 0$, while the left-hand side of (19) is anticipated to decay as $\mathcal{O}(h^{2s})$ with some $s \in (0, 1/2]$, we expect to be able to set $d' := \mathrm{dist}(\partial\Omega', \partial\Omega) = \mathcal{O}(h^s)$ as $h \to 0$.

This is the desired *a posteriori* bound on the error between the analytical solution $u \in \mathrm{BV}(\Omega)$ and its finite element approximation $u_h \in \mathcal{V}_{hp}$, in terms of the computable domain and boundary residuals, the numerical solution u_h and the data.

Acknowledgements: I wish to express my sincere gratitude to Ricardo Nochetto (University of Maryland), Christoph Ortner (University of Oxford) and Maria Schonbek (University of California at Santa Cruz) for helpful comments and stimulating discussions on the subject of this paper.

References

[1] M. Ainsworth, J.T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley Interscience, New York (2000).

[2] G. Anzellotti, M. Giaquinta, *Funzioni BV e tracce*, Rend. Sem. Mat. Padova, **60** (1978), pp. 1–21.

[3] R. Becker, R. Rannacher, *An optimal control approach to a posteriori error estimation in finite element methods*, Acta Numerica 2001, pp. 1–101, Cambridge University Press, Cambridge (2001).

[4] W. Bangerth, R. Rannacher, *Adaptive Finite Element Methods for Differential Equations*, Birkhauser Verlag, Basel (2003).

[5] K. Eriksson, D. Estep, P. Hansbo, C. Johnson, *Introduction to adaptive methods for differential equations,* Acta Numerica 1995, pp. 105–158, Cambridge University Press, Cambridge (1995).

[6] M. B. Giles, E. Süli, *Adjoint methods for PDEs: a posteriori error analysis and postprocessing by duality*, Acta Numerica 2002, pp. 145-236, Cambridge University Press, Cambridge (2002).

[7] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, Vol. **80**, Birkhäuser Verlag, Boston, Basel, Stuttgart (1984).

[8] Y. Meyer, *Oscillating patterns in image processing and in some nonlinear evolution equations*, The Fifteenth Dean Jacqueline B. Lewis Memorial Lectures, November 1, 2002.

[9] H. Nessyahu, E. Tadmor, *The convergence rate of approximate solutions for nonlinear scalar conservation laws*, SIAM J. Numer. Anal. **29** (1992), pp. 1505–1519.

[10] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey (1970).

[11] E. Tadmor, *Local error estimates for discontinuous solutions of nonlinear hyperbolic equations*, SIAM J. Numer. Anal., **28** (1991), pp. 891–906.

[12] R. Verfürth, *A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Teubner-Verlag and J. Wiley, Stuttgart (1996).

[13] W. P. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Mathematics, Springer-Verlag, Berlin (1989).