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RECONSTRUCTION OF THE RIGHT-HAND SIDE OF THE ELLIPTICAL EQUATION FROM OBSERVATION DATA OBTAINED AT THE BOUNDARY

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Abstract. We consider the classical inverse problem in the potential theory in which it is required to determine the unknown right-hand side of the elliptic equation in the case in which additional data are set on the boundary of the calculation domain. To simplify the consideration, restrict ourselves to the two-dimensional Poisson equation.

1. STATEMENT OF THE INVERSE PROBLEM

Consider a model inverse problem in which it is required to determine the unknown right-hand side from observation data obtained at the domain boundary. To simplify the consideration, restrict ourselves to the two-dimensional Poisson equation. Consider first the formulation of the direct problem.

In a bounded domain Ω the function $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2)$ satisfies the equation

$$-\Delta u \equiv -\sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2} = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega. \tag{1}$$

Consider a Dirichlet problem in which equation (1) is supplemented with the following first-kind homogeneous boundary conditions:

$$u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\Omega. \tag{2}$$

The direct problem is formulated in the form (1), (2), with known right-hand side $f(\boldsymbol{x})$ in (1).

Among the inverse problems for elliptic equation, consider the right-hand side identification problem. We assume that additional measurements are feasible only on the domain boundary. In addition to (2), the following second-kind boundary conditions are also set:

$$\frac{\partial u}{\partial n}(\boldsymbol{x}) = \mu(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega, \tag{3}$$

where n is the external normal to Ω.

In this general formulation the solution of the inverse problem in which it is required to determine the pair of functions $\{u(\boldsymbol{x}), f(\boldsymbol{x})\}$ from conditions (1)–(3) is not unique [1]. The latter statement requires no special comments: it suffices to consider the inverse problem in a circle with the right-hand side dependent on the distance from the center of the circle. The non-uniqueness stems from the fact that we are trying to reconstruct a two-dimensional function (the right-hand side $f(\boldsymbol{x})$) from a function with lower dimensionality ($\mu(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$).

2. UNIQUENESS OF THE INVERSE-PROBLEM SOLUTION

Unique determination of the right-hand side is possible in the case in which the unknown right-hand side is independent of one of the variables [2]. Not trying to consider the general case, turn to a typical example [3]. We assume that the right-hand side (1) can be represented as

$$f(\boldsymbol{x}) = \varphi_1(x_2) + x_1\varphi_2(x_2). \tag{4}$$

We pose a problem in which it is required to determine two functions $\varphi_\alpha(x_2)$, $\alpha = 1, 2$, independent of one of the variables (namely, of the variable x_1), from (1)–(3).

We reformulate the inverse problem (1)–(4) by eliminating the unknown functions $\varphi_\alpha(x_2)$, $\alpha = 1, 2$. Double differentiation of (1) with respect to x_1 with allowance for (4) gives:

$$\frac{\partial^2}{\partial x_1^2} \Delta u = 0, \qquad \boldsymbol{x} \in \Omega. \tag{5}$$

In this way, we arrive at a boundary-value problem for the composite equation (2), (3), (5).

Let us show that the solution of problem (2), (3), (5) is unique. For this to be shown, it suffices to prove that the solution of the problem with homogeneous boundary conditions

$$\frac{\partial u}{\partial n}(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\Omega, \tag{6}$$

is $u(\boldsymbol{x}) \equiv 0$, $\boldsymbol{x} \in \Omega$.

We multiply equation (5) by $u(\boldsymbol{x})$ and perform integration over the whole domain Ω; this yields

$$\int_\Omega \frac{\partial^2}{\partial x_1^2} \Delta u \, u \, d\boldsymbol{x} = 0.$$

Taking into account the homogeneous boundary conditions (2) and permutability of the operators $\partial/\partial x_1$ and Δ, we obtain:

$$\int_\Omega v \, \Delta v \, d\boldsymbol{x} = 0, \quad v = \frac{\partial u}{\partial x_1}.$$

Conditions (2), (6) guarantees that

$$v(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\Omega.$$

Under these conditions, we have

$$\int_\Omega v \, \Delta v \, d\boldsymbol{x} = \sum_{\alpha=1}^{2} \int_\Omega \Big(\frac{\partial v}{\partial x_\alpha} \Big)^2 d\boldsymbol{x} = 0$$

and, hence, $v(\boldsymbol{x}) = 0$ throughout the whole domain Ω. From

$$\frac{\partial u}{\partial x_1} = 0, \qquad \boldsymbol{x} \in \Omega$$

and boundary conditions (2) it follows that the only solution of problem (2), (5), (6) is $u(\boldsymbol{x}) \equiv 0$, $\boldsymbol{x} \in \Omega$.

More informative a priori estimates for the solution of the boundary-value problem (2), (3), (5) can also be obtained. This matter will be considered on the difference level below.

3. DIFFERENCE PROBLEM

We assume the calculation domain to be a rectangle:

$$\Omega = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1, x_2),\ 0 < x_\alpha < l_\alpha,\ \alpha = 1, 2\}.$$

For the sides of Ω we use designations indicated in Figure 1, so that

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4.$$

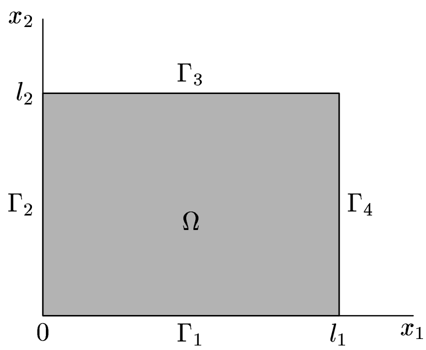

Figure 1: Calculation domain

We seek the right-hand side of (1) in class (4) under the following additional conditions posed on the sides Γ_2 and Γ_4 of the rectangle:

$$\frac{\partial u}{\partial x_1}(0, x_2) = \mu_1(x_2), \qquad \frac{\partial u}{\partial x_1}(l_1, x_2) = \mu_2(x_2). \tag{7}$$

With boundary conditions set on Γ_1 and/or Γ_3 (see (3)), the problem becomes over-specified.

Along both direction x_α, $\alpha = 1, 2$, we introduce a uniform grid

$$\overline{\omega}_\alpha = \{x_\alpha \mid x_\alpha = i_\alpha h_\alpha,\ i_\alpha = 0, 1, \dots, N_\alpha,\ N_\alpha h_\alpha = l_\alpha\},$$

so that

$$\omega_\alpha = \{x_\alpha \mid x_\alpha = i_\alpha h_\alpha,\ i_\alpha = 1, 2, \dots, N_\alpha - 1,\ N_\alpha h_\alpha = l_\alpha\},$$

$$\partial\omega_\alpha = \{x_\alpha \mid x_\alpha = 0, l_\alpha\}.$$

For the grid in the rectangle Ω we use the designations

$$\overline{\omega} = \overline{\omega}_1 \times \overline{\omega}_2 = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1, x_2),\ x_\alpha \in \overline{\omega}_\alpha,\ \alpha = 1, 2\},$$

$$\omega = \omega_1 \times \omega_2.$$

In the standard notation adopted in the theory of difference schemes[4], at internal nodes we define the difference Laplace operator

$$\Lambda y = y_{\bar{x}_1 x_1} + y_{\bar{x}_2 x_2}, \qquad \boldsymbol{x} \in \omega.$$

We put into correspondence to the direct problem (1), (2) the difference problem

$$-\Lambda y = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \omega, \tag{8}$$

$$y(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\omega. \tag{9}$$

In the inverse problem, the right-hand side is sought in the class (4) from the additional conditions (7). To pass to a difference analogue of (2), (4), (7), we define the mesh function $v = -\Lambda y$ not only at internal nodes (see (8)), but also on the set of boundary nodes.

We can conveniently introduce fictitious nodes with $i_1 = -1$ $i_1 = N_1 + 1$ to extend the grid over the variable x_1 by one node from either side. We approximate boundary conditions (7) on the extended grid. Accurate to $O(h_1^2)$, we have:

$$\frac{y(h_1, x_2) - y(-h_1, x_2)}{2h_1} = \mu_1(x_2), \tag{10}$$

$$\frac{y(l_1 + h_1, x_2) - y(l_1 - h_1, x_2)}{2h_1} = \mu_2(x_2). \tag{11}$$

Taking into account the boundary conditions (2) at the left boundary, we obtain:

$$v(0, x_2) = -\Lambda y(0.x_2) = -\frac{y(h_1, x_2) - 2y(0, x_2) + y(-h_1, x_2)}{h_1^2}.$$

With (10) taken into account, we arrive at the expression

$$v(0, x_2) = -\frac{2}{h_1^2}\, y(h_1, x_2) + \frac{2}{h_1}\, \mu_1(x_2). \tag{12}$$

In a similar way, on Γ_4 we obtain:

$$v(l_1, x_2) = -\frac{2}{h_1^2}\, y(l_1 - h_1, x_2) - \frac{2}{h_1}\, \mu_1(x_2). \tag{13}$$

Double difference differentiation (8) yields the equation

$$v_{\bar{x}_1 x_1} = 0, \qquad \boldsymbol{x} \in \omega. \tag{14}$$

The boundary conditions for this equations have the form (11), (12). With known v, the solution is to be determined (see (8), (9)) from

$$-\Lambda y = v(\boldsymbol{x}), \qquad \boldsymbol{x} \in \omega, \tag{15}$$

$$y(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\omega. \tag{16}$$

In this manner, we arrive at a system of two difference Poisson equations for the pair $\{y, v\}$. These two equations are interrelated via boundary conditions (12), (13).

We can conveniently reformulate the boundary-value problem with non-homogeneous boundary conditions (12)–(14) as a problem with homogeneous boundary conditions for a non-homogeneous equation at the internal nodes. With (12), at near-boundary nodes we have:

$$\frac{2v(h_1, x_2) - v(2h_1, x_2)}{h_1^2} = \frac{v(0, x_2)}{h_1^2} = -\frac{2}{h_1^4}\, y(h_1, x_2) + \frac{2}{h_1^3}\, \mu_1(x_2).$$

From (13), we obtain:

$$\frac{2v(l_1 - h_1, x_2) - v(l_1 - 2h_1, x_2)}{h_1^2} = -\frac{2}{h_1^4}\, y(l_1 - h_1, x_2) - \frac{2}{h_1^3}\, \mu_2(x_2).$$

Let us define difference operators A_α, $\alpha = 1, 2$ on the set of mesh functions vanishing at the boundary nodes:

$$A_\alpha y = -y_{\bar{x}_\alpha x_\alpha}, \qquad \boldsymbol{x} \in \omega.$$

Then, the boundary-value problem (12)–(14) can be written as

$$A_1 v = -A_0 y + \phi, \qquad \boldsymbol{x} \in \omega. \tag{17}$$

Here, the difference operator A_0 is defined by

$$A_0 y = \begin{cases} \dfrac{2}{h_1^4}\, y(h_1, x_2), & x_1 = h_1, \\ 0, & h_1 < x_1 < l_1 - h_1, \\ \dfrac{2}{h_1^4}\, y(l_1 - h_1, x_2), & x_1 = l_1 - h_1. \end{cases}$$

The right-hand side ϕ in nonzero only at near-boundary nodes:

$$\phi = \begin{cases} -\dfrac{2}{h_1^3}\, \mu_1(x_2), & x_1 = h_1, \\ 0, & h_1 < x_1 < l_1 - h_1, \\ \dfrac{2}{h_1^3}\, \mu_2(x_2), & x_1 = l_1 - h_1. \end{cases}$$

In the introduced notation, the boundary-value problem (15), (16) assumes the form

$$(A_1 + A_2) y = v, \qquad \boldsymbol{x} \in \omega. \tag{18}$$

In this way, we pass from (12)–(16) to system (17), (18). The latter system can be conveniently written as the following single operator equation:

$$A y = \phi. \tag{19}$$

Here,

$$A = (A_1 + A_2) A_1 + A_0. \tag{20}$$

The mesh pattern used in this difference scheme is shown in Figure 2.

In the ordinary way, in the Hilbert space $H = L_2(\omega)$ we introduce the scalar product and the norm:

$$(y, w) \equiv \sum_{\boldsymbol{x} \in \omega} y(\boldsymbol{x}) w(\boldsymbol{x}) h_1 h_2, \qquad \|y\| \equiv (y, y)^{1/2}.$$

In H,

$$A_\alpha = A_\alpha^* > 0, \qquad A_0 = A_0^* \geq 0$$

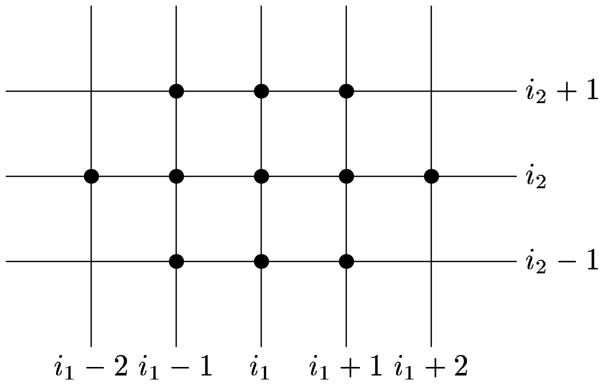

Figure 2: The mesh pattern for the difference scheme

and, hence, in (19) we have $A = A^* > 0$. By virtue of this, the difference problem (19) has a unique solution.

We gave a sufficiently general scheme for constructing a discrete analogue to the non-classical boundary-value problem (2), (5), (7) suitable for solving even more complex problems. In the case under consideration, we can substantially simplify the problem by explicitly writing the solution of problem (12)–(14). It should be noted that such transformation allows for specific features of the inverse problem in the greatest possible extent and most clearly on the difference level.

The general solution of the difference equation (14) is a linear function of x_1:

$$v(x_1, x_2) = \Big(1 - \frac{x_1}{l_1}\Big) v(0, x_2) + \frac{x_1}{l_1}\, v(l_1, x_2).$$

With boundary conditions (12), (13) taken into account, we obtain:

$$v(x_1, x_2) = -\Big(1 - \frac{x_1}{l_1}\Big) \frac{2}{h_1^2}\, y(h_1, x_2) - \frac{x_1}{l_1}\, \frac{2}{h_1^2}\, y(l_1 - h_1, x_2) + \psi(x_1, x_2), \qquad (21)$$

where

$$\psi(x_1, x_2) = \Big(1 - \frac{x_1}{l_1}\Big)\, \frac{2}{h_1}\, \mu_1(x_2) - \frac{x_1}{l_1}\, \frac{2}{h_1}\, \mu_2(x_2).$$

Substitution of (21) into (8) leads us to the difference equation

$$-\Lambda y + \Big(1 - \frac{x_1}{l_1}\Big) \frac{2}{h_1^2}\, y(h_1, x_2) + \frac{x_1}{l_1}\, \frac{2}{h_1^2}\, y(l_1 - h_1, x_2) = \psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \omega. \qquad (22)$$

In this way, the solution of the inverse right-hand side identification problem for equation (8) in class (4) has reduced to the solution of the boundary-value problem (9), (22). Equation (22) is a loaded difference equation.

4. SOLUTION OF THE DIFFERENCE PROBLEM

To find the solution of the difference problem, we use the variable separation method [5]. This approach can be applied to the difference problem (9) with the composite difference operator A defined by (10). Let us dwell on a second possibility, when to be sought is the solution of the boundary-layer problem for the loaded difference elliptic equation (9), (22).

We write problem (9), (22) as the equation

$$\begin{gathered}(A_1 + A_2)y + q_1(x_1)y(h_1, x_2) + q_2(x_1)y(l_1 - h_1, x_2) = \psi(\boldsymbol{x}), \\ \boldsymbol{x} \in \omega,\end{gathered} \tag{23}$$

in which $q_\alpha \geq 0$, $\alpha = 1, 2$.

We denote as λ_k, $v_k(x_2)$, $k = 1, 2, \dots, N_2 - 1$ the eigenvalues and eigenfunctions of A_2:

$$A_2 v = \lambda v.$$

The solution of this difference spectral problem is well known:

$$\lambda_k = \frac{4}{h_2^2} \sin^2 \frac{k\pi h_2}{2l_2}, \quad v_k(x_2) = \sqrt{\frac{2}{l_2}} \sin \frac{k\pi x_2}{l_2}.$$

The eigenfunctions are orthonormal functions:

$$(v_k, v_m)^{(2)} = \delta_{km}, \qquad \delta_{km} = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases}$$

where

$$(v, w)^{(2)} = \sum_{i_2=1}^{N_2-1} v(x_2)w(x_2)h_2$$

is the scalar product in $L_2(\omega_2)$.

We seek the solution of problem (23) as an expansion in the eigenfunctions of A_2:

$$y(x_1, x_2) = \sum_{k=1}^{N_2-1} c_k(x_1) v_k(x_2).$$

Substitution into (23) leads us to the necessity to solve the difference problems

$$(A_1 + \lambda_k) c_k(x_1) + q_1(x_1) c_k(h_1) + q_2(x_1) c_k(l_1 - h_1) = \psi_k(x_1), \tag{24}$$

where

$$\psi_k(x_1) = (\psi, v_k)^{(2)}, \qquad k = 1, 2, \ldots, N_2 - 1.$$

The matter of solution of these $N_2 - 1$ one-dimensional difference problems should be given particular attention. In the case of (24) it is required to find the solution of the difference boundary-value problem

$$-w_{\bar{x}_1 x_1} + \lambda w + q_1(x_1) w(h_1) + q_2(x_1) w(l_1 - h_1) = r(x_1), \tag{25}$$

$$w(0) = 0, \qquad w(l_1) = 0. \tag{26}$$

We represent the solution of problem (25), (26) as

$$w(x_1) = s(x_1) + s^{(1)}(x_1) w(h_1) + s^{(2)}(x_1) w(l_1 - h_1). \tag{27}$$

We insert this expression into (25) and isolate the functions s_α, $\alpha = 1, 2$, collecting the terms with $w(h_1)$, $w(l_1 - h_1)$ and equating them to zero. This gives us three three-point difference equations for the auxiliary functions $s(x_1)$, $s_\alpha(x_1)$, $\alpha = 1, 2$. With allowance for (26), we assume the boundary conditions to be homogeneous, so that

$$-s_{\bar{x}_1 x_1} + \lambda s = r(x_1), \tag{28}$$

$$s(0) = 0, \qquad s(l_1) = 0, \tag{29}$$

$$-s^{(1)}_{\bar{x}_1 x_1} + \lambda s^{(1)} = -q_1(x_1), \tag{30}$$

$$s^{(1)}(0) = 0, \qquad s^{(1)}(l_1) = 0, \tag{31}$$

$$-s^{(2)}_{\bar{x}_1 x_1} + \lambda s^{(2)} = -q_2(x_1). \tag{32}$$

$$s^{(2)}(0) = 0, \qquad s^{(2)}(l_1) = 0. \tag{33}$$

After solving the three standard problems (28)–(33), the functions $w(h_1)$ and $w(l_1 - h_1)$ can be found. From representation (27), it readily follows that

$$w(h_1) = s(h_1) + s^{(1)}(h_1)w(h_1) + s^{(2)}(h_1)w(l_1 - h_1), \tag{34}$$

$$w(l_1 - h_1) = s(l_1 - h_1) + s^{(1)}(l_1 - h_1)w(h_1) + s^{(2)}(l_1 - h_1)w(l_1 - h_1). \tag{35}$$

Solvability of this system is controlled by the determinant

$$D = (1 - s^{(1)}(h_1))(1 - s^{(2)}(l_1 - h_1)) - s^{(2)}(h_1)s^{(1)}(l_1 - h_1),$$

whose nonzero value can easily be guaranteed under certain constraints. Taking the inequalities $\lambda > 0$ and $q_\alpha \geq 0$, $\alpha = 1, 2$ into account, we have: $s^{(\alpha)}(x_1) \geq 0$, $0 \leq x_1 \leq l_1$. Hence, we have $D > 0$ at sufficiently small l_1, for instance.

As a matter of fact, the determinant of (34), (35) is always positive. To show this, we have to recall (see (21)) the expressions for the mesh functions $q_\alpha(x_1)$, $\alpha = 1, 2$:

$$q_1(x_1) = \left(1 - \frac{x_1}{l_1}\right)\frac{2}{h_1^2}, \qquad q_2(x_1) = \frac{x_1}{l_1}\,\frac{2}{h_1^2}.$$

By virtue of this, for the solutions of the boundary-value problems (30), (31) and (32), (33) there holds the relations

$$s^{(1)}(x_1) = s^{(2)}(l_1 - x_1), \qquad 0 \leq x_1 \leq l_1.$$

The latter means, in particular, that it is unnecessary for us to solve (in the case of the uniform computational grid used) one of the two difference boundary-value problem, (30), (31) or (32), (33). Hence,

$$D = 1 - 2s^{(1)}(h_1) + (s^{(1)}(h_1))^2 - (s^{(1)}(l_1 - h_1))^2$$

and, with allowance for $s^{(1)}(h_1) > s^{(1)}(l_1 - h_1)$, we obtain that $D > 1$.

With the mesh functions $s(x_1)$, $s^{(\alpha)}$, $\alpha = 1, 2$ found from (28)–(33) and with the mesh functions $w(h_1)$ and $w(l_1 - h_1)$ found from (34), (35), the solution of problem (25), (26) can be found in the form (27). The program realization of this algorithm is discussed below.

5. CALCULATION DATA

The developed program solves the inverse problem (1), (2), (7), in which to be found is the right-hand side (4) with

$$\varphi_1(x_2) = \begin{cases} 1, & 0 < x_2 < 0.5, \\ 0, & 0.5 < x_2 < 1, \end{cases} \qquad \varphi_1(x_2) = x_2.$$

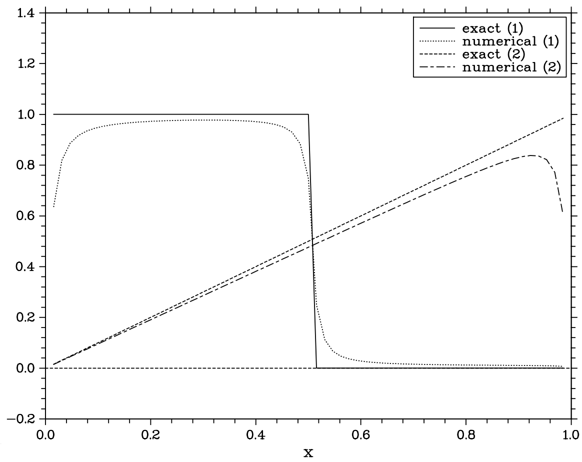

Figure 3: Solution of the problem obtained on the grid $N_1 = N_2 = 65$

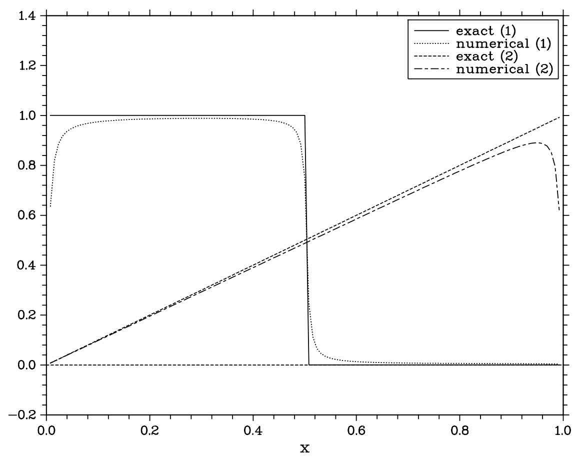

Figure 4: Solution of the problem obtained on the grid $N_1 = N_2 = 129$

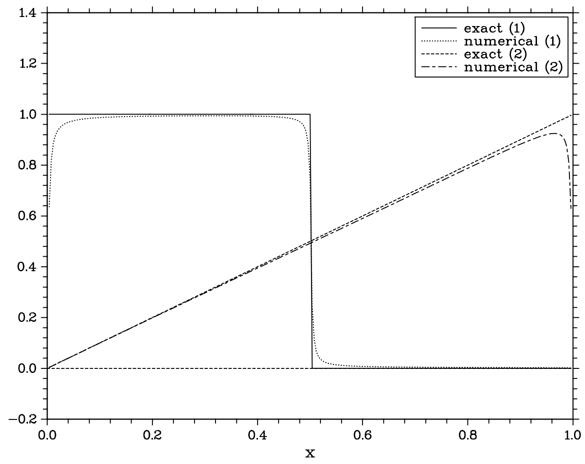

Figure 5: Solution of the problem obtained on the grid $N_1 = N_2 = 257$

The problem is solved in the unit square ($l_1 = l_2 = 1$). To obtain the input data for the inverse problem, the direct problem (1), (2) is preliminarily solved at a given right-hand side.

First of all, consider the solution data obtained for the inverse problem with undisturbed input data. Of interest here are numerical data obtained on a sequence of progressively refined grids (see Figures 3–5). The approximate solution is seen to converge to the exact solution. A sufficiently high accuracy can be obtained using refined grids.

More sensitive to input-data inaccuracies are calculation data obtained at different levels of boundary-condition inaccuracies (7). These inaccuracies were modeled in the ordinary way, for instance, as

$$\tilde{\mu}_1(x_2) = \mu_1(x_2) + 2\delta(\sigma(x_2) - 1/2), \qquad x_2 \in \omega_2,$$

where $\sigma(x_2)$ is a random function normally distributed over the interval $[0, 1]$, and the parameter δ defines the inaccuracy level. Figure 6 shows data obtained by solving the inverse problem with $\delta = 0.0003$. This inaccuracy level corresponds to a relative inaccuracy of 0.1%. The calculation grid $N_1 = N_2 = 129$ was used.

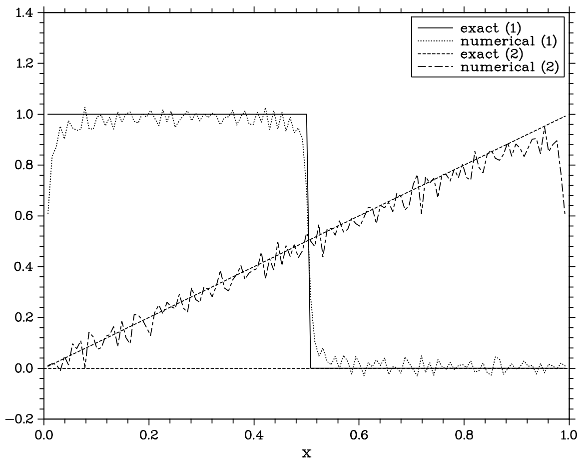

Figure 6: Solution of the problem obtained at the inaccuracy level $\delta = 0.0003$

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